Volume polynomials and the Heine–Shephard problem

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Abstract. We give necessary and sufficient conditions for a homogeneous polynomial to be the volume polynomial of compact convex sets, resolving the Heine–Shephard realization problem. For n sets in dimension $d \geq 3$, we prove that the coefficient locus is semialgebraic exactly when $n \leq 2$, ruling out any finite algebraic characterization for $n \geq 3$. For all n, d, we nevertheless characterize realizability by coefficient nonnegativity and an explicitly enumerable family of homogeneous quadratic inequalities with rational coefficients. For computable coefficient vectors, these conditions yield finite certificates of nonrealizability.

The nonsemialgebraicity obstruction occurs on an affine slice of ternary cubics whose boundary departs from an algebraic curve by a term of order $s^3 \log(1/s)$. The proof uses matrix-valued slope measures to represent planar mixed volume and a logarithmic energy estimate. We also show that finitely many mixed-volume relations can force a nonsemialgebraic realizing set: an irrational-rotation Minkowski series with infinitely many exposed edges, whose area function has the unit circle as a natural boundary. Consequently, some volume polynomials admit no realization by semialgebraic compact convex sets, and hence none by polytopes.

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1. Introduction

Given nonempty compact convex sets $K_1, \dots, K_n \subset \mathbb{R}^d$, Minkowski's theorem makes

$$F_K(t_1, \dots, t_n) = \mathrm{vol}_d(t_1 K_1 + \cdots + t_n K_n)$$

a homogeneous polynomial of degree d with nonnegative coefficients. The Heine–Shephard realization problem asks which polynomials arise in this way, or equivalently which numerical arrays can occur as all the mixed volumes of a fixed number of convex sets. It originates in work of Heine (1938) and Shephard (1960): Heine characterized three bodies in the plane, and Shephard characterized two bodies in every dimension [5, 9]. These classical characterizations are algebraic.

We resolve the realization problem for every number of sets and every dimension by giving a countable coefficient characterization. We first prove that in dimension at least three the passage from two sets to three precludes any finite algebraic characterization. We then construct an explicitly enumerable family of homogeneous quadratic inequalities with rational coefficients which, together with coefficient nonnegativity, is necessary and sufficient for realizability. The nonsemialgebraicity theorem and the countable characterization are separate results: the former rules out finite algebraic descriptions, while the latter supplies a complete criterion beyond that obstruction.

Let $\mathcal{V}_{n,d}$ denote the coefficient locus of these volume polynomials, including realizations by lower-dimensional sets. We keep every coefficient, including pure powers and repeated-index mixed volumes. A subset of a real vector space is *semialgebraic* if it is a finite Boolean combination of polynomial equalities and inequalities, with arbitrary real coefficients.

Theorem 1.1 (Sharp algebraic threshold)**.** *For every $d \geq 3$ and $n \geq 1$,*

$$\mathcal{V}_{n,d} \text{ is semialgebraic} \quad \Longleftrightarrow \quad n \leq 2.$$

The same nonsemialgebraicity holds when every realizing body is required to be a full-dimensional polytope. Thus no finite collection of polynomial inequalities, even with Boolean operations or finitely many auxiliary real variables, describes volume-polynomial realizability in these dimensions. This answers the semialgebraicity question of Averkov, Borger and Soprunov [1]. It also places a limit on descriptions by stronger log-concavity inequalities; the obstruction concerns every finite algebraic description, rather than a particular proposed family of inequalities. For the relation between volume polynomials and Lorentzian polynomials, see Brändén–Huh [4], Menges [7] and Huh [6].

The essential case is a ternary cubic. We specify its six repeated mixed volumes so that every realization consists of planar bodies with independent plane normals. After normalization, only the common area of two unit-width planar bodies and their interaction remain to be optimized. The lower boundary $b(s)$ on this affine coefficient slice satisfies

$$r(s) - b(s) \asymp s^3 \log(1/s), \qquad r(s) = \frac{1 + s + \sqrt{1 + s^2}}{2}, \qquad s \downarrow 0. \tag{1.1}$$

Here and below $f \asymp g$ means that their ratio is bounded above and below by positive constants. A semialgebraic boundary cannot have the logarithmic gap in (1.1).

The logarithm comes from the geometry of the planar bodies. Their four corner caps are encoded by positive measures on logarithmic slope space, with area given by the energy kernel $e^{-|x-y|/2}$. The operation representing triple mixed volume becomes multiplication of 2×2 Fourier matrices. A measure with small energy can retain appreciable total mass only by spreading over a long interval; the available exponential moments limit that interval to logarithmic length. Hyperbolic arcs, whose logarithmic slope measures are uniform, attain the resulting order.

To retain three bodies in higher dimensions, we raise the degree of an existing variable. The coefficient support forces the enlarged body to have planar fibers, whose Minkowski average satisfies a sharp area inequality; a cone construction attains the two moments entering that inequality. This preserves the logarithmic boundary comparison in every dimension.

The coefficients also constrain the shapes of realizing bodies. In the cubic slice they force three independent planes and leave a planar optimization problem. Additional equalities in the planar Minkowski inequality identify projected bodies up to translation; making three such identifications around a cycle of planes prescribes a linear return map, which we call *projection holonomy*.

Theorem 1.2 (Shape forcing)**.** *There is a finite system of polynomial conditions with rational coefficients on (λ, a, F), where $F \in \mathcal{V}_{9,3}$ and $0 < \lambda < 1$, whose projection to (λ, a) is exactly the graph*

$$a(\lambda) = \frac{1}{(1-\lambda^2)\sin\theta} \sum_{j\geq 1} |\sin(j\theta)|\lambda^j, \qquad \cos\theta = \frac{1}{3}.$$

In every realization, a specified body is affinely equivalent to

$$K_\lambda = \sum_{j\geq 0} \lambda^j M^j[-e_1/2, e_1/2], \qquad M = \begin{pmatrix} 0 & 1 \\ -1 & 2/3 \end{pmatrix}.$$

It has infinitely many exposed edges and is not semialgebraic. The area function a has rational Taylor coefficients and the unit circle as a natural boundary; in particular, it is transcendental over $\mathbb{R}(\lambda)$.

Consequently some volume polynomials have no realization by semialgebraic convex sets, and hence none by polytopes of any complexity. The obstruction persists in every higher dimension after adding one body. A second specialization of the same projection pattern forces ellipses, so the rigidity is already visible before the contraction series is introduced.

An exact coefficient criterion remains possible when infinitely many inequalities are allowed, and it can be given using only rational arithmetic.

Theorem 1.3 (Countable coefficient characterization)**.** *For every $n, d \geq 1$, there is an explicitly enumerable family of homogeneous rational quadratic forms Q_j such that*

$$c \in \mathcal{V}_{n,d} \quad \Longleftrightarrow \quad c_\alpha \geq 0 \text{ for every coefficient}, \qquad Q_j(c) \geq 0 \text{ for all } j.$$

Every nonrealizable vector strictly violates one of these inequalities. For computable input coefficients, nonrealizability therefore has a finite rational certificate. Realizable inputs admit a convergent construction by rational polytopes.

The construction rests on a uniform affine normalization and finite rational nets for convex bodies. Nonrealizability can be detected by a finite search, whereas the realizing polytopes are obtained by a convergent limiting construction; Section 8 gives the precise computational statements.

The two-body realization theorem supplies the algebraic side of the threshold; a proof is included in Appendix A.

2. Coefficients, projections and compactness

Our normalization of mixed volume is

$$\operatorname{vol}_d\Big(\sum_{i=1}^{n} t_i K_i\Big) = \sum_{|\alpha|=d} \frac{d!}{\alpha!} V(K_1[\alpha_1], \dots, K_n[\alpha_n]) t^\alpha,$$

where $K_i[\alpha_i]$ means that K_i occurs α_i times. In dimension three we write V_3, and in the plane we use

$$\operatorname{area}(sA + tB) = s^2 \operatorname{area}(A) + 2st V_2(A, B) + t^2 \operatorname{area}(B).$$

We use symmetry, multilinearity, nonnegativity, Hausdorff continuity and translation invariance of mixed volume, together with its covariance by the absolute determinant of a common linear map [8]. Independent translations of the individual bodies leave the volume polynomial unchanged.

2.1. Reading geometry from coefficients

The support of a volume polynomial records the dimensions of all partial sums. This elementary observation lets zero coefficients impose subspace constraints without any assumptions on the realizing bodies.

Lemma 2.1 (Degree and dimension)**.** *Suppose $K_1 + \cdots + K_n$ is full dimensional. For every nonempty $I \subseteq \{1, \dots, n\}$, the largest total exponent of the variables $(t_i)_{i \in I}$ in F_K is $\dim \sum_{i \in I} K_i$.*

Proof. Put $K = \sum_{i \in I} K_i$ and $L = \sum_{i \notin I} K_i$; when I is the whole index set, take $L = \{0\}$. Identifying the variables in I and setting the others equal to one gives $\operatorname{vol}_d(tK + L)$. Its degree is the stated largest exponent because the coefficients are nonnegative.

Translate K into a linear subspace U of dimension $r = \dim K$. For $0 < r < d$, the projection of L to $U^\perp$ is full dimensional there, and every nonempty fiber of $tK + L$ contains a translate of tK. Fubini therefore bounds the volume below by

$$t^r \operatorname{vol}_r(K) \operatorname{vol}_{d-r}(\pi_{U^\perp} L).$$

Boxes adapted to $U \oplus U^\perp$ give an upper bound $C(1 + t)^r$. Hence its degree is r. The cases $r = 0$ and $r = d$ follow directly from translation invariance and volume homogeneity. □

For a unit vector u, write $w_u(K)$ for the width of K in that direction. Widths and projected areas occur as coefficients through the following extrusion identity.

Lemma 2.2 (Extrusion)**.** *If $C \subset \mathbb{R}^d$ is compact and convex and $v \neq 0$, then*

$$\mathrm{vol}_d(C + t[0, v]) = \mathrm{vol}_d(C) + t|v| \,\mathrm{vol}_{d-1}(\pi_{v^\perp} C).$$

In particular, if $A \subset \mathbb{R}^3$ is planar with unit normal u, then

$$3V_3(A, A, B) = \mathrm{area}(A) w_u(B), \tag{2.1}$$

and for arbitrary compact convex $A, B \subset \mathbb{R}^3$,

$$3V_3(A, B, [0, v]) = |v| V_2(\pi_{v^\perp} A, \pi_{v^\perp} B). \tag{2.2}$$

Proof. Every nonempty fiber of C parallel to v is an interval, possibly a point, and adding $t[0, v]$ increases its length by $t|v|$. Integration proves the first assertion. Comparing coefficients after replacing C by a Minkowski sum proves (2.2); slicing perpendicular to the plane of A gives (2.1). □

For planar convex sets A, B of positive area, the Minkowski inequality gives

$$V_2(A, B)^2 \geq \mathrm{area}(A) \,\mathrm{area}(B), \tag{2.3}$$

with equality precisely when A and B are positively homothetic [8, Theorem 7.2.1]. Thus equality of both areas and their mixed area identifies the two bodies up to translation. This form applies to all compact convex sets of positive area; no boundary regularity is needed.

2.2. **A bounded model for realization**

The coefficient locus has a compact model after one normalization. We use the same model both to pass to limits in the geometric constructions and to obtain finite rational approximations later.

Lemma 2.3 (Uniform normalization)**.** *If $\mathrm{vol}_d(K_1 + \cdots + K_n) = 1$, independent translations and a common linear map of determinant modulus one put the tuple in the form*

$$0 \in L_i \subseteq [-d, d]^d, \qquad \sum_i L_i \subseteq [-d, d]^d.$$

Proof. Choose a simplex $\mathrm{conv}(v_0, \ldots, v_d)$ of maximum volume in $K = \sum_i K_i$, and set $E = (v_1 - v_0, \ldots, v_d - v_0)$ and $D = |\det E|$. Replacing v_j by any $x \in K$ shows that each coordinate of $E^{-1}(x - v_0)$ has absolute value at most one. Since the simplex lies in a set of volume one, $D \leq d!$; consequently

$$T = D^{1/d} E^{-1}, \qquad |\det T| = 1, \qquad T(K - v_0) \subseteq [-(d!)^{1/d}, (d!)^{1/d}]^d.$$

Write $v_0 = \sum_i p_i$ with $p_i \in K_i$ and take $L_i = T(K_i - p_i)$. Each L_i contains zero and lies in their sum, which is contained in $[-d, d]^d$. □

Let $\mathcal{K}_d$ be the compact metric space of nonempty compact convex subsets of $[-d, d]^d$ containing zero, with Hausdorff distance, and let $\Phi : \mathcal{K}_d^n \to \mathbb{R}^{\binom{n+d-1}{d}}$ be the coefficient map. Compactness follows from the uniformly bounded Lipschitz support functions on the unit sphere, and Φ is continuous by mixed-volume continuity. The image

$$\mathcal{D}_{n,d} = \Phi(\mathcal{K}_d^n) \tag{2.4}$$

is therefore compact. The normalization lemma says that every nonzero volume polynomial, divided by its value at $(1, \dots, 1)$, belongs to this image.

Proposition 2.4. *The locus $\mathcal{V}_{n,d}$ is closed. If a class of tuples is dense for the product Hausdorff metric, the closure of its coefficient image is $\mathcal{V}_{n,d}$.*

Proof. Suppose $c_j \in \mathcal{V}_{n,d}$ converges to c and write $s_j = \sum_\alpha (c_j)_\alpha$, $s = \sum_\alpha c_\alpha$. All coefficients are nonnegative, so $s = 0$ implies $c = 0$, which is realized by points. If $s > 0$, then eventually $c_j/s_j \in \mathcal{D}_{n,d}$; compactness puts c/s in the same image, and dilating its realizing bodies by $s^{1/d}$ realizes c. The second assertion now follows from continuity and density. □

3. **An affine slice of ternary cubics**

The obstruction will be found in the following family

$$\begin{aligned} P_{\alpha,s,z}(X_1, X_2, X_3) = {} & 3\alpha X_1^2 X_2 + 3\alpha X_1^2 X_3 + 3X_1 X_2^2 + 3X_2^2 X_3 \\ & + 3sX_1 X_3^2 + 3X_2 X_3^2 + 6zX_1 X_2 X_3, \end{aligned} \tag{3.1}$$

where $\alpha > 0$ and $0 < s < 1$. The pure coefficients vanish, while all six repeated mixed volumes are positive. This forces each body to be planar, and the asymmetry between the two cyclic products of repeated coefficients forces the planes into general position. The following normalization describes every realization of the slice.

Proposition 3.1. *The polynomial $P_{\alpha,s,z}$ belongs to $\mathcal{V}_{3,3}$ if and only if there are planar compact convex sets $A \subset xy$, $B \subset yz$, $C \subset xz$ and $T > 0$ satisfying*

body	*coordinate widths*	*area*	
A	$(1, s)$	αT	$z = \dfrac{6V_3(A, B, C)}{2T}.$
B	$(1, 1)$	T	
C	$(1, 1)$	T	

(3.2)

Proof. In any realization, a zero pure coefficient excludes dimension three, and a positive repeated coefficient excludes dimensions zero and one. The planes are pairwise distinct by (2.1). Translate them through the origin. If their normals were dependent, the three planes would share a line. Projecting perpendicular to this line, write u_i for a unit vector along the image of the ith plane, w_i for the projected width of its body and A_i for its area. Formula (2.1) would give

$$m_{iij} = \frac{1}{3} A_i w_j |\det(u_i, u_j)|, \qquad m_{ijk} = V_3(K_i, K_j, K_k),$$

and hence

$$m_{112} m_{223} m_{133} = m_{113} m_{122} m_{233}.$$

The two sides for (3.1) are αs and α, respectively. Since $s \neq 1$, the normals are independent.

A common linear map of determinant modulus one can therefore take the planes to xy, yz, xz. Denote the resulting widths of the three bodies by $(a, b), (c, \ell), (e, f)$. Comparing repeated coefficients using (2.1) yields

$$\ell = f, \qquad a = e, \qquad b = sc.$$

Apply the diagonal map $\operatorname{diag}(1/a, 1/c, 1/\ell)$. The widths now have the stated values, and the new areas are $\alpha T, T, T$ with $T = 3/(ac\ell)$. The triple mixed volume is divided by $ac\ell$, which gives (3.2).

Conversely, a normalized triple as in the proposition has repeated mixed volumes equal to $T/3$ times the six prescribed values, and its triple mixed volume is $Tz/3$. A common dilation by $(3/T)^{1/3}$ therefore gives exactly (3.1). □

We shall set $\alpha = 2s$, so the coefficient map $(s, z) \mapsto P_{2s,s,z}$ is affine. This choice restricts the common area to $T \leq 1/2$, since A has area $2sT$ inside a rectangle of area s. Define

$$b(s) = \inf\{z : P_{2s,s,z} \in \mathcal{V}_{3,3}\}, \qquad 0 < s < 1. \tag{3.3}$$

These fibers are nonempty; take B and C to be unit right triangles and A to be the rectangle of widths $(1, s)$. Then $T = 1/2$ and (3.2) gives $z = 1 + s$. Nonnegativity of mixed volume shows $0 \leq b(s) \leq 1 + s$.

The normalization has reduced the problem to planar geometry, but it has not yet separated the three bodies. The next section replaces the interaction of B and C by a single planar body Q. Once that replacement is available, optimizing A will be an equality case of one Cauchy–Schwarz inequality.

4. **Corner measures and triple mixed volume**

4.1. **The energy of a corner**

For a planar convex set in its bounding rectangle, the area deficit is the sum of four corner areas. There is a useful quadratic expression for each of them. Set

$$k(x) = e^{-|x|/2}, \qquad d\omega(\xi) = \frac{d\xi}{2\pi(\xi^2 + 1/4)},$$

so that ω is a probability measure and

$$k(x-y) = \int_{\mathbb{R}} e^{i\xi(x-y)}\, d\omega(\xi). \tag{4.1}$$

For finite real signed measures put

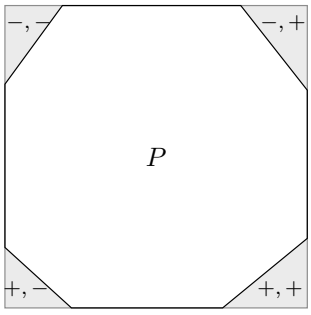
$$\langle \mu, \nu \rangle_k = \iint k(x-y)\, d\mu(x)\, d\nu(y), \qquad I(\mu) = \langle \mu, \mu \rangle_k.$$

With the Fourier convention $\widehat{\mu}(\xi) = \int e^{i\xi x}\, d\mu(x)$, formula (4.1) identifies $I(\mu)$ with $\int |\widehat{\mu}|^2\, d\omega$. In particular this energy is nonnegative and satisfies Cauchy–Schwarz.

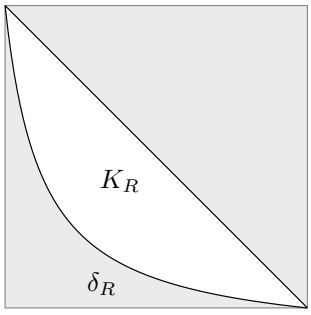
Orient the boundary of a convex polygon P counterclockwise. A non-axis edge (a, b) contributes the atom

$$\sqrt{|ab|}\, \delta_{\log|b/a|} \tag{4.2}$$

to a measure $\rho^P_{\sigma\tau}$, where $\sigma = \operatorname{sgn} a$ and $\tau = \operatorname{sgn} b$. Arrange these four measures in a 2×2 matrix, with rows and columns ordered $+, -$, and write $F_P = (\widehat{\rho}^P_{\sigma\tau})$. Axis edges contribute no atom. The exponential moments of an atom recover its horizontal and vertical lengths, which is the reason for both the logarithmic coordinate and the square-root weight in (4.2).



Four corner measures



A uniform slope measure

FIGURE 1. The bounding rectangle separates area into four corner energies. The hyperbolic arc on the right has constant mass density in logarithmic slope; it will supply the logarithmic lower bound.

Lemma 4.1 (Corner identities)**.** *If P has coordinate widths (w, h), then*

$$\operatorname{area}(P) = wh - \frac{1}{2}\sum_{\sigma,\tau} I(\rho^P_{\sigma\tau}), \tag{4.3}$$

$$\sum_{\tau}\int e^{-x/2}\, d\rho^P_{\sigma\tau}(x) \le w, \qquad \sum_{\sigma}\int e^{x/2}\, d\rho^P_{\sigma\tau}(x) \le h. \tag{4.4}$$

Corner measures add under Minkowski addition. Consequently, for polygons P, Q of widths (w_P, h_P) and (w_Q, h_Q),

$$2V_2(P, Q) = w_P h_Q + h_P w_Q - \sum_{\sigma,\tau}\langle \rho^P_{\sigma\tau}, \rho^Q_{\sigma\tau}\rangle_k. \tag{4.5}$$

For arbitrary planar compact convex sets,

$$2V_2(P, Q) \ge w_P h_Q + h_P w_Q - 2\sqrt{(w_P h_P - \operatorname{area}(P))(w_Q h_Q - \operatorname{area}(Q))}. \tag{4.6}$$

Proof. Translate the bounding rectangle to $[0, w] \times [0, h]$. Its complement consists of four caps, each bounded by a chain on which the signs of the edge components are fixed. After reflecting one cap, write the absolute edge components as (u_j, v_j) in slope order. The shoelace formula gives its area as

$$\frac{1}{2}\sum_j u_j v_j + \sum_{i<j}\min(u_i v_j, u_j v_i) = \frac{1}{2}\sum_{i,j}\min(u_i v_j, u_j v_i).$$

The identity

$$\min(u_i v_j, u_j v_i) = \sqrt{u_i v_j u_j v_i}\, e^{-\frac{1}{2}|\log(v_i/u_i) - \log(v_j/u_j)|}$$

makes this precisely $I(\rho)/2$. Summing the caps proves (4.3). The two exponential moments of (4.2) are $|a|$ and $|b|$, and each signed horizontal or vertical variation of a convex boundary equals the corresponding width. Omitting axis edges gives (4.4).

The edges of a Minkowski sum are obtained by merging edge directions and adding parallel edge vectors. At a fixed direction the atom location is fixed and its mass is linear in edge length; thus $\rho^{P+tQ} = \rho^P + t\rho^Q$. Comparing the coefficient of t in (4.3) proves (4.5). Cauchy–Schwarz in the direct sum of the four energy spaces yields (4.6). Degenerate polygons follow by adding a vanishing axis-parallel rectangle, and polygon approximation extends the inequality to all compact convex sets. □

4.2. **Admissible Fourier matrices**

Call a 2×2 matrix $\rho = (\rho_{ij})$ of finite positive measures *admissible* if

$$\sum_j \int e^{-x/2}\, d\rho_{ij}(x) \le 1, \qquad \sum_i \int e^{x/2}\, d\rho_{ij}(x) \le 1$$

for each row and column, respectively. Unit-width polygons give admissible matrices, and their area identity becomes

$$I_2(F_P) := \int \|F_P\|_{\mathrm{HS}}^2\, d\omega = 2(1 - \mathrm{area}(P)). \tag{4.7}$$

The moment constraints have a particularly useful operator consequence.

Lemma 4.2. *The Fourier matrix of an admissible matrix satisfies* $\|F(\xi)\|_{\mathrm{op}} \le 1$ *for every real* ξ.

Proof. For $u, v \in \mathbb{C}^2$, Cauchy–Schwarz with the reciprocal weights $e^{-x/2}$ and $e^{x/2}$ gives

$$\begin{aligned} |u^* F(\xi) v| &\le \sum_{i,j} \int |u_i||v_j|\, d\rho_{ij} \\ &\le \left(\sum_i |u_i|^2 \sum_j \int e^{-x/2}\, d\rho_{ij} \right)^{1/2} \left(\sum_j |v_j|^2 \sum_i \int e^{x/2}\, d\rho_{ij} \right)^{1/2} \\ &\le \|u\|\|v\|. \end{aligned}$$

□

4.3. **A planar body for the interaction of two planes**

We identify bodies in yz with their (y, z) coordinates and bodies in xz with their (x, z) coordinates. The next construction turns the triple mixed volume of A, B, C into a mixed area involving A.

Proposition 4.3. *Let* $B \subset yz$ *and* $C \subset xz$ *have coordinate widths* $(1, 1)$ *and positive area. There is a planar compact convex set* $Q \subset xy$ *of widths* $(1, 1)$ *such that, for every compact convex* $A \subset xy$,

$$6V_3(A, B, C) = 2V_2(A, Q). \tag{4.8}$$

Its area q *satisfies*

$$q \ge \max\{\mathrm{area}(B), \mathrm{area}(C)\}. \tag{4.9}$$

For polygonal inputs, its corner Fourier matrix is

$$F_Q = JF_B^* F_C, \qquad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.10}$$

Proof. Suppose first that B, C are polygons. For each pair of non-axis counterclockwise edges $b = (b_1, b_2)$ and $c = (c_1, c_2)$ with $b_1 c_1 > 0$, prescribe an edge of Q by

$$e_Q = (-|c_1| b_2,\ |b_1| c_2). \tag{4.11}$$

Each of its four signed coordinate totals is at most one. For example, the positive horizontal total is

$$\sum_{b : b_2 < 0} |b_2| \sum_{c : b_1 c_1 > 0} |c_1| \le \sum_{b : b_2 < 0} |b_2| \le 1.$$

Add axis edges until each signed total is exactly one. These edges sum to zero, so their concatenation in angular order is the boundary of a convex polygon, possibly degenerate, of widths $(1, 1)$. Indeed, for any fixed edge the successive cross products with the remaining edges change sign at most once; since their sum is zero, every partial sum lies on the same side of that edge line.

To identify its mixed-area functional, put $S = B + C$. The facet first-variation formula, followed by planar extrusion, gives

$$6V_3(A, B, C) = \sum_{H \text{ facet of } S} \operatorname{area}(H) h_A(n_H) - \operatorname{area}(B) w_x(A) - \operatorname{area}(C) w_y(A), \tag{4.12}$$

where n_H is the outward unit normal. A facet whose normal has non-axis projection to xy must be the sum of two edges; neither whole planar summand can be exposed in such a direction. The two edges cannot both be vertical, so the common normal has nonzero z-component, which forces $b_1 c_1 > 0$. Its area-weighted normal is

$$N = (|b_1| c_2,\ |c_1| b_2,\ -\operatorname{sgn}(b_1) |b_1 c_1|).$$

Conversely this vector exposes both edges; its restrictions are $|c_1|(b_2, -b_1)$ on yz and $|b_1|(c_2, -c_1)$ on xz. The contribution of this parallelogram facet to (4.12) is therefore exactly the contribution of (4.11) to $2V_2(A, Q)$.

The difference between the two functionals is supported on the four axis normals. Translation invariance makes it $c_1 w_x(A) + c_2 w_y(A)$, and it vanishes on both coordinate unit segments. For example, with $S_x = [0, e_x]$,

$$6V_3(S_x, B, C) = 2V_2(B, [0, e_z]) = 1 = 2V_2(S_x, Q).$$

Thus $c_1 = c_2 = 0$, proving (4.8).

The logarithmic slope of (4.11) is the difference of the input logarithmic slopes, its mass is their product, and its signs are $(-\operatorname{sgn} b_2, \operatorname{sgn} c_2)$. Summing over the common row sign gives (4.10). Hence

$$2(1-q) = \int \|F_B^* F_C\|_{\mathrm{HS}}^2 \, d\omega \le \min\{I_2(F_B), I_2(F_C)\}, \tag{4.13}$$

where the inequality uses Lemma 4.2. Together with (4.7), this proves (4.9).

For arbitrary B, C, approximate them by polygons and correct their widths by diagonal maps tending to the identity. Translate the resulting Q's into the unit square and take a Hausdorff convergent subsequence. The limit has widths $(1,1)$, and continuity proves (4.8) and (4.9). □

The remaining body can now be optimized explicitly. Write $S_x = [0, e_x]$ and $S_y = [0, e_y]$.

Lemma 4.4. *Suppose A has widths $(1,s)$ and area αT, while B, C have widths $(1,1)$ and common area T. For the body Q above, with $q = \operatorname{area}(Q)$, put $D = 6V_3(A,B,C)$. Then*

$$D \ge 1 + s - 2\sqrt{(s-\alpha T)(1-q)}.$$

If $q < 1$ and

$$\lambda = \sqrt{\frac{s-\alpha T}{1-q}} \le \min\{1, s\}, \tag{4.14}$$

equality is attained by

$$A = \lambda Q + (1-\lambda) S_x + (s-\lambda) S_y. \tag{4.15}$$

Proof. Apply (4.6) to (4.8). The set in (4.15) has the required widths; adding the coordinate segments fills the missing widths without changing the non-axis corner directions. Extrusion gives

$$\operatorname{area}(A) = s - \lambda^2(1-q), \qquad 2V_2(A,Q) = 1 + s - 2\lambda(1-q),$$

which proves both its area condition and equality. □

5. **The logarithmic boundary gap**

The corner calculus reduces the sharp threshold to one estimate. When B and C have area $1/2 - \delta$, the area of their mixed body Q can fall below $1/2$, but only by order $\delta^2 \log(1/\delta)$. We first prove this upper bound and then construct bodies attaining its

order. Throughout this section c, C denote positive absolute constants, whose values may change between occurrences; all small-parameter assertions hold below a fixed positive absolute threshold.

5.1. **Energy controls mass logarithmically**

For $R > 0$ define the positive measure

$$\sigma_R = \frac{1}{4}\mathbf{1}_{[-R,R]}\,dx + \frac{1}{2}(\delta_{-R} + \delta_R).$$

A direct integration gives

$$(k * \sigma_R)(x) = \begin{cases} 1, & |x| \le R, \\ e^{-(|x|-R)/2}, & |x| > R, \end{cases} \qquad I(\sigma_R) = 1 + R/2. \tag{5.1}$$

Thus the energy needed to test total mass on an interval grows linearly with its length. Exponential tail control turns this length into a logarithm.

Lemma 5.1. *Let ν be a finite real signed measure satisfying $\int e^{|x|/2}\,d|\nu|(x) \le 4$. If $0 < I(\nu) \le 1/2$, then*

$$|\nu(\mathbb{R})|^2 \le C I(\nu) \log \frac{e}{I(\nu)}.$$

If $I(\nu) = 0$, then $\nu(\mathbb{R}) = 0$.

Proof. Writing $E = I(\nu)$, the potential in (5.1) and energy Cauchy–Schwarz give

$$|\nu(\mathbb{R})| \le |\langle \nu, \sigma_R \rangle_k| + |\nu|\{x : |x| > R\} \le \sqrt{E(1 + R/2)} + 4e^{-R/2}.$$

For $E > 0$ choose $R = \log(1/E)$ and square; for $E = 0$ let $R \to \infty$. □

5.2. **The fourth moment of a Fourier matrix**

For an admissible Fourier matrix F, set

$$I_4(F) = \int \operatorname{tr}((F^*F)^2)\,d\omega.$$

For a unit right triangle, the single corner measure is a unit atom, so $I_2 = I_4 = 1$. The contraction bound alone gives $I_4 \le I_2$. The following estimate gains an additional power of the excess $I_2 - 1$, because the determinant is itself the Fourier transform of a measure with controlled exponential moments.

Lemma 5.2. *If F is admissible and $I_2(F) = 1 + \varepsilon$, with $\varepsilon > 0$ sufficiently small, then*

$$I_4(F) \le 1 + C\varepsilon^2 \log(1/\varepsilon).$$

Proof. Let $a(\xi) \ge b(\xi)$ be the squared singular values of $F(\xi)$, and let $a_0 \ge b_0$ be those of its nonnegative mass matrix $M = F(0)$. Contraction and entrywise domination give

$$0 \le b \le a \le 1, \qquad a + b \le a_0 + b_0 \le 1 + b_0. \tag{5.2}$$

There is nothing to prove when $I_4 \le 1$, so assume $I_4 > 1$. We first bound b_0 away from one, so that the fourth-moment defect controls the energy of $\det F$. The mass estimate will then turn this energy bound into a bound on b_0 that tends to zero with ε.

To obtain the initial uniform gap when $I_2 \le 3/2$, consider an entry measure μ of mass m and put

$$J(\mu) = \iint e^{|x-y|/2}\, d\mu(x)\, d\mu(y).$$

Its two exponential moments are at most one, so

$$I(\mu) + J(\mu) \le 2, \qquad m^4 \le I(\mu)J(\mu), \qquad I(\mu) \ge 1 - \sqrt{1 - m^4}.$$

One of the two permutation products xy in $\det M$ is at least $|\det M| = \sqrt{a_0 b_0} \ge b_0$. Applying the last inequality to these two entries and using concavity of the square root yields

$$I_2 \ge 2 - \sqrt{1 - x^4} - \sqrt{1 - y^4} \ge 2 - 2\sqrt{1 - b_0^2}.$$

Thus $I_2 \le 3/2$ implies $b_0 \le \sqrt{15}/4 =: \kappa < 1$.

By (5.2),

$$a(1 - a) + b(1 - b) \ge b(2 - a - b) \ge (1 - b_0)b.$$

Since $\int b\, d\omega \ge I_2 - 1 = \varepsilon$, integration gives

$$I_2 - I_4 \ge (1 - b_0) \int b\, d\omega, \qquad I_4 - 1 \le b_0 \varepsilon. \tag{5.3}$$

Since $I_4 > 1$, the first inequality shows

$$E := \int |\det F|^2\, d\omega = \int ab\, d\omega \le \int b\, d\omega \le \frac{\varepsilon}{1 - \kappa}.$$

Now

$$\det F = \widehat{\nu}, \qquad \nu = \rho_{11} * \rho_{22} - \rho_{12} * \rho_{21}.$$

Each entry has both exponential moments at most one; the same is true of each convolution, so $\int e^{|x|/2}\,d|\nu| \le 4$. Lemma 5.1 therefore gives

$$a_0 b_0 = |\det M|^2 = |\nu(\mathbb{R})|^2 \le C\varepsilon \log(1/\varepsilon).$$

Finally $2a_0 \ge a_0 + b_0 \ge I_2 > 1$, so $b_0 \le C\varepsilon\log(1/\varepsilon)$. Substitute this in the second inequality of (5.3). □

Corollary 5.3 (Upper bound for the area deficit)**.** *If B, C have widths $(1,1)$ and area $1/2 - \delta$, and $q = \operatorname{area}(Q)$ for the body of Proposition 4.3, then*

$$\frac{1}{2} - q \le C\delta^2 \log(1/\delta)$$

for all sufficiently small $\delta > 0$.

Proof. For polygons, pointwise Hilbert–Schmidt Cauchy–Schwarz followed by scalar Cauchy–Schwarz gives

$$\int \|F_B^* F_C\|_{\mathrm{HS}}^2\,d\omega = \int \operatorname{tr}(F_B F_B^* F_C F_C^*)\,d\omega \le \sqrt{I_4(F_B) I_4(F_C)}.$$

Both second moments equal $1 + 2\delta$, so the result follows from Lemma 5.2 and (4.13). For general bodies use the normalized polygon approximations in Proposition 4.3. Their two area deficits may differ, but both tend to δ; applying the fourth-moment bound to each separately and passing to the limit proves the claim. □

5.3. **Hyperbolic arcs attain the logarithmic order**

The test measure σ_R suggests using a corner measure spread uniformly over a long interval. Such a measure has a direct geometric realization. For $R \ge 2$ put

$$a_R = \frac{1}{e^R - 1}, \qquad c_R = \frac{1}{4\sinh(R/2)}, \qquad \rho_R = c_R \mathbf{1}_{[-R,R]}\,dx,$$

and let

$$K_R = \left\{ (x,y) : 0 \le x \le 1, \quad \frac{a_R(1+a_R)}{x + a_R} - a_R \le y \le 1 - x \right\}.$$

The lower boundary is a convex hyperbolic arc from $(0,1)$ to $(1,0)$, so K_R is convex and has widths $(1,1)$. Its logarithmic slope parameter t satisfies

$$x + a_R = 2c_R e^{-t/2}, \qquad y + a_R = 2c_R e^{t/2}, \qquad -R \le t \le R.$$

Thus the lower arc has corner measure ρ_R, while the closing diagonal has corner measure δ_0. These identities can also be obtained by inscribing polygons in the arc and taking their Riemann-sum limits.

Proposition 5.4 (Lower bound for the area deficit)**.** *For every sufficiently small $\delta > 0$, there are unit-width polygons B, C of area $1/2 - \delta$ for which the associated body Q has area $1/2 - \eta$ with*

$$\eta \geq c\delta^2 \log(1/\delta). \tag{5.4}$$

Proof. First take identical copies of K_R for B, C. Their Fourier matrices and (4.10) give

$$F = \begin{pmatrix} 0 & \widehat{\rho}_R \\ 1 & 0 \end{pmatrix}, \qquad F_Q = \begin{pmatrix} 0 & |\widehat{\rho}_R|^2 \\ 1 & 0 \end{pmatrix}.$$

Since ρ_R is even, the two area deficits are

$$\begin{aligned} \delta_R &= \frac{1}{2} I(\rho_R) = 4c_R^2(R - 1 + e^{-R}) = Ra_R(1 + a_R) - a_R, \\ \eta_R &= \frac{1}{2} I(\rho_R * \rho_R). \end{aligned}$$

The first formula follows either by integrating the hyperbola or by integrating the kernel on $[-R, R]^2$.

The convolution is supported on $[-2R, 2R]$ and has mass $(2Rc_R)^2$. Testing it against σ_{2R} in (5.1) gives

$$\eta_R \geq \frac{8c_R^4 R^4}{1 + R}, \qquad \frac{\eta_R}{\delta_R^2} \geq \frac{R^2}{2(1 + R)} \geq \frac{R}{3}.$$

Moreover $\delta_R \geq (R/2)e^{-R}$ for $R \geq 2$, so $\log(1/\delta_R) \leq R$. Consequently

$$\eta_R \geq \tfrac{1}{3}\delta_R^2 \log(1/\delta_R). \tag{5.5}$$

The positive continuous function δ_R tends to zero as $R \to \infty$, and therefore takes every sufficiently small positive value.

To obtain polygons with exactly the desired deficit, fix such an R and take a nested sequence of polygons P_j inscribed in its lower arc, retaining the two endpoints. Their areas increase to $\operatorname{area}(K_R)$, so their deficits δ_j tend to δ_R from above. Let $\Delta = \operatorname{conv}(0, e_x, e_y)$ and set

$$\theta_j = \sqrt{\delta_R/\delta_j}, \qquad B_j = C_j = \theta_j P_j + (1 - \theta_j)\Delta.$$

The diagonal corner measure remains δ_0, while the other corner measure is multiplied by θ_j. Thus B_j, C_j have deficit exactly δ_R, and the deficit of their mixed body tends to η_R. The measures involved have uniformly bounded slope support, so this convergence follows directly from the continuous kernel energy. For sufficiently large j, (5.5) retains, for example, the constant $1/4$ in place of $1/3$. This proves (5.4). □

5.4. **Optimizing the coefficient slice**

We now return to $b(s)$ in (3.3). The algebraic comparison curve is

$$r(s) = \frac{1 + s + \sqrt{1 + s^2}}{2}, \qquad \delta_*(s) = \frac{s}{4r(s)^2}.$$

Here $\delta_*(s)$ is the area deficit at the comparison minimum. The function r is the larger root of $r^2 - (1 + s)r + s/2 = 0$, and the identities below show why that quadratic is the appropriate reference.

Theorem 5.5 (Logarithmic gap)**.** *For all sufficiently small $s > 0$,*

$$cs^3 \log(1/s) \le r(s) - b(s) \le Cs^3 \log(1/s). \tag{5.6}$$

The lower bound is witnessed by triples of polygons.

Proof. Normalize any realization of $P_{2s,s,z}$ by Proposition 3.1, and write

$$T = \frac{1}{2} - \delta, \qquad q = \frac{1}{2} - \eta.$$

Since $\operatorname{area}(A) = 2sT \le s$ and $q \ge T$, we have $0 \le \delta < 1/2$ and $\eta \le \delta$. Lemma 4.4 gives

$$z \ge f_s(\delta, \eta) := \frac{1 + s - 2\sqrt{s\delta(1 + 2\eta)}}{1 - 2\delta}. \tag{5.7}$$

Put $r = r(s)$ and $\delta_* = \delta_*(s)$. With $v = 4s\delta$ and $w = s/(2r) + 2r\delta$, the defining quadratic for r yields

$$r - f_s(\delta, \eta) = \frac{\sqrt{v(1 + 2\eta)} - w}{1 - 2\delta}, \qquad w^2 - v = 4r^2(\delta - \delta_*)^2. \tag{5.8}$$

In particular $f_s(\delta, 0) \ge r$, with equality at $\delta = \delta_*$. A deficit $\eta > 0$ is the only possible improvement over this algebraic comparison.

Suppose $z < r$. Squaring the positive comparison in (5.8) gives

$$(\delta - \delta_*)^2 < 8\delta_*\delta\eta.$$

It follows that $\delta, \eta > 0$, and since $\eta \le \delta$ and $\delta_* = O(s)$, for small s this forces $\delta < 2\delta_* \le s/2$. Corollary 5.3 now applies and gives

$$\eta \le C\delta^2 \log(1/\delta) \le Cs^2 \log(1/s).$$

Using $\sqrt{v} \le w$ and $\sqrt{1 + 2\eta} - 1 \le \eta$ in (5.8), we obtain

$$0 < r - z \le \frac{\eta\sqrt{v}}{1 - 2\delta} \le Cs\eta \le Cs^3 \log(1/s).$$

Realizations with $z \geq r$ satisfy the same lower bound on z automatically. Taking the infimum proves the upper bound in (5.6).

For the lower bound choose $\delta = \delta_*$ and take the polygons of Proposition 5.4. They satisfy

$$\eta \geq c\delta_*^2 \log(1/\delta_*), \qquad 0 < \eta \leq \delta_*.$$

The scale in (4.14) is now

$$\lambda^2 = \frac{4s\delta_*}{1+2\eta} = \frac{s^2}{r^2(1+2\eta)} < s^2.$$

Thus (4.15) supplies a polygon A attaining (5.7), and for its coefficient z,

$$r - z = \frac{s}{r}\,\frac{\sqrt{1+2\eta}-1}{1-2\delta_*} \geq cs\eta.$$

Since $r \to 1$ and $\delta_* \sim s/4$, this is at least $cs^3 \log(1/s)$, as required. □

Corollary 5.6. *The locus $\mathcal{V}_{3,3}$ is not semialgebraic.*

Proof. If it were, its inverse image under the affine map $(s, z) \mapsto P_{2s,s,z}$ would be semialgebraic. The graph of the finite fiber infimum $b(s)$ would then be semialgebraic by the Tarski–Seidenberg theorem; being a lower bound and being approached arbitrarily closely from above are first-order conditions over the real field [2]. The same would hold for

$$g(s) = \frac{r(s) - b(s)}{s^3}, \qquad c\log(1/s) \leq g(s) \leq C\log(1/s).$$

A semialgebraic function graph is contained in the zero set of a nonzero polynomial. Indeed, in a finite polynomial sign description, at least one defining polynomial must vanish at each graph point; otherwise the signs would be constant on a neighborhood and the graph would have interior. The product of the nonzero defining polynomials therefore vanishes on the graph. Write such a relation as

$$0 = P(s, g(s)) = \sum_{j=j_0}^{m} s^j p_j(g(s)), \qquad p_{j_0} \not\equiv 0.$$

After division by s^{j_0}, all terms with $j > j_0$ tend to zero because $g(s) = O(\log(1/s))$. But $g(s) \to \infty$, so the nonzero polynomial $p_{j_0}(g(s))$ cannot tend to zero. This is a contradiction. □

6. The threshold in every dimension

Multiplying a cubic volume polynomial by a power of a new variable gives a higher-dimensional example, but it also adds a body. To keep the sharp threshold at three bodies we must instead raise the degree of an existing variable. In an arbitrary realization that body may have varying planar sections, so a direct product construction alone does not describe the resulting coefficient slice. The appropriate correction is the sharp second-moment factor for concave functions.

6.1. Averaging planar sections

For $k \geq 1$ put

$$c_k = \frac{2(k+1)}{k+2}.$$

If $Z \subset \mathbb{R}^k$ is a convex body, write $\mathbb{E}$ for uniform averaging on Z.

Lemma 6.1. *Every bounded nonnegative concave function f on Z satisfies*

$$\mathbb{E}f^2 \leq c_k(\mathbb{E}f)^2. \tag{6.1}$$

If $A_y \subset \mathbb{R}^2$ are the fibers of a compact convex set over Z, and $\overline{A}$ is their Minkowski average, defined by $h_{\overline{A}} = \mathbb{E}h_{A_y}$, then

$$\operatorname{area}(\overline{A}) \geq c_k^{-1}\mathbb{E}\operatorname{area}(A_y).$$

Widths and mixed area with a fixed body commute with this average.

Proof. Inequality (6.1) is the $p = 1, q = 2$ case of Berwald's inequality [3]; the following level-set proof also identifies its sharp constant. We may assume $f \not\equiv 0$. Let $M = \operatorname{ess\,sup} f$ and $v(t) = |\{f > t\}|/|Z|$. Brunn–Minkowski makes $v^{1/k}$ concave on $[0, M)$, with right limit one at zero. Dilating Z towards a point where f approaches M gives $v(t) \geq (1 - t/M)^k$, so $M \leq a := (k+1)\mathbb{E}f$.

Compare v with $w(t) = (1 - t/a)^k$ on $[0, a]$, extended by zero. They have the same integral. Concavity implies that $[v(t)^{1/k} - (1 - t/a)]/t$ is nonincreasing where $v > 0$; therefore $v - w$ has at most one sign change, from positive to negative. If τ is a crossing point, then

$$\int_0^\infty t(v - w)\,dt = \int_0^\infty (t - \tau)(v - w)\,dt \leq 0.$$

Layer-cake integration now gives

$$\mathbb{E}f^2 \leq 2\int_0^a t(1 - t/a)^k\,dt = \frac{2a^2}{(k+1)(k+2)},$$

which is (6.1). A plateau at the maximum only adds a downward jump to v and leaves the comparison unchanged.

Convexity of the total set gives $A_{(1-t)y+ty'} \supseteq (1-t)A_y + tA_{y'}$. Thus $f(y) = \sqrt{\operatorname{area}(A_y)}$ is concave, and planar Brunn–Minkowski applied to Minkowski averages yields $\sqrt{\operatorname{area}(\overline{A})} \geq \mathbb{E}f$. Combine this with (6.1). The averages can be formed as limits of finite Minkowski averages; fiber support functions are bounded and concave in the interior of Z, and have uniform Lipschitz bounds in their direction variable. This also justifies the asserted linearity of widths and mixed area. □

6.2. **The lifted coefficient slice**

Fix $d \geq 4$, put $k = d-3$, and define

$$P^{(d)}_{\alpha,s,z} = X_1^k P_{c_k\alpha,s,z}. \tag{6.2}$$

Proposition 6.2. *Every realization of* (6.2) *gives planar bodies B, C, Q of widths* $(1,1)$ *and a planar body $\overline{A}$ of widths* $(1,s)$ *such that, for some $T > 0$,*

$$\operatorname{area}(B) = \operatorname{area}(C) = T, \qquad \operatorname{area}(\overline{A}) \geq \alpha T,$$
$$2Tz = 2V_2(\overline{A}, Q), \qquad q := \operatorname{area}(Q) \geq T.$$

In particular,

$$2Tz \geq 1 + s - 2\sqrt{(s-\alpha T)(1-q)}, \qquad \alpha T \leq s. \tag{6.3}$$

Conversely, every normalized cubic triple in Proposition 3.1 lifts to a realization of (6.2) *with the same* (α, s, z)*. Polygonal triples lift to polytopes.*

Proof. Let K_1, K_2, K_3 realize (6.2). Lemma 2.1 reads its monomial support as

$$\dim K_1 = d-1, \qquad \dim K_2 = \dim K_3 = 2, \qquad \dim(K_2 + K_3) = 3.$$

Translate the affine hulls through zero and put $H = \operatorname{span} K_1$, $W = \operatorname{span}(K_2 + K_3)$. Since the total sum is full dimensional, $E = H \cap W$ is a plane. The positive coefficients of $X_1^{d-1}X_2$ and $X_1^{d-1}X_3$ ensure that neither of the other two planes is contained in H.

Choose a complement U to E in H, so $\mathbb{R}^d = W \oplus U$. Let $Z \subset U$ be the projection of K_1 and write its fibers as $A_y + y$, with $A_y \subset E$. Set $B = K_2$, $C = K_3$. In product coordinates of determinant modulus one, Fubini gives

$$\operatorname{vol}_d(t_1K_1 + t_2B + t_3C) = t_1^k \int_Z \operatorname{vol}_3(t_1A_y + t_2B + t_3C)\, dy. \tag{6.4}$$

The factor t_1^k is the Jacobian in the quotient coordinate.

The three planes $E, \operatorname{span} B, \operatorname{span} C$ have independent normals within W. Otherwise they would share a line, and the cyclic product calculation in Proposition 3.1 would still apply to the averaged coefficients. Indeed, the relevant directional widths of every fiber A_y are a single width transverse to the common line, multiplied by fixed angular factors; averaging therefore preserves their ratio and the cyclic-product identity. Replacing the area and width of A by their respective averages would make the two cyclic products equal, whereas their prescribed ratio is $s \neq 1$.

We may therefore take these planes to xy, yz, xz and use the remaining coordinates for U. Write $m = \operatorname{vol}_k(Z)$ and

$$\begin{aligned} \mathbb{E}(w_x(A_y), w_y(A_y)) &= (a, b), \qquad \tau = \mathbb{E}\operatorname{area}(A_y), \\ (w_y(B), w_z(B)) &= (c, \ell), \qquad (w_x(C), w_z(C)) = (e, f). \end{aligned}$$

Comparing the six repeated coefficients in (6.4) with (6.2) gives

$$\begin{aligned} m\tau\ell = m\tau f &= 3c_k\alpha, \\ m\operatorname{area}(B)a = m\operatorname{area}(B)e &= 3, \\ m\operatorname{area}(C)b = 3s, &\qquad m\operatorname{area}(C)c = 3. \end{aligned}$$

Hence $\ell = f$, $a = e$ and $b = sc$. Apply the diagonal map $\operatorname{diag}(1/a, 1/c, 1/\ell)$ on W, leaving U fixed, and put $T = 3/(mac\ell)$. In the new coordinates,

$$\begin{aligned} \mathbb{E}(w_x(A_y), w_y(A_y)) &= (1, s), & \mathbb{E}\operatorname{area}(A_y) &= c_k\alpha T, \\ \operatorname{area}(B) = \operatorname{area}(C) &= T, & \mathbb{E}[6V_3(A_y, B, C)] &= 2Tz. \end{aligned}$$

Both B, C have widths $(1, 1)$. Lemma 6.1 puts $\operatorname{area}(\overline{A}) \geq \alpha T$, while Proposition 4.3 and linearity of mixed area give

$$2Tz = \mathbb{E}[2V_2(A_y, Q)] = 2V_2(\overline{A}, Q).$$

The corner inequality (4.6) proves (6.3) and the other forward assertions.

For the converse, start with a normalized cubic triple A, B, C. On the standard simplex $Z = \{y_j \geq 0 : \sum_j y_j \leq 1\} \subset \mathbb{R}^k$, set

$$\rho(y) = (k+1)\Big(1 - \sum_j y_j\Big).$$

Its first two normalized moments are

$$|Z| = 1/k!, \qquad \mathbb{E}\rho = 1, \qquad \mathbb{E}\rho^2 = c_k.$$

Define the cone

$$\widetilde{K}_1 = \{(\rho(y)a, y) : a \in A,\ y \in Z\} = \operatorname{conv}\big(((k+1)A)\times\{0\}, (0,e_1), \dots, (0,e_k)\big), \quad (6.5)$$

and embed B, C at quotient coordinate zero. Formula (6.4) and the two moments give

$$\operatorname{vol}_d(X_1\widetilde{K}_1 + X_2B + X_3C) = \frac{T}{3k!}P^{(d)}_{\alpha,s,z}.$$

A common dilation by $(3k!/T)^{1/d}$ realizes the required polynomial. The cone is a polytope whenever A is a polygon. □

The proof shows the role of the correction factor; averaging arbitrary sections gives the universal inequality, and the cone in (6.5) attains both moments that enter it. Thus the lift preserves the boundary comparison even though an arbitrary realization need not have constant sections.

Proof of Theorem 1.1. The case $d = 3, n = 3$ is Corollary 5.6. For $d \geq 4$, restrict (6.2) to $\alpha = 2s$ and define

$$b_d(s) = \inf\{z : P^{(d)}_{2s,s,z} \in \mathcal{V}_{3,d}\}.$$

The lifted rectangle-and-triangles example shows that every fiber is nonempty. The forward part of Proposition 6.2 supplies exactly (5.7), with $T \leq 1/2$ and $q \geq T$; therefore the upper estimate in Theorem 5.5 holds for $r(s) - b_d(s)$. Its polygonal lower witnesses lift by the converse part of the proposition, with unchanged z. Consequently

$$cs^3\log(1/s) \leq r(s) - b_d(s) \leq Cs^3\log(1/s).$$

The same semialgebraic-graph argument proves that $\mathcal{V}_{3,d}$ is not semialgebraic.

Restriction to three variables maps $\mathcal{V}_{n,d}$ onto $\mathcal{V}_{3,d}$ for every $n \geq 3$; one direction discards bodies, and the other adjoins points. A semialgebraic source would have a semialgebraic image. The one-body locus is $\mathbb{R}_{\geq 0}$, and the two-body locus is semialgebraic by Shephard's theorem, proved in Appendix A.

Finally, full-dimensional polytope tuples are Hausdorff dense in all compact convex tuples; approximate by polytopes and add a vanishing cube. Proposition 2.4 identifies the closure of their coefficient image with $\mathcal{V}_{n,d}$. Since the closure of a semialgebraic set is semialgebraic, their coefficient locus is nonsemialgebraic whenever $n, d \geq 3$. □

Remark 6.3. The same boundary gap excludes definability in any polynomially bounded expansion of the ordered real field, even with real parameters. If the

locus were definable, so would be $g(x) = x^3(r(1/x) - b_d(1/x))$ for large x, with $c \log x \le g(x) \le C \log x$. Hence

$$G(y) = \inf\{x \ge x_0 : g(x) \ge y\}$$

would be a finite definable function satisfying $G(y) \ge e^{y/C}$, contrary to polynomial boundedness. No monotonicity of g is required. The assertion also holds for any Hausdorff-dense realization class, because topological closure is definable.

7. **Projection holonomy and forced shapes**

The sharp threshold was obtained by optimizing a coefficient slice. We now use equalities among coefficients to determine a body itself. The point is that (2.2) turns a mixed-volume equality into equality of two planar projections, while the equality case of Minkowski's inequality removes all freedom except translation. Following such identifications around three planes produces a return map on the first one.

7.1. **A three-plane return map**

Put $\tau = 2/3$ and fix the segment directions

$$v = (0, 1, 1), \qquad w = (1, 1, 0), \qquad z = (1, \tau, 1). \tag{7.1}$$

Parallel projection along these directions, successively between xy, xz, yz, xy, gives

$$\begin{aligned} T_{01}(x, y, 0) &= (x, 0, -y), \\ T_{12}(x, 0, z') &= (0, -x, z'), \\ T_{20}(0, y, z') &= (-z', y - \tau z', 0). \end{aligned}$$

Their planar determinants have modulus one, and their return map is

$$T_{20}T_{12}T_{01}(x, y, 0) = (y, -x + \tau y, 0), \qquad M = \begin{pmatrix} 0 & 1 \\ -1 & \tau \end{pmatrix}.$$

Thus, if the first two projection comparisons identify B with $T_{01}A$ and C with $T_{12}B$, a third comparison of A with $\lambda C + \varepsilon[0, e_1]$ imposes

$$A = \lambda M A + \varepsilon[0, e_1] + b$$

for a translation b. The rest of the construction makes these directions and comparisons recoverable from the coefficients alone.

Use nine body labels

$$E_1, E_2, E_3, S_v, S_w, S_z, A, B, C,$$

and write m_{IJK} for the mixed-volume entry with the indicated labels. These entries are fixed rational multiples of the ordinary cubic coefficients. First impose

$$m_{HHJ} = 0 \quad (H \in \{E_1, E_2, E_3, S_v, S_w, S_z\},\ J \text{ arbitrary}), \qquad 6m_{E_1E_2E_3} = 1. \tag{7.2}$$

The following table specifies nine further coefficient values

H	$6m_{HE_2E_3}$	$6m_{HE_1E_3}$	$6m_{HE_1E_2}$
S_v	0	1	1
S_w	1	1	0
S_z	1	τ	1

$$6m_{S_vS_wS_z} = 2 - \tau. \tag{7.3}$$

Lemma 7.1 (Calibration)**.** *Every realization of* (7.2)–(7.3) *can be normalized by independent translations and a common linear map of determinant modulus one so that*

$$E_j = [0, e_j], \qquad S_v = [0, v], \quad S_w = [0, w], \quad S_z = [0, z],$$

with the vectors in (7.1)*.*

Proof. The positive basis coefficient makes the total sum full dimensional. By Lemma 2.1, the six labels in (7.2) are segments or points, and the nonzero coefficients in the table exclude points. Their triple coefficients are absolute determinants, so the basis coefficient normalizes E_1, E_2, E_3 as stated.

The table gives the absolute coordinates of the three remaining segment vectors. Coordinate reflections and reversals of segment vectors can simultaneously arrange $v = (0, 1, 1)$ and $w = (1, 1, 0)$; fix the second coordinate, then orient the third and first coordinates in turn. Reverse the last vector to make its third coordinate positive; it has the form $(s, \epsilon\tau, 1)$ with $s, \epsilon \in \{\pm 1\}$. Its determinant with v, w is $\epsilon\tau - s - 1$. The possible absolute values are $2 - \tau, 2 + \tau, \tau, \tau$, so the last coefficient in (7.3) selects $(1, \tau, 1)$ uniquely. Reversing a segment vector merely translates that segment. □

Next impose the three zero-width conditions

$$m_{AE_1E_2} = m_{BE_1E_3} = m_{CE_2E_3} = 0, \tag{7.4}$$

and the two positive comparison equalities

$$m_{AAS_v} = m_{ABS_v} = m_{BBS_v} > 0, \tag{7.5}$$
$$m_{BBS_w} = m_{BCS_w} = m_{CCS_w} > 0.$$

For parameters $0 < \lambda \leq 1$ and $\varepsilon \geq 0$, the final comparison is

$$\begin{aligned} m_{AAS_z} &= \lambda m_{ACS_z} + \varepsilon m_{AE_1S_z} \\ &= \lambda^2 m_{CCS_z} + 2\lambda\varepsilon m_{CE_1S_z} > 0. \end{aligned} \tag{7.6}$$

All the conditions are polynomial with rational coefficients; entries not listed are free.

Proposition 7.2 (Exact functional equation)**.** *The coefficient conditions* (7.2)–(7.6) *have a realization if and only if there is a positive-area planar compact convex set K satisfying*

$$K = \lambda MK + \varepsilon S, \qquad S = [-e_1/2, e_1/2]. \tag{7.7}$$

In every realization, after the normalization of Lemma 7.1 and further independent translations, the last three bodies are $K, T_{01}K, T_{12}T_{01}K$ in their respective coordinate planes. Moreover

$$\operatorname{area}(K) = 3m_{AAE_3}. \tag{7.8}$$

Proof. Calibrate the six segments. Extrusion turns (7.4) into zero widths, so translations put $A \subset xy$, $B \subset xz$, $C \subset yz$. The positive self-entries in (7.5) ensure that all three have positive relative area. By (2.2) and the equality case of (2.3), the first two comparisons identify the corresponding projected bodies up to translation. Centering each body at its area centroid removes these translations simultaneously, since an invertible linear map between two planes carries area centroids to area centroids. Hence $B = T_{01}A$ and $C = T_{12}B$.

Let $D = \lambda C + \varepsilon E_1$. Multilinearity and $m_{E_1E_1S_z} = 0$ identify (7.6) with

$$V_3(A, A, S_z) = V_3(A, D, S_z) = V_3(D, D, S_z) > 0.$$

The projected bodies are therefore translates. Lifting that relation to xy along z gives $A = \lambda T_{20}C + \varepsilon E_1 + b = \lambda MA + \varepsilon E_1 + b$. The matrix M preserves the positive definite quadratic form

$$Q_\tau = \begin{pmatrix} 1 & -\tau/2 \\ -\tau/2 & 1 \end{pmatrix}, \qquad M^T Q_\tau M = Q_\tau, \tag{7.9}$$

and its eigenvalues are nonreal. Thus $I - \lambda M$ is invertible for $0 < \lambda \leq 1$. After centering E_1 and translating A by the unique solution of the resulting linear translation equation, we obtain (7.7). Translate B, C by the corresponding images to retain their relations with A.

Conversely, place a solution K in xy, its two indicated images in xz, yz, and the six calibrated segments in $\mathbb{R}^3$. The first two projection comparisons are identities,

and the third follows from (7.7), since S is a translate of E_1. The determinant and zero-width conditions hold by construction, so the actual volume polynomial of this tuple satisfies every condition. Extrusion in direction e_3 proves (7.8). □

7.2. **Irrational rotation and contraction**

The eigenvalues of M are $e^{\pm i\theta}$, where $\cos\theta = 1/3$. This angle is an irrational multiple of π. Indeed, if $e^{i\theta}$ were a root of unity, then $2\cos\theta = 2/3$ would be an algebraic integer; the recurrence

$$P_0(t) = 2, \quad P_1(t) = t, \quad P_{j+1}(t) = tP_j(t) - P_{j-1}(t)$$

gives monic integral polynomials P_j for $j \geq 1$ with $P_j(2\cos\theta) = 2\cos(j\theta)$. A suitable $P_j - 2$ would have the nonintegral rational root $2/3$, a contradiction.

For $(\lambda, \varepsilon) = (1, 0)$, equation (7.7) says $K = MK$. By (7.9), a linear change of coordinates makes M an irrational rotation. Its powers are dense in the rotation group, so continuity of the support function makes K a disk in those coordinates. Every positive-area solution is therefore an ellipse. This gives a coefficient pattern whose every realization has three elliptic bodies, and in particular no all-polytope realization.

For $0 < \lambda < 1$ and $\varepsilon = 1$, the same equation instead determines a unique body, with an area that records every iterate of the return map.

Theorem 7.3 (The forced Minkowski series)**.** *For $0 < \lambda < 1$, the unique compact convex solution of $K = S + \lambda MK$ is*

$$K_\lambda = \sum_{j \geq 0} \lambda^j M^j S. \tag{7.10}$$

It has positive area and infinitely many exposed edges in distinct directions. Its area is

$$a(\lambda) = \frac{1}{(1-\lambda^2)\sin\theta} \sum_{j \geq 1} |\sin(j\theta)| \lambda^j, \qquad \cos\theta = \frac{1}{3}. \tag{7.11}$$

The function has rational Taylor coefficients and a holomorphic extension to the unit disk with the unit circle as a natural boundary. It is transcendental over $\mathbb{R}(\lambda)$, and K_λ is not a semialgebraic set.

Proof. In the norm defined by Q_τ, M is an isometry, so the tails of (7.10) are uniformly bounded by $C\lambda^m/(1-\lambda)$. The sum is consequently a compact convex set; its first two summands form a nondegenerate parallelogram. Separating the first term gives

the functional equation. Conversely, iterating that equation for any compact solution gives

$$K = \sum_{j=0}^{m-1} \lambda^j M^j S + \lambda^m M^m K.$$

The last term converges to zero in Hausdorff distance, proving uniqueness.

For planar vectors $u_0, \dots, u_{m-1}$, the area of their zonotope is

$$\operatorname{area}\Big(\sum_{j=0}^{m-1} [-u_j/2, u_j/2]\Big) = \sum_{i<j} |\det(u_i, u_j)|.$$

This follows by expanding its quadratic area polynomial; the pure terms vanish and each mixed coefficient is a parallelogram area. Cayley–Hamilton gives

$$\det(e_1, M^j e_1) = \det(e_1, Me_1)\frac{\sin(j\theta)}{\sin\theta}, \qquad |\det(e_1, Me_1)| = 1.$$

Apply the zonotope formula to $u_j = \lambda^j M^j e_1$ and pass to the Hausdorff limit. Summing first over $j - i$ gives (7.11).

The lines $\mathbb{R}M^j e_1$ are pairwise distinct, because a repeated line would make a nonzero integral multiple of θ a multiple of π. A normal perpendicular to $M^j e_1$ exposes the entire jth segment of the sum and a unique endpoint of every other segment. The endpoint series converges, so the exposed face is a translate of $\lambda^j M^j S$, a genuine edge. A semialgebraic planar set can have boundary segments on only finitely many lines; its boundary lies in the union of finitely many polynomial zero sets, and any polynomial vanishing on a nontrivial line segment has that line as a factor. Thus K_λ is not semialgebraic.

To study the area, put $H(z) = \sum_{j\ge1} |\sin(j\theta)| z^j$. The absolutely convergent Fourier series

$$|\sin t| = \sum_{m\in\mathbb{Z}} \beta_m e^{2imt}, \qquad \beta_0 = \frac{2}{\pi}, \qquad \beta_m = -\frac{2}{\pi(4m^2-1)} \quad (m \neq 0)$$

gives, for $|z| < 1$,

$$H(z) = \sum_{m\in\mathbb{Z}} \beta_m \frac{z e^{2im\theta}}{1 - z e^{2im\theta}}. \tag{7.12}$$

The points $\zeta_\ell = e^{-2i\ell\theta}$ are distinct and dense on the unit circle. Since $|(1-r)re^{i\phi}/(1-re^{i\phi})| \le 1$, absolute summability of (β_m) allows dominated convergence in (7.12), giving

$$\lim_{r\uparrow1} (1-r) H(r\zeta_\ell) = \beta_\ell \neq 0.$$

Thus H is unbounded near a dense set of boundary points and cannot be continued across any boundary arc. The rational factor $((1-z^2)\sin\theta)^{-1}$ in (7.11) does not change this conclusion.

For completeness, this also rules out a polynomial relation with arbitrary real coefficients without invoking general facts about algebraic functions. If $P(z, a(z)) = 0$, substitute (7.11), clear the power of $1 - z^2$, and divide by the highest power of $H(z)$. Along $z = r\zeta_\ell$ with $\ell \neq 0$, all lower terms tend to zero. The leading coefficient polynomial must therefore vanish at every ζ_ℓ, which is impossible unless it is zero.

Finally, $\sin(j\theta)/\sin\theta = U_{j-1}(1/3)$, where $U_0 = 1$, $U_1 = 2t$ and $U_{j+1} = 2tU_j - U_{j-1}$. These values are rational, so (7.11) has rational Taylor coefficients. □

Proof of Theorem 1.2. Take $0 < \lambda < 1$, $\varepsilon = 1$, the finite coefficient conditions above, and $a = 3m_{AAE_3}$. Proposition 7.2 and Theorem 7.3 show both that every realizing tuple has the stated body and area, and that every λ occurs. This proves the exact projection statement and all its asserted properties. □

7.3. **Persistence under direct-sum suspension**

For shape forcing, one additional body can carry all the new dimensions. A monomial in its variable forces every realization to split into complementary subspaces, preserving the three-dimensional obstruction.

Proposition 7.4. *For every cubic polynomial F in n variables and every $d \geq 4$,*

$$F \in \mathcal{V}_{n,3} \quad \Longleftrightarrow \quad u^{d-3}F \in \mathcal{V}_{n+1,d}.$$

The equivalence also holds when both sides require realizations by polytopes, or by semialgebraic compact convex sets.

Proof. Embed a realizing tuple for F in $\mathbb{R}^3$ and a unit-volume $(d-3)$-cube in a complementary subspace. Volumes multiply, giving the forward implication.

Conversely, suppose $u^{d-3}F$ is realized by $H_1, \ldots, H_n, T$. The zero polynomial is immediate, so assume $F \neq 0$ and put $S = \sum_i H_i$. The two-body polynomial is

$$\mathrm{vol}_d(tS + uT) = F(1, \ldots, 1)t^3u^{d-3}.$$

Lemma 2.1 gives $\dim S = 3$ and $\dim T = d - 3$; their spans are complementary because the total sum is full dimensional. Each H_i lies in the span of S after translation, and product volume in these complementary coordinates gives $F = c\,\mathrm{vol}_3(\sum_i t_i H_i)$ for some $c > 0$. A common dilation absorbs c. Restriction to the three-dimensional subspace, linear maps and dilations preserve both polytopality and semialgebraicity. □

It follows that arrays with no all-semialgebraic realization occur for $n \geq 9$ in dimension three and for $n \geq 10$ in every higher dimension. These are shape obstructions for particular arrays, distinct from the sharp three-body obstruction to a finite algebraic description of the entire coefficient locus.

8. **A countable arithmetic characterization**

The preceding obstructions rule out finite algebraic descriptions and exact replacement by polytopes. Approximation behaves differently; the bounded model in Section 2 has explicit finite rational nets. Their error bounds give both an exact infinite coefficient criterion and finite certificates outside the locus.

Fix $n, d \geq 1$ and put

$$\mathcal{A} = \{\alpha \in \mathbb{Z}^n_{\geq 0} : |\alpha| = d\}, \quad N = |\mathcal{A}| = \binom{n+d-1}{d}, \quad B_d = (4d^2)^d.$$

For a coefficient vector $c = (c_\alpha)_{\alpha \in \mathcal{A}}$, write $s = \sum_\alpha c_\alpha$. All constructions below depend only on n, d; the input vector enters only when the resulting inequalities are evaluated.

8.1. **Finite rational nets**

For an integer $m \geq 1$, let

$$G_{d,m} = \{z/m : z \in \mathbb{Z}^d,\ -dm \leq z_j \leq dm\}.$$

For each tuple $S = (S_1, \ldots, S_n)$ of subsets of this grid containing zero, compute the coefficient vector $q(S)$ of the rational polytopes $P_i = \operatorname{conv}(S_i)$. Let

$$\mathcal{Q}_m = \{q(S) : 0 \in S_i \subseteq G_{d,m}\}.$$

This is a finite rational set that can be generated by rational arithmetic. We give an explicit formula and error estimate, so the definition involves no realization test.

For a finite rational point set T, its convex-hull volume is computed as follows. Return zero if its affine rank is less than d. Otherwise enumerate the supporting hyperplanes determined by affinely independent d-subsets of T, retaining those for which all points lie on one side, and record their complete active point sets. These are precisely the facets. Recursively dissect each facet by coning its facets to the average of its input points, and cone the resulting boundary simplices to the average of T. At each stage, affine coordinates and supporting equations are obtained by rational

Gaussian elimination. The average lies in the relative interior, so these cones cover the hull with disjoint relative interiors. Its volume is therefore the rational number

$$W_d(T) = \frac{1}{d!} \sum_{(v_0,\dots,v_d)} |\det(v_1 - v_0, \dots, v_d - v_0)|,$$

where the sum is over the resulting simplices. Duplicate facet point sets are discarded. This finite procedure includes coplanar and lower-dimensional inputs.

For a homogeneous polynomial $H(t) = \sum_{|\alpha|=d} h_\alpha t^\alpha$, finite differences give

$$h_\alpha = \frac{1}{\alpha!} \sum_{0 \le \beta \le \alpha} (-1)^{d-|\beta|} \binom{\alpha}{\beta} H(\beta). \tag{8.1}$$

Indeed, the difference operator of multi-order α kills every other monomial of total degree d and takes t^α to $\alpha!$. Since convex hull commutes with Minkowski addition, the required grid coefficients are explicitly

$$q_\alpha(S) = \frac{1}{\alpha!} \sum_{0 \le \beta \le \alpha} (-1)^{d-|\beta|} \binom{\alpha}{\beta} W_d\left(\left\{\sum_i \beta_i x_i : x_i \in S_i\right\}\right). \tag{8.2}$$

Lemma 8.1 (Quantitative nets)**.** *For the compact image $\mathcal{D}_{n,d}$ in* (2.4)*,*

$$\mathcal{Q}_m \subseteq \mathcal{D}_{n,d}, \qquad \sup_{x \in \mathcal{D}_{n,d}} \min_{q \in \mathcal{Q}_m} \|x - q\|_\infty \le \frac{B_d}{m}.$$

Consequently, for every $x \in \mathbb{R}^N$,

$$\operatorname{dist}_\infty(x, \mathcal{D}_{n,d}) \le \min_{q \in \mathcal{Q}_m} \|x - q\|_\infty \le \operatorname{dist}_\infty(x, \mathcal{D}_{n,d}) + \frac{B_d}{m}.$$

Proof. Round every point of $L_i \in \mathcal{K}_d$ to a nearest grid point, with a fixed rule at ties, and let P_i be the convex hull of the resulting finite set. Then $d_{H,\infty}(L_i, P_i) \le 1/(2m)$ and $P_i \in \mathcal{K}_d$.

We use the elementary estimate

$$|\operatorname{vol}_d(C) - \operatorname{vol}_d(D)| \le (2b + 2\eta)^d - (2b)^d \tag{8.3}$$

when $C, D \subseteq [-b, b]^d$ have sup-norm Hausdorff distance at most η. To prove it, add the coordinate segments of $[-\eta, \eta]^d$ one at a time. Extrusion bounds the successive volume increments by $2\eta(2b + 2\eta)^{j-1}(2b)^{d-j}$; their sum is the right side of (8.3). The two Hausdorff inclusions give the assertion.

For $|\beta| \le d$, the sums $\sum_i \beta_i L_i$ and $\sum_i \beta_i P_i$ lie in $[-d^2, d^2]^d$ and are within $d/(2m)$ in Hausdorff distance. The sum of the absolute weights in (8.1) is at most 2^d. Thus every coefficient changes by at most

$$\begin{aligned} 2^d\big((2d^2 + d/m)^d - (2d^2)^d\big) &= B_d \left(\left(1 + \frac{1}{2dm}\right)^d - 1 \right) \\ &\le B_d \sum_{j \ge 1} (2m)^{-j} \le B_d/m. \end{aligned}$$

This proves the net assertion, including its independence from n. The distance comparison follows by choosing a closest point in the compact set $\mathcal{D}_{n,d}$ and then a net point. □

Theorem 8.2 (Finite tests at every scale)**.** *A vector c belongs to $\mathcal{V}_{n,d}$ if and only if either $c = 0$, or $s > 0$ and, for every $m \ge 1$,*

$$\min_{q \in \mathcal{Q}_m} \|c - sq\|_\infty \le \frac{B_d s}{m}.$$

Each test is a finite Boolean combination of rational linear inequalities. A nonrealizable vector with $s > 0$ fails every sufficiently fine test.

Proof. If c is nonzero and realizable, Lemma 2.3 puts c/s in $\mathcal{D}_{n,d}$, and Lemma 8.1 gives every test. Conversely, the tests put c/s at distance zero from the compact set $\mathcal{D}_{n,d}$. A realizing tuple for c/s, dilated by $s^{1/d}$, realizes c. If $c/s \notin \mathcal{D}_{n,d}$, its positive distance from that set exceeds B_d/m for all sufficiently large m, so the corresponding tests fail. Finally, the minimum over a finite set is the finite disjunction, over $q \in \mathcal{Q}_m$, of the inequalities $-B_d s/m \le c_\alpha - sq_\alpha \le B_d s/m$ for all α. □

8.2. **Homogeneous quadratic inequalities**

The finite tests can be expressed as a single countable family of quadratic inequalities by excluding rational balls outside the compact image. Admit a triple

$$(a, r, m) \in \mathbb{Q}^N \times \mathbb{Q}_{>0} \times \mathbb{N}_{>0}$$

when the finite rational test

$$\|a - q\|_2^2 > \left(r + \frac{NB_d}{m}\right)^2 \qquad \text{for every } q \in \mathcal{Q}_m \tag{8.4}$$

holds, and associate to it the homogeneous quadratic form

$$Q_{a,r,m}(c) = \sum_\alpha (c_\alpha - s a_\alpha)^2 - r^2 s^2. \tag{8.5}$$

The rational constants in these forms are computed once from (8.2); clearing a positive common denominator makes each form integral if desired.

Theorem 8.3. *For every $n, d \geq 1$,*

$$c \in \mathcal{V}_{n,d} \quad \Longleftrightarrow \quad c_\alpha \geq 0 \ (\alpha \in \mathcal{A}), \quad Q_{a,r,m}(c) \geq 0 \text{ for every admitted triple.}$$

The family is effectively enumerable by rational arithmetic, and every nonrealizable vector strictly violates a displayed inequality.

Proof. The zero vector satisfies all the inequalities. For a nonzero realizable vector, $s > 0$ and $x = c/s \in \mathcal{D}_{n,d}$. Choose $q \in \mathcal{Q}_m$ with $\|x - q\|_2 \leq \sqrt{N} B_d/m \leq N B_d/m$. For every admitted triple, (8.4) gives $\|a - x\|_2 > r$, and multiplication by s^2 proves the quadratic inequality. Coefficient nonnegativity is standard.

Conversely, suppose the coefficients are nonnegative and $c \neq 0$, so $s > 0$. If $x = c/s \notin \mathcal{D}_{n,d}$, let $\delta = \operatorname{dist}_2(x, \mathcal{D}_{n,d}) > 0$. Choose rational a, r with

$$\|a - x\|_2 < \delta/8, \qquad \delta/4 < r < \delta/2,$$

and then m with $N B_d/m < \delta/8$. Every $q \in \mathcal{Q}_m \subseteq \mathcal{D}_{n,d}$ is at distance greater than $7\delta/8$ from a, so this triple is admitted. But $\|a - x\|_2 < r$, and (8.5) is strictly negative, a contradiction. Hence $x \in \mathcal{D}_{n,d}$ and c is realizable.

To enumerate the family, enumerate rational a, r and integers m, generate the finite list $\mathcal{Q}_m$, and retain the triples passing (8.4). Every step is finite rational arithmetic. Every nonrealizable vector therefore has either a negative coefficient or a strictly negative value of an admitted quadratic form. □

For computable coefficients, certified rational approximations and interval arithmetic eventually certify any fixed strict violation. Dovetailing the enumeration of inequalities with increasing precision therefore rejects every nonrealizable input after a finite computation and never rejects a realizable one. This procedure works directly on c, including when it is not known whether $s = 0$; no equality test or division by s is required.

8.3. **Constructing realizing bodies**

The tests also organize positive witnesses. Let x be a normalized input, $\sum_\alpha x_\alpha = 1$, supplied with rational approximations $\widehat{x}_j$ satisfying

$$\|\widehat{x}_j - x\|_\infty \leq 2^{-j}.$$

At level j, a state is a tuple of grid polytopes $P_{i,j} = \operatorname{conv}(S_{i,j})$ with $0 \in S_{i,j} \subseteq G_{d,2^j}$, whose coefficient vector q_j satisfies

$$\|q_j - \widehat{x}_j\|_\infty \leq (B_d + 1)2^{-j}. \tag{8.6}$$

Join consecutive states when

$$d_{H,\infty}(P_{i,j}, P_{i,j+1}) \leq 3 \cdot 2^{-j-2} \qquad (1 \leq i \leq n). \tag{8.7}$$

A node is a path of states starting at level one. Every level is finite, and all its tests are rational; for rational polytopes, sup-norm Hausdorff distance at most η is equivalent to the two inclusions into the other polytope plus $\eta[-1,1]^d$. Each inclusion reduces to finitely many rational convex-hull membership tests, solved by enumerating affinely independent subsets and checking their barycentric coordinates.

Proposition 8.4. *The state tree has nodes at every height if and only if $x \in \mathcal{D}_{n,d}$. Every infinite path converges to an exact realizing tuple, with Hausdorff error at level j at most $3 \cdot 2^{-j-1}$.*

If the approximations $\widehat{x}_j$ are computable, selecting the terminal state of the lexicographically least path at each height gives a rational algorithm that eventually rejects a nonrealizable input and otherwise produces an entire sequence converging to a realizing tuple. No computable rate for this latter sequence is asserted.

Proof. For $x = \Phi(L)$, round the same realizing tuple L at every level. Its Hausdorff errors are at most 2^{-j-1} and its coefficient errors at most $B_d 2^{-j}$, so these rounded tuples form an infinite path satisfying (8.6) and (8.7).

Conversely, a finitely branching tree with nodes at every height has an infinite path; a node with arbitrarily long extensions has a child with that property, and this observation can be iterated from the root. Along any infinite path, the successive Hausdorff bounds have tail sum $3 \cdot 2^{-j-1}$. The path therefore converges to compact convex bodies in $\mathcal{K}_d$, with the stated error. Its coefficients tend to x by (8.6), so continuity makes the limit an exact realization.

Order grid subsets, states and paths lexicographically. If a height has no node, the input is rejected; otherwise output the terminal state of its least path. The preceding equivalence proves the rejection assertion. For a realizable input, fix a depth j. Among the finitely many depth-j nodes, those extending to height h form a decreasing nonempty family as $h \to \infty$, so eventually this family stabilizes. The depth-j prefix of the chosen path therefore stabilizes as well. All sufficiently late outputs are within $3 \cdot 2^{-j-1}$ of that prefix. Since this tends to zero with j, the whole output sequence is Hausdorff Cauchy and converges to a realizing tuple. The

argument does not bound the height at which a prefix stabilizes, which explains the absence of an effective convergence rate. □

Theorem 8.3 and Proposition 8.4 prove Theorem 1.3.

Appendix A. The two-body realization theorem

We include a simplex proof of the two-body theorem of Shephard [9], with zero coefficients allowed. Besides completing the threshold statement, the proof constructs the realizing bodies directly from the ordered ratios of consecutive mixed volumes.

Proposition A.1. *A homogeneous polynomial*

$$F(x,y)=\sum_{i=0}^{d}\binom{d}{i}a_i x^{d-i}y^i$$

belongs to $\mathcal{V}_{2,d}$ if and only if

$$a_i\ge 0\quad(0\le i\le d),\qquad a_ia_j\ge a_{i-1}a_{j+1}\quad(1\le i\le j\le d-1). \tag{A.1}$$

Every such polynomial is realized by two simplices, possibly lower dimensional.

Proof. For necessity, add ε times a ball to each realizing body. All resulting mixed volumes are positive, and Alexandrov–Fenchel makes their successive ratios nonincreasing. They therefore satisfy (A.1); let $\varepsilon\downarrow 0$.

For sufficiency, the inequalities with $i=j$ give adjacent log-concavity. The remaining inequalities exclude internal zero blocks; if $a_i=\cdots=a_j=0$ lies between positive entries, then $a_ia_j\ge a_{i-1}a_{j+1}$ fails. Thus the positive support, if nonempty, is an interval $[l,r]$, and the ratios a_i/a_{i-1} decrease on that interval.

The basic construction uses

$$\Delta_d=\operatorname{conv}(0,e_1,\dots,e_d),\qquad \Lambda_d=\operatorname{conv}(0,\lambda_1e_1,\dots,\lambda_de_d),\qquad \lambda_1\ge\cdots\ge\lambda_d\ge 0.$$

We claim that

$$\operatorname{vol}_d(t\Delta_d+s\Lambda_d)=\frac{1}{d!}\sum_{i=0}^{d}\binom{d}{i}(\lambda_1\cdots\lambda_i)t^{d-i}s^i. \tag{A.2}$$

For positive λ_d, consider a section at last coordinate z. Allocating an additional height η to the first simplex replaces $\eta\Delta_{d-1}$ by $(\eta/\lambda_d)\Lambda_{d-1}$, which contains it. Thus the largest section uses the maximal possible first height $\min\{t,z\}$. Integrating sections gives the recurrence

$$\operatorname{vol}_d(t\Delta_d+s\Lambda_d)=\int_0^t\operatorname{vol}_{d-1}(u\Delta_{d-1}+s\Lambda_{d-1})\,du+\frac{\lambda_ds^d}{d}\operatorname{vol}_{d-1}(\Lambda_{d-1}).$$

Induction proves (A.2); zero values of λ_i follow by continuity.

When all a_i are positive, choose $\lambda_i = a_i/a_{i-1}$ and scale both simplices by $(d!a_0)^{1/d}$. For support $[l, r]$, put $C = (d!a_l)^{1/d}$ and $\lambda_i = a_i/a_{i-1}$ for $l < i \le r$. The required pair is

$$\begin{aligned} K &= C\,\mathrm{conv}(0, e_{l+1}, \dots, e_d), \\ L &= C\,\mathrm{conv}(0, e_1, \dots, e_l, \lambda_{l+1}e_{l+1}, \dots, \lambda_r e_r). \end{aligned} \tag{A.3}$$

To check the boundary case without separate zero-pattern computations, extend the sequence by $a_i^{(\varepsilon)} = a_l\varepsilon^{l-i}$ for $i < l$ and $a_i^{(\varepsilon)} = a_r\varepsilon^{i-r}$ for $i > r$. For small ε, its ratios decrease from ε^{-1} through the prescribed λ_i to ε. The positive construction uses common scale $C\varepsilon^{l/d}$. Apply to both simplices the diagonal map with factors $\varepsilon^{1-l/d}$ on the first l coordinates and $\varepsilon^{-l/d}$ on the others; its determinant is one. The simplices converge to (A.3), so continuity gives the desired polynomial. Empty vertex lists are omitted, and points realize the zero polynomial. □

In degree three, the criterion is simply

$$\begin{aligned} & ax^3 + 3bx^2y + 3cxy^2 + dy^3 \in \mathcal{V}_{2,3} \\ \iff \quad & a, b, c, d \ge 0, \qquad b^2 \ge ac, \quad c^2 \ge bd, \quad bc \ge ad. \end{aligned}$$

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