Unimodular configurations and complete fans

Caius Folkerts

Abstract. We describe all complete fans on a uniformly unimodular ray configuration by integral facet equations and a single degree condition. Nonnegative integral solutions automatically have multiplicity one and meet face to face, giving a finite system of primitive cone replacements that connects every pair of compatible fans. Together with regular-matroid decomposition, this yields an integral construction of all such fans, including nonsimplicial and nonprojective ones. Deep monotonicity imposes a much stronger restriction, forcing the ray matroid to be series-parallel; conversely, every series-parallel matroid occurs as a minor of a deeply monotone ray matroid. Every regular matroid occurs as a minor of a uniformly unimodular smooth Fano ray matroid, and an explicit eight-dimensional example shows that the exceptional R_{10} block cannot be omitted. We also determine all fans on the largest possible ray configurations and obtain consequences for Thomsen intersections and section rings.

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1. Introduction

A unimodular vector configuration controls determinants, but a toric fan also requires a compatible choice of cones. The first problem belongs to the theory of regular matroids, where Seymour's decomposition supplies graphic, cographic and exceptional building blocks. The second is geometric, because even cones generated by lattice bases need not meet in faces or cover the ambient space. Keeping these two problems separate makes it possible both to construct all complete fans on a fixed configuration and to identify the additional restrictions imposed by positivity.

Let N be a lattice of rank $n \geq 1$. A finite spanning configuration $U \subset N$ is *uniformly unimodular* if every independent n-tuple is a lattice basis. We retain the orientations of its primitive vectors, allowing both directions on a line but no repeated positive ray. A complete fan with rays U can exist only when U positively spans $N_{\mathbb{R}}$. Classifying complete fans with this ray property is also motivated by their role in toric syzygies [13].

For a positively spanning ray configuration, consider the pointed full-dimensional cones whose extreme rays lie in U and which contain no other prescribed ray. There are finitely many such cones. Record their oriented facets in a boundary matrix ∂, and let λ indicate which cones contain one fixed generic direction. The crucial fact is that these local data determine the global fan condition.

Theorem 1.1. *For every finite positively spanning ray configuration U, the complete fans with rays U are exactly the nonnegative integral solutions of*

$$\partial c = 0, \qquad \lambda c = 1.$$

Every solution is a zero-one vector. Primitive integer relations among these equations connect every pair of fans through fans using only cones from the two endpoints.

Facet cancellation first makes the generic covering multiplicity constant. Positivity and degree one make the interiors disjoint; applying the same cancellation in quotient stars then forces the boundaries to agree. The last local step is what turns a covering statement into a fan theorem.

For uniformly unimodular U, the construction starts canonically. Its convex hull is terminal reflexive and has exactly U as its nonzero lattice points, so the radial face fan gives an initial solution. Integral rigidity identifies each oriented regular matroid with a unique lattice configuration after a basis is fixed. Consequently the graphic, cographic and R_{10} lattice blocks, their signed sums, and the primitive cone replacements give all complete uniformly unimodular fans, together with an integral

isomorphism criterion. Every such configuration also admits a smooth projective weak Fano realization, although other compatible fans can be nonprojective.

Positivity becomes more restrictive when it is imposed at every corner. A smooth reflexive polytope is *deeply monotone* if the unit parallelepiped spanned by the primitive inward edge directions at each vertex lies in the polytope. In the normal fan, this means that the sum of the positive coordinates of each ray, expressed in every maximal-cone basis, is at most one. It is also equivalent to nefness of every divisor obtained by rounding the ray pairings upward.

Theorem 1.2. *The matroid of the primitive facet normals of a deeply monotone polytope is a direct sum of series-parallel networks. The matroids occurring as minors of these normal configurations are exactly the series-parallel matroids.*

The proof explains why this graph class appears. The corner condition survives every contraction of a ray-generated subspace, while a rank-three configuration containing the six lines of K_4 necessarily exposes a quadrilateral in its ray hull. This excludes $M(K_4)$ as a minor. Conversely, an elementary fan extension adds a triangle along any selected ray and realizes both series and parallel extensions inside the deeply monotone class.

Within the uniformly unimodular class, Fano positivity places no further restriction on the possible matroid minors. Every regular matroid occurs, and the exceptional block already appears in an eight-dimensional example with thirteen rays. This corrects an exclusion of R_{10} proposed in [14]. A separate four-dimensional example with six facet normals shows that excluding unimodular triangular faces does not imply uniform unimodularity. These examples distinguish the global shape of a Fano polytope from the stronger constraints of its individual corners.

Two further consequences make the structure explicit. Heller's extremal theorem bounds the number of rays by $n(n+1)$; equality gives the full type-A_n root system, on which every complete fan is a compatible subdivision of products of simplices. In rank three, one linear relation counts all 729 labelled fans and identifies exactly which are projective. For deeply monotone varieties, the series-parallel representation gives the Thomsen estimate $d_\sigma \cdot C \leq \dim \sigma$. The graphic representation also turns nef section polytopes into difference-inequality polytopes, whose regular unimodular flag triangulations give Koszul section rings.

Conventions

We write $M = \mathrm{Hom}(N, \mathbb{Z})$ and use outward normals, so a divisor with coefficients a_ρ has reflected polytope $\{m \mid \langle m, u_\rho \rangle \leq a_\rho\}$. The positive Thomsen class of an arrangement stratum is

$$d_\sigma = \left[\sum_\rho \lceil \langle \theta, u_\rho \rangle \rceil D_\rho \right], \qquad \theta \in \sigma \subset M_\mathbb{R}/M.$$

Its sign is opposite to the usual Thomsen line bundle. Standard fan, divisor and invariant-curve conventions follow [3]. Matroids may have loops and parallel elements when minors or intermediate constructions are discussed, while ray configurations themselves have no zero vectors or repeated positive directions.

2. Integral ray configurations

2.1. Rigidity and the three regular blocks

Lemma 2.1 (Integral rigidity)**.** *Let U be a unimodular system, and let $B_0 = (b_1, \ldots, b_n)$ be an ordered column basis. Normalize the lattice coordinates so that $u_{b_j} = e_j$ and choose $\chi(B_0) = 1$. Then*

$$(u_e)_j = \chi(b_1, \ldots, b_{j-1}, e, b_{j+1}, \ldots, b_n). \tag{2.1}$$

Consequently, two labelled unimodular systems with the same oriented matroid are integrally lattice-isomorphic. The isomorphism is unique once the images of a labelled basis are specified.

Proof. Cramer's rule expresses $(u_e)_j$ as the determinant of the displayed basis replacement. A nonzero such determinant has absolute value one, so it equals its sign. Formula (2.1) determines every column. Matching two labelled lattice bases therefore matches all columns by an element of $\mathrm{GL}_n(\mathbb{Z})$. □

Choosing a column basis turns a unimodular system into a matrix containing I_n. Every square minor of its remaining columns can be completed, by identity columns, to a maximal minor. Thus the normalized matrix is totally unimodular. Its underlying matroid is regular. We use two classical structural theorems, Seymour's decomposition of regular matroids into graphic, cographic and R_{10} pieces [12], and the uniqueness of the orientation class of a regular matroid up to reorientation [1]. The following integral description specifies how their actual vector realizations enter the classification.

2.1.1. **Graphic blocks.** Orient the edges of a finite graph G. In the lattice of integer vertex vectors having sum zero on each connected component, take $e_{\text{head}} - e_{\text{tail}}$ for each edge. A spanning forest is a lattice basis. Hence every nonzero maximal minor is ± 1. Zero columns of graph loops are omitted.

2.1.2. **Cographic blocks.** Let D_G be an oriented incidence matrix with one vertex row removed from each connected component. Take the images of the standard edge vectors in

$$N_G^* = \mathbb{Z}^{E(G)} / \operatorname{im}(D_G^{\mathsf{T}}).$$

A spanning-forest submatrix of D_G is unimodular, so this quotient is free. Complements of spanning forests give lattice bases of the quotient; this is the integral cographic representation. Equivalently, eliminate the forest columns integrally so that the nonforest columns become an identity matrix. Bridge columns have zero image and are omitted.

2.1.3. **The exceptional block.** In the lattice

$$D_5 = \{z \in \mathbb{Z}^5 \mid \textstyle\sum_i z_i \text{ is even}\},$$

take the ten vectors $e_i + e_j$, $1 \le i < j \le 5$. A full-rank selection of five columns is the signless incidence matrix of a graph whose components are unicyclic with odd cycles. Since each odd cycle uses at least three vertices, there is only one component. Removing its pendant tree edges reduces the ordinary determinant to an odd-cycle determinant, of absolute value two. Since $[\mathbb{Z}^5 \mid D_5] = 2$, the determinant in D_5 has absolute value one.

This configuration represents R_{10}. One direct identification with its usual binary representation is as follows. In the basis $(e_1 - e_5, \dots, e_4 - e_5, 2e_5)$, reduce the ten columns modulo two. The invertible linear change sending the first four coordinate vectors to $e_i + e_5$ and the fifth to $(1, 1, 1, 1, 1)$ sends these columns to the ten five-bit vectors of weight three.

Column sign changes are allowed at every stage. They specify actual ray orientations and are retained in the output data.

The three sum operations have a direct lattice realization. A 1-sum is simply the union of the embedded configurations in $N_1 \oplus N_2$. For a 2-sum, choose primitive non-coloop port columns p_i in the two systems, identify them in

$$N = (N_1 \oplus N_2)/\mathbb{Z}(p_1, -p_2),$$

and delete the port labels.

For a 3-sum, choose a triangle of port columns (a_i, b_i, c_i) on each side. Align their signs so that $c_i = a_i + b_i$, and form

$$N = (N_1 \oplus N_2)/\langle (a_1, -a_2), (b_1, -b_2) \rangle_{\mathbb{Z}}.$$

Delete the three port labels. Require that deleting the triangle leaves each side spanning, equivalently that the triangle contains no cocircuit of either side.

Proposition 2.2 (Signed lattice sums)**.** *These operations preserve uniform unimodularity and realize the corresponding regular-matroid sums. All quotient lattices are free. For the* 2- *and* 3*-sums, the stated hypotheses ensure that the output spans the displayed quotient lattice.*

Proof. Any independent subset of a unimodular system extends to a column lattice basis. In particular, the port lattice in either operation is saturated and has a direct-sum complement. The quotient lattices and the asserted ranks follow. Every primitive circuit in a unimodular system has coefficients in $\{1, -1\}$. Extend a circuit with one element removed to a lattice basis and apply Cramer's rule. This justifies the sign alignment of triangle ports.

The 1-sum assertion follows by block determinants. For a 2-sum write

$$N_1 = \mathbb{Z}p \oplus A, \qquad N_2 = \mathbb{Z}p \oplus B, \qquad \operatorname{rank} A = p_0, \quad \operatorname{rank} B = q_0.$$

A full output basis uses either p_0 columns from the first side and $q_0 + 1$ from the second, or $p_0 + 1$ from the first and q_0 from the second. Its determinant factors into a full side determinant and a quotient-projection determinant. The latter is the determinant obtained by appending the port column in the original side. Both nonzero factors have absolute value one.

For a 3-sum write

$$N_1 = P \oplus A, \qquad N_2 = P \oplus B, \qquad N = P \oplus A \oplus B, \qquad P = \mathbb{Z}a \oplus \mathbb{Z}b,$$

and put $\operatorname{rank} A = p_0$, $\operatorname{rank} B = q_0$. A full output basis has $k_1 + k_2 = p_0 + q_0 + 2$ columns from the two sides. Its projections span A and B, so $k_1 \geq p_0$ and $k_2 \geq q_0$. There are three cases, $k_1 = p_0, p_0 + 1, p_0 + 2$. In the first and last, the determinant again factors into a full side determinant and a projection determinant; append a, b on the other side to see that the latter also has absolute value one.

In the middle case, choose p_0 first-side columns whose projections form a basis of A, and q_0 second-side columns whose projections form a basis of B. Each projection determinant has absolute value one, because appending the two port basis columns gives a full side minor. Integral column elimination reduces the two remaining columns

to $v, w \in P$. If $v = v_1 a + v_2 b$, the original maximal-minor conditions, using in turn the additional port a, b, or $c = a + b$, give

$$|v_1| \le 1, \qquad |v_2| \le 1, \qquad |v_1 - v_2| \le 1.$$

Thus a nonzero v is in $\{\pm a, \pm b, \pm(a+b)\}$. The same holds for w. Their two-dimensional determinant is 0 or ± 1, and is nonzero for the selected output basis. This proves uniform unimodularity.

Finally, reduction modulo two identifies the underlying sums. An output dependence separates into two side sums equal to the same vector in the common port space. Express that vector using the same subset of port labels on both sides. The resulting two side cycles have symmetric difference equal to the output cycle, with the ports deleted. Conversely, matched side cycles give an output dependence. This is precisely the binary 2- or 3-sum rule. Reduction modulo two preserves independence in a unimodular system, so it identifies the underlying matroids. □

Seymour's theorem and Proposition 2.2 construct every regular line matroid by actual lattice operations. The orientation theorem and Lemma 2.1 show that final column sign choices give every integral oriented realization. No nonzero column requires rescaling, because it extends to a lattice basis and is primitive. Repeated positive columns describe the same ray and are removed; opposite columns describe distinct rays and are retained.

2.2. **Exactly which ray orientations admit complete fans**

It is convenient to separate a simple line matroid from the choice to use both directions on some of its lines. This also gives a constructive description of the allowed orientations.

Proposition 2.3 (Flow construction of the ray systems)**.** *Construct a simple rank-n regular configuration $U_0 = (v_e)_{e \in E_0}$ by the preceding lattice blocks and sums. Choose a subset $\mathcal{D} \subseteq E_0$ containing every coloop, and choose*

$$r \in \ker_{\mathbb{Z}} U_0, \qquad r_e \neq 0 \quad (e \notin \mathcal{D}). \tag{2.2}$$

Form the ray system

$$U = \{\operatorname{sgn}(r_e) v_e \mid e \notin \mathcal{D}\} \ \cup \ \{v_e, -v_e \mid e \in \mathcal{D}\}.$$

Every such output is uniformly unimodular and positively spans $N_{\mathbb{R}}$. Conversely, up to integral lattice isomorphism, every positively spanning uniformly unimodular ray

system is obtained this way. A relation satisfying (2.2) *exists exactly when* $\mathcal{D}$ *contains every coloop.*

Proof. Sign changes and antipodal duplication preserve the maximal-minor condition. Give an undoubled output ray the coefficient $|r_e|$. For a doubled pair give $v_e, -v_e$ the respective positive coefficients

$$1 + \max(r_e, 0), \qquad 1 + \max(-r_e, 0).$$

Their difference is r_e, so these coefficients give a strictly positive relation among all output rays. A spanning configuration with a strictly positive relation positively spans its whole space, since the negative of any one generator is a nonnegative combination of the others.

Conversely, simplify a given ray system to one reference vector on each unoriented line. Uniform unimodularity makes this a simple regular configuration. A line occurs either once or as an antipodal pair; let $\mathcal{D}$ record the latter. Full positive span gives a strictly positive rational relation among the rays, and clearing denominators gives an integral one. On a doubled line take the difference of its two coefficients; on an undoubled line take its signed coefficient relative to the reference vector. These entries form the required r. The regular orientation theorem and integral rigidity identify the reference realization with one constructed from the lattice blocks.

A coordinate vanishes identically on $\ker_{\mathbb{R}} U_0$ exactly when its element is a coloop. If no undoubled element is a coloop, its coordinate hyperplane is a proper subspace of the relation space. Finitely many proper rational hyperplanes cannot cover that space, so an integral relation avoids their union. Explicitly, for an integral relation basis $g_1, \ldots, g_t$, take $r(T) = \sum_{j=1}^{t} T^{j-1} g_j$ and choose an integer T outside the finitely many roots of its relevant nonzero coordinate polynomials. When there are no undoubled elements, the zero relation is allowed. □

Only the signs of r_e for $e \notin \mathcal{D}$ affect the ray system. Thus the orientation parameter is a chamber of a finite coordinate hyperplane arrangement on the regular relation space. There are no continuous realization parameters. Equivalently, the final oriented matroid is ray-simple and totally cyclic, meaning that it has no positively parallel pair, and each nonzero covector has both signs.

3. **Positive cycles and complete fans**

Fix a finite positively spanning ray configuration $U = (u_e)_{e \in E}$, with no repeated positive ray. The construction in this section applies without a unimodularity assumption. We seek its complete fans by recording their maximal cones, while

keeping the intersection condition in the equations rather than checking pairs of cones afterwards.

3.1. **Allowed cones and their boundary matrix**

Call $S \subseteq E$ an *allowed block* if $K_S = \mathrm{cone}(u_e \mid e \in S)$ is pointed, its extreme rays are exactly those listed in S, and

$$K_S \cap U = \{u_e \mid e \in S\}.$$

Let $\mathcal{K}_j$ be the allowed blocks of rank j, with $\mathcal{K}_0 = \{\varnothing\}$. The last condition is necessary for a fan using every ray of U, since any additional prescribed ray lying in a cone must itself be a face. Allowed blocks are closed under subsets and faces.

These conditions depend only on signed circuits. Pointedness excludes positive circuits; extremality excludes a signed circuit of the restriction with a singleton sign class; and the final condition excludes a signed circuit with one sign class $\{e\}$ outside S and the other contained in S. Each equivalence follows by taking a minimal nonnegative dependence or a minimal nonnegative expression for the offending vector. In particular, the blocks can be built from independent sets by successive ray insertion, with the circuit conditions imposed at each step. Nonsimplicial blocks are retained.

A subset $F \subset S$ is a face precisely when a covector of the restriction is zero on F and strictly positive on $S \setminus F$; it is a facet if its rank is one less. Choose an orientation for the span of each $F \in \mathcal{K}_{n-1}$. If F is a facet of $S \in \mathcal{K}_n$, put $\partial_{F,S} = 1$ or -1 according to the side of its span containing K_S, and put $\partial_{F,S} = 0$ otherwise. Thus

$$\partial \in \{0, 1, -1\}^{\mathcal{K}_{n-1} \times \mathcal{K}_n}$$

is an oriented facet boundary matrix. Changing a facet orientation merely negates its row.

Choose a direction w outside all ray-generated hyperplanes and set

$$\lambda_S = \mathbf{1}_{w \in \operatorname{int} K_S}.$$

This choice can also be made from the oriented matroid. Given an ordered ray basis $b_1, \ldots, b_n$, take

$$w = u_{b_1} + \varepsilon u_{b_2} + \cdots + \varepsilon^{n-1} u_{b_n}$$

for sufficiently small positive ε. Its side of any facet span is the first nonzero chirotope sign on that ordered basis. Both ∂ and λ are therefore determined by finite oriented-matroid data.

Theorem 3.1 (Positive degree-one gluing)**.** *The nonnegative integral solutions of*

$$\partial c = 0, \qquad \lambda c = 1 \tag{3.1}$$

are exactly the indicators of the maximal cones of complete fans with ray configuration U. In particular, every solution has $c_S \in \{0, 1\}$, and no separate intersection or ray-usage conditions are needed.

The facet equations express local cancellation, while the degree equation fixes the total covering multiplicity. The step requiring care is that cancellation must also exclude mismatched subdivisions along the boundaries of the cones. This is a polyhedral version of the cocircuit description of triangulations in [5, Theorem 1.1], with quotient-star arguments as in the theory of multifans [7].

Proof. Away from the finitely many facet hyperplanes, define

$$k(x) = \sum_{S \in \mathcal{K}_n} c_S \, \mathbf{1}_{\operatorname{int} K_S}(x).$$

Cross a facet hyperplane at a point lying on no other distinct facet hyperplane and on no proper face of a facet in that hyperplane. For each facet block containing the crossing point, its contribution to the jump is a common sign times the corresponding entry of ∂c, hence zero. Generic polygonal paths join any two generic points using only such crossings, so k is constant. Its value is $\lambda c = 1$. In dimension one, the empty-facet equation gives the same assertion for the two half-lines.

Since c_S is a nonnegative integer summand of $k(x)$ at a generic interior point of K_S, it is zero or one. The selected cones have disjoint interiors, and their closed union contains every generic point, so they cover the ambient space.

To control their boundaries, let F be a face of a selected cone and put $L = \operatorname{span} K_F$. Project all selected cones having F as a face to $N_{\mathbb{R}}/L$. Their facets are the images of the original facets G containing F. After compatible choices of orientations, the projected incidence sign differs from $\partial_{G,S}$ by a factor depending only on F and G. Every selected cone having G as a facet also has F as a face, so the original row equation cancels all projected contributions. Coincident projected facets cause no difficulty, since cancellation holds before their contributions are combined.

The same crossing argument now shows that the projected cones have constant nonnegative integral multiplicity. This multiplicity is positive because at least one cone contains F, so they cover the quotient. Consequently the original star of F covers a neighborhood of every $x \in \operatorname{relint} K_F$. Indeed, the facet inequalities containing F descend to the quotient, whereas all other facet inequalities are strict at x and remain strict in a sufficiently small common neighborhood.

Suppose x lies in two selected cones and let F, G be their minimal faces containing x. If $F \neq G$, no selected cone can have both as faces, since their relative interiors meet at x. Their stars would therefore be disjoint collections of selected cones, each covering a neighborhood of x. This contradicts generic multiplicity one, so $F = G$.

For two selected cones, choose x in the relative interior of their intersection and let F be this common minimal face. A functional exposing F is nonnegative on the intersection and vanishes at its relative-interior point, hence on the entire intersection. The intersection is therefore exactly F. The selected cones and all their faces form a complete fan.

Completeness places every u_e in a selected maximal cone. By the definition of an allowed block, u_e must be a listed extreme ray of that cone. Thus every prescribed ray occurs. Conversely, a complete fan on U has allowed maximal blocks, two maximal cones on opposite sides of each facet, and covering degree one. Its indicator satisfies (3.1). □

The crossing argument also applies to signed integer cycles. It follows that all generic-direction degree rows agree on $\ker_{\mathbb{Z}} \partial$, so the solution set and the integer kernel of

$$\mathsf{A}_U = \begin{pmatrix} \partial \\ \lambda \end{pmatrix}$$

are independent of the chosen direction. Integrality is essential, however, as nonnegative real solutions may be fractional. Exact facet cancellation is also stronger than an almost-everywhere cover, which alone would permit nonconforming boundaries.

3.2. **A canonical initial fan**

Uniform unimodularity supplies an initial solution without a search through the fan equations.

Lemma 3.2. *If U is a positively spanning uniformly unimodular ray configuration, then $P = \operatorname{conv}(U)$ is terminal reflexive, every member of U is a vertex, and*

$$P \cap N = \{0\} \cup U.$$

Its radial face fan is a canonical complete projective fan with exactly the rays U.

Proof. For $x \in P \cap N$, minimize $\sum_e \alpha_e$ subject to $U\alpha = x$ and $\alpha \geq 0$. The optimum is at most one and is attained at a basic solution with independent support. Extend that support to a ray basis. Uniform unimodularity makes all basic coefficients integral, so their sum is zero or one and x is either zero or a member of U. Applying the same

argument with one column removed shows that this column cannot lie in the convex hull of the others.

Positive spanning puts the origin in the interior of P. Every vertex of

$$P^\circ = \{m \in M_{\mathbb{R}} \mid \langle m, u_e \rangle \leq 1 \text{ for every } e\}$$

is determined by independent active ray normals, which form a lattice basis. The polar is therefore integral, proving reflexivity; the displayed lattice-point statement proves terminality. The radial face fan of P is the normal fan of P° and has the required rays. □

3.3. **Primitive replacements and connectivity**

For integer vectors, write $h \sqsubseteq g$ when their coordinates have compatible signs and $|h_i| \leq |g_i|$ for every i. The minimal nonzero integer vectors in $\ker \mathsf{A}_U$ for this order form the *Graver basis*. We call them primitive replacements, since their positive and negative parts specify the two cone assemblies to be exchanged.

Lemma 3.3. *There are finitely many primitive replacements, effectively determined by U. They connect every pair of complete fans on U. A replacement $c \mapsto c + g$ is allowed exactly when $c + g \geq 0$, and the intermediate fans can be chosen so that every maximal cone belongs to one of the two endpoint fans.*

Proof. Finiteness follows from Dickson's lemma in each orthant. For effectivity, intersect the rational kernel with an orthant and triangulate the resulting pointed rational cone. Choose integral primitive generators for each simplicial cone. An indecomposable lattice point is either one of these generators or lies in their bounded half-open fundamental parallelepiped, because a coefficient at least one permits subtraction of a generator. Enumerating these finite sets and retaining the conformally minimal nonzero vectors gives the Graver basis.

An allowed replacement preserves (3.1); Theorem 3.1 therefore gives another complete fan. For connectivity, let c, d be two fan indicators. If $d - c \neq 0$, choose a primitive vector $g \sqsubseteq d - c$. Then $c + g$ lies coordinatewise between c and d, and

$$\|d - (c + g)\|_1 < \|d - c\|_1.$$

Iteration reaches d. A coordinate absent from both endpoints never appears, proving the last assertion. □

The two assemblies in a replacement cover the same closed region. Their signed indicator has zero boundary and degree, so the crossing argument makes it identically

zero off the boundaries. This region can be nonconvex, which is why primitive replacements are more general than circuit flips; disconnected flip graphs occur even for totally unimodular configurations [10].

For a primitive replacement the region has connected interior. If its interior had several components, assign each changed cone to the component containing its interior. Cones incident along a changed facet lie in the same component, so boundary cancellation holds separately in each component. The two indicators agree there as well, giving degree zero. The replacement would split into proper conformal kernel vectors, contrary to primitivity.

Theorem 3.4 (Integral classification)**.** *Every complete uniformly unimodular fan is obtained by assembling graphic, cographic and R_{10} lattice configurations by signed* 1*-,* 2*- and* 3*-sums, choosing a positively spanning ray orientation as in Proposition 2.3, and applying primitive replacements to the radial face fan of its ray hull. Conversely, every such construction gives a complete uniformly unimodular fan. Nonsimplicial and nonprojective fans are included.*

Two outputs are integrally lattice-isomorphic exactly when an isomorphism of their oriented ray matroids, allowing reversal of the global chirotope sign, carries their maximal-cone families to one another.

Proof. Seymour's decomposition, regular orientation uniqueness and integral rigidity give precisely the ray configurations in Section 2. Lemma 3.2 gives their initial fans, and Lemma 3.3 reaches every solution of (3.1) from those fans. The gluing theorem identifies these solutions with exactly the required complete fans.

A lattice isomorphism preserves the oriented ray matroid and the maximal cones. Conversely, match an ordered ray basis under an oriented-matroid isomorphism. Integral rigidity extends that match uniquely to all rays; matching the maximal blocks then matches every cone and face. □

Different decomposition trees and replacement paths can describe the same fan. The isomorphism criterion accounts for that redundancy while retaining the actual ray orientations, which cannot be discarded by arbitrary reorientation once the cones are chosen.

4. **Products and smooth realizations**

The first lattice operation has a stronger geometric consequence, since its compatible fan structures also split. This reduces the classification to connected regular line matroids.

Proposition 4.1 (Product decomposition)**.** *Suppose a positively spanning ray system is a direct sum $U = U_1 \sqcup \cdots \sqcup U_t$ in $N = N_1 \oplus \cdots \oplus N_t$, with $U_i \subset N_i$. Every complete fan with ray system U is uniquely a product $\Sigma_1 \times \cdots \times \Sigma_t$, where Σ_i is a complete fan in N_i with ray system U_i.*

For a uniformly unimodular ray system, the connected components of its line matroid give such an integral direct sum. Thus every fan in the classification has a canonical product decomposition, up to permutation of the factors, indexed by those components.

Proof. Every cone γ generated by rays in U has the direct-sum form $\gamma = \gamma_1 \oplus \cdots \oplus \gamma_t$, where $\gamma_i = \gamma \cap (N_i)_{\mathbb{R}}$. Each γ_i is a face. Choose functionals strictly positive on the nonzero vectors of the other pointed cones and zero on N_i. Consequently, for a complete fan Σ, its cones lying in $(N_i)_{\mathbb{R}}$ form a complete fan Σ_i. Completeness follows by applying completeness of Σ to points in that summand.

Every cone of Σ is a product of cones of these factor fans. Conversely, choose one full-dimensional cone τ_i of each factor and take $x_i \in \operatorname{int} \tau_i$. A full-dimensional cone of Σ contains $\sum_i x_i$. Its factor cones must be the τ_i, since distinct full-dimensional cones of a factor fan have disjoint interiors. Therefore every product of full-dimensional factor cones belongs to Σ. Taking faces proves the claimed equality and uniqueness.

For the final assertion, the usual matroid component decomposition splits the real spans. Choose a column basis in each component. Their union is a column basis of the entire configuration, hence an integral basis by uniform unimodularity. The real direct sum therefore splits the lattice as well. □

Proposition 4.2 (Smooth projective weak Fano realization)**.** *Every ray system produced by the classification is the exact ray system of a smooth projective weak Fano fan. Its fan is a subdivision, without new rays, of the canonical terminal reflexive seed.*

Proof. Write the seed's polar polytope as

$$Q = \{m \mid \langle m, u_e \rangle \leq 1 \text{ for every } e\}.$$

Choose a generic rational vector (h_e) and perturb the right-hand sides to $1 + \varepsilon h_e$, for sufficiently small positive rational ε. Every original inequality remains a facet

inequality, because a relative-interior point of its original facet has strict slack in all the other inequalities, and this property persists under a small perturbation. The perturbed polytope remains bounded and full-dimensional. Genericity makes it simple, since an intersection of more than n facet hyperplanes imposes a nontrivial rational linear equation on the perturbation parameters.

For sufficiently small ε, each vertex's active normals are a subset of the normals active at one vertex of Q. Indeed, otherwise a sequence of such vertices as $\varepsilon \to 0$ would contradict compactness and the limiting active equations. Thus the new normal fan subdivides the seed and has exactly the same rays. Every maximal cone is generated by an independent ray basis, so it is smooth. Its defining rational polytope, after integral dilation, proves projectivity.

The anticanonical support function of the seed takes value one on every ray and is convex. It remains the anticanonical support function on the subdivision, because no ray was added. Hence the new anticanonical divisor is nef. Its section polytope is the full-dimensional Q, so it is big. The resulting smooth variety is therefore weak Fano. □

Corollary 4.3 (Smooth patch connectivity)**.** *For a fixed allowed ray system, any two compatible complete fans are joined by primitive replacements whose intermediate maximal cones all belong to one of the two endpoint fans. In particular, two smooth complete fans are joined through smooth complete fans, without any assumption of projectivity.*

Restricting the allowed cone family to independent rank-n blocks and starting at the smooth projective seed of Proposition 4.2 gives a constructive classification of precisely the smooth complete fans on those rays.

Proof. The conformal-decomposition proof of Lemma 3.3 keeps every block coordinate between its two endpoint values. A block absent at both endpoints can never appear, and every changing block is inserted or removed only once. Uniform unimodularity makes each independent rank-n block smooth. The same positive-gluing and primitive-replacement proofs apply after all other block columns are deleted; the stated seed survives this restriction. This proves both assertions. □

5. Fano configurations and exceptional minors

5.1. An eight-dimensional exceptional configuration

The R_{10} block is essential even in the smooth Fano subfamily. The following explicit construction also clarifies why positivity of a whole ray configuration cannot exclude an exceptional matroid minor.

Proposition 5.1. *There is an* 8*-dimensional smooth Fano polytope with* 13 *vertices whose vertex configuration is uniformly unimodular and has an* R_{10} *matroid minor. Its ray matroid cannot be constructed from graphic and cographic matroids using only matroid-preserving elementary operations, duality, and* 1*-,* 2*-, and* 3*-sums.*

Proof. Use the representation $a_{ij} = e_i + e_j$ of R_{10} in $D_5 = \{x \in \mathbb{Z}^5 \mid \sum_i x_i \text{ is even}\}$, with edge order

$$12, 13, 14, 15, 23, 24, 25, 34, 35, 45.$$

Negate the two columns 23 and 24, and call the resulting matrix A. Repeat its ten columns with multiplicities

$$h = (2, 2, 2, 1, 1, 1, 1, 1, 1, 1).$$

The resulting map $A_h \colon \mathbb{Z}^{13} \to D_5$ is surjective and unimodular, and

$$A_h\mathbf{1} = Ah = b = (7, 1, 3, 3, 4).$$

Put $L = \ker(A_h)$, a lattice of rank 8, and define

$$Q = \{z \in L_{\mathbb{R}} \mid z_i \geq -1 \text{ for } 1 \leq i \leq 13\}. \tag{5.1}$$

The origin is an interior point. The covector $w = (3, -2, 1, 1, 3)$ is strictly positive on every signed column of A. Thus $\sum_i \langle w, (A_h)_i \rangle z_i = 0$ has strictly positive coefficients, so the lower bounds in (5.1) bound Q.

Translate by $\mathbf{1}$ to identify Q with $\{x \geq 0 \mid A_h x = b\}$. A vertex has independent positive-support columns, so it uses at most one copy of each edge. On the original five labels its support graph has no isolated vertex, because every b_i is positive. Column signs do not change linear independence. A disconnected independent support would have components of sizes two and three. The two-vertex component forces equal positive margins and a positive column, so it must be the edge 34. The other three vertices would then have margins $(7, 1, 4)$, but their three columns are positive and $7 > 1 + 4$, making that component infeasible. Thus the support is connected. Independent columns of the unsigned incidence matrix give either a tree or an odd unicyclic graph. A tree would require its two bipartition classes to have equal margin sums, namely 9, independently of the column signs. No subset of $(7, 1, 3, 3, 4)$ sums to 9. No singleton or pair does, and complements cover the remaining possibilities. Consequently every vertex has exactly five positive coordinates, on an odd unicyclic graph. These five columns form a basis of D_5, so every vertex is integral and nondegenerate.

All thirteen inequalities in (5.1) are facets. For a doubled edge, start at $x = \mathbf{1}$, set one copy to zero and the other to two. For the singleton edges, write E_{ij} for the coordinate unit vectors in $\mathbb{R}^{10}$ and use the kernel vectors

$$\begin{aligned}
g_{15} &= E_{14} - E_{15} + E_{24} + E_{25},\\
g_{23} &= -E_{13} + E_{14} - E_{23} + E_{24},\\
g_{25} &= E_{12} - E_{13} - E_{25} + E_{35},\\
g_{34} &= -E_{12} - E_{34} + \tfrac{1}{2}(E_{13} + E_{14} - E_{23} - E_{24}),\\
g_{45} &= -E_{12} + E_{14} + E_{25} - E_{45}.
\end{aligned}$$

The signed incidence relations give $Ag_e = 0$, while $h + g_e$ has zero in coordinate e and is positive elsewhere. Taking $g_{24} = -g_{23}$ and $g_{35} = -g_{25}$ covers the remaining two edges. Splitting the first three group totals between their two copies gives a point on precisely the required coordinate hyperplane. Thus each coordinate inequality is a facet.

At a vertex, the five positive-coordinate columns form a lattice basis of D_5. Hence the remaining eight coordinate functionals form a basis of L^*, by the exact sequence

$$0 \longrightarrow L \longrightarrow \mathbb{Z}^{13} \xrightarrow{A_h} D_5 \longrightarrow 0.$$

Therefore Q is smooth, with primitive facet normals and lattice distance one from the origin. Its polar P is a smooth Fano polytope with thirteen vertices. The complete facet-normal configuration is a Gale dual of A_h, so it is uniformly unimodular, since complementary maximal minors agree up to sign. Its matroid is the dual of a parallel extension of R_{10}. Contracting the extra series elements recovers $R_{10}^* \cong R_{10}$.

For completeness, R_{10}-minor avoidance is preserved by the stated operations. The matroid R_{10} is self-dual, 4-connected, and neither graphic nor cographic; these are standard properties of the exceptional block in Seymour's theorem [12]. Here is the relevant minor-locality argument. Represent a binary 3-sum by two spaces meeting in the port plane. After contractions on both sides, their images still meet in an image of that plane, of dimension at most two. If an R_{10} minor survives, its induced partition into the two sides therefore has connectivity at most two. Its 4-connectivity forces one surviving side to have at most two elements. Any dimension of that side outside the span of the other would produce a cocircuit of size at most two, which R_{10} does not have. Thus these elements lie in the image of the port plane. Over $\mathbb{F}_2$, its nonzero vectors are images of the three ports; any collapsed port subspace can itself be contracted using port elements. Simplicity excludes duplicate surviving port vectors. The minor is therefore already a minor of the other summand. The same argument with port spaces of dimensions zero and one handles 1- and 2-sums.

Duality preserves avoidance because R_{10} is self-dual. Graphic and cographic matroids have no such minor, so constructions starting only with them cannot give the ray matroid just obtained. □

The example contradicts the exclusion of R_{10} in [14, Theorem 4]. The projection argument there treats an exceptional subconfiguration as the entire projected configuration. The additional projected columns, together with the orientations of the rays, are essential for the positivity of the complete ray hull.

5.2. **Every regular matroid occurs as a Fano ray-matroid minor**

Corollary 5.2. *The minors of ray matroids of smooth projective uniformly unimodular Fano varieties are exactly the finite regular matroids.*

Proof. The ray matroid of a uniformly unimodular system is regular, and regularity is preserved under minors. We prove the converse by an explicit construction.

Let $(u_e)_{e\in E}$ be an integral unimodular representation of a regular matroid in a rank-r lattice N; loops and repeated columns are allowed at this stage. Introduce a private coordinate f_e for every label and take

$$a_e = (u_e, f_e), \qquad b_e = (0, -f_e) \quad \text{in } N \oplus \mathbb{Z}^E.$$

Every vector is nonzero and primitive, and no two lie on the same positive ray. When $u_e = 0$, its two vectors are opposite, which is allowed. A full independent subset must use at least one vector from each pair, and exactly r pairs are used in full. Adding the two columns in a full pair and eliminating the private coordinates reduces its determinant, up to sign, to the corresponding r-column minor of (u_e). Thus the new ray system is uniformly unimodular. Contracting all the b_e recovers the original represented matroid, including its loops and parallel elements.

Add any missing opposite rays, keeping only one vector on each positive ray. The resulting uniformly unimodular system U positively spans. Proposition 4.2 realizes it as the rays of a smooth projective weak Fano fan. Choose positive integral ample support numbers h_ρ. The Fano multiwedge construction of [6] replaces each ray u_ρ by

$$(u_\rho, -e_{\rho,1} - \cdots - e_{\rho,h_\rho-1}), \qquad (0, e_{\rho,j}) \quad (1 \le j < h_\rho).$$

The result is a smooth projective uniformly unimodular Fano fan. Contract its pure extra vectors, delete the newly added opposite rays, and contract the retained b_e. The resulting matroid is exactly the prescribed regular matroid. The empty matroid is obtained by deleting all rays of the fan of $\mathbb{P}^1$. □

Corollary 5.3 (Rational matroids without the unimodularity restriction)**.** *For every integer $d \geq 2$, the minors of ray matroids of smooth projective Fano varieties whose anticanonical class is divisible by d and whose anticanonical polytope is UT-free are exactly the finite matroids representable over $\mathbb{Q}$.*

Proof. Necessity holds because every ray matrix is integral and rational representability is preserved under minors. Conversely, choose an integral representation (u_e) of a rational matroid and use the same private-coordinate vectors $a_e = (u_e, f_e)$ and $b_e = (0, -f_e)$ as in the preceding proof. They are distinct primitive positive rays, including when the original matroid has loops or parallel elements. Beginning with the fan of a product of projective lines, successively insert each prescribed ray by a star subdivision. These subdivisions preserve projectivity and completeness. A projective toric resolution by further star subdivisions makes the fan smooth and retains all prescribed rays [3, §11.1].

Apply the divisible-index form of the same Fano multiwedge construction [6] to this smooth projective fan. Contract the pure extra rays, delete every retained ray other than the labeled a_e, b_e, and contract the b_e. The resulting matroid is the original one, while the ambient Fano variety has the required divisibility and UT-free properties. The empty case follows from any such Fano variety by deletion. □

5.3. **A six-normal obstruction to uniform unimodularity**

A monotone polytope is a smooth reflexive lattice polytope. It is *UT-free* if none of its two-dimensional faces is a unimodular triangle. The following segment–simplex bundle gives a direct counterexample to the proposed inclusion in [14].

Proposition 5.4. *The four-dimensional polytope*

$$Q = \{(x_1, x_2, x_3, z) \mid -1 \leq z \leq 1, \quad x_i \leq 1, \quad -x_1 - x_2 - x_3 + 2z \leq 1\}$$

is smooth, reflexive and UT-free, but its six facet normals are not uniformly unimodular.

Proof. Set $y_i = 1 - x_i$. Then $y_i \geq 0$ and $\sum_i y_i \leq 4 - 2z$, with $-1 \leq z \leq 1$. Thus Q is combinatorially a tetrahedron times an interval, with vertices given by $z = \pm 1$ and either $y = 0$ or $y = (4 - 2z)e_i$. At each vertex the active normals are one of $\pm e_4$ and three members of

$$e_1, e_2, e_3, \qquad v = -e_1 - e_2 - e_3 + 2e_4.$$

Every such set is a lattice basis, proving smoothness. All inequalities are facets at lattice distance one from the interior origin, so the integral polytope is reflexive.

Every vertex has all four original coordinates odd. Consequently Q is an integral translate of twice a lattice polytope, and every edge contains an interior lattice point. It has no unimodular triangular face. On the other hand, e_1, e_2, e_3, v have determinant two. □

This is the bundle SSB$(4, 2)$ of [4]. More generally, for $n \geq 2$, replace the three x-coordinates by $n-1$ coordinates and the coefficient two by an integer $0 \leq a \leq n-1$. The resulting smooth reflexive bundle is UT-free exactly when $n = 2$ or $a \leq n-2$ [4, Proposition 5.14]. Its only potentially nonunit maximal minor has determinant a, so its facet normals are uniformly unimodular exactly when $a \leq 1$. The obstruction is therefore carried by one determinant throughout the family.

6. **Deep monotonicity and series-parallel structure**

A monotone polytope is *deeply monotone* when the unit parallelepiped spanned by the primitive inward edge directions at each vertex is contained in the polytope. Unlike the exclusion of small triangular faces, this is a condition on every corner and every facet. Its effect on the ray configuration is much stronger than uniform unimodularity.

Lemma 6.1 (A positive-coordinate criterion)**.** *Let Σ be a complete smooth fan. Suppose that, relative to the ordered primitive generators of each maximal cone, every ray has coordinates whose positive entries sum to at most one. Then its entire ray configuration is uniformly unimodular.*

Proof. Choose any independent n rays and an interior point of their cone lying on no wall of Σ. This point lies in the interior of a maximal cone σ. Express the chosen rays in the lattice basis of σ, obtaining an integral matrix T. There are positive vectors λ, w with $T\lambda = w$. Every row of T must contain a positive entry. Each column has at most one such entry, equal to one. Since T is square, there is exactly one in every row and column. Permuting columns therefore gives $T = I - A$ with A a nonnegative integral matrix with zero diagonal.

The inequality $\lambda > A\lambda$ prohibits a directed cycle in the support graph of A, since along such a cycle it would imply successively $\lambda_i > \lambda_j$ and eventually $\lambda_i > \lambda_i$. Thus a simultaneous row and column permutation makes A strictly triangular. It follows that $|\det T| = 1$. □

Corollary 6.2. *The primitive facet normals of every deeply monotone polytope are uniformly unimodular.*

Proof. Let v be a vertex of a smooth reflexive polytope, with active primitive outward normals $b_1, \ldots, b_n$ and dual basis $b_1^*, \ldots, b_n^*$. Reflexivity gives $v = \sum_i b_i^*$, and its

inward primitive edge directions are $-b_i^*$. Write another facet normal as $u = \sum_i a_i b_i$. The maximum of $\langle u, \cdot \rangle$ over the corner parallelepiped at v is exactly $\sum_i \max(a_i, 0)$. Deep monotonicity puts this parallelepiped inside the polytope, whose u-inequality is $\langle u, x \rangle \leq 1$. Apply Lemma 6.1 to the normal fan. □

Proposition 6.3. *For a smooth complete toric variety, the positive-coordinate criterion is equivalent both to nefness of every positive Thomsen class and to the requirement that the negative coefficients in every normalized wall relation have total absolute value at most one. These conditions force the variety to be Fano, with deeply monotone anticanonical polytope, and every deeply monotone polytope satisfies them.*

Proof. In a maximal-cone basis b_i, write $\langle \theta, b_i \rangle = c_i - \delta_i$ with $c_i \in \mathbb{Z}$ and $0 \leq \delta_i < 1$. If $u = \sum_i a_i b_i$ is another ray, then the difference between its rounded coefficient and the value prescribed by that cone's Cartier point is

$$\left\lceil \sum_i a_i (c_i - \delta_i) \right\rceil - \sum_i a_i c_i = \left\lceil - \sum_i a_i \delta_i \right\rceil .$$

Here smoothness makes the a_i integral. This quantity is nonnegative for every $\delta \in [0, 1)^n$ exactly when $\sum_i \max(a_i, 0) \leq 1$. The toric support-function criterion for nefness proves the equivalence of the coordinate and nefness conditions.

Applying the coordinate criterion to the outside generator across a wall gives the wall condition, since its positive coordinates are the negatives of the negative wall coefficients. Conversely, under the wall condition a wall relation has either no negative term, or exactly one term $-v$. For every θ, the intersection with d_σ is nonnegative by subadditivity of the ceiling function, applied to the positive side of that relation. The invariant-curve criterion then gives nefness.

Each wall has two outside coefficients equal to one. The wall condition therefore gives $(-K_X) \cdot C \geq 1$ for every invariant curve. The toric strict-convexity criterion makes $-K_X$ ample, even though projectivity was not initially assumed. The corner computation in Corollary 6.2 now identifies the coordinate condition with deep monotonicity. □

The wall condition is often called Bondal's numerical criterion. In the Frobenius terminology of [2, Theorem 5.10], the equivalent nefness condition describes precisely the *extremal Fano* toric varieties, whose extremal contractions are projective bundles or smooth blowups. Their contraction theorem [2, Corollaries 5.16–5.17] consequently gives a sequence of smooth toric blowdowns, through deeply monotone varieties, ending at a product of projective spaces. These facts are compatible with the stronger local criterion above, rather than with the UT-free condition alone.

Lemma 6.4 (Contraction closure)**.** *Suppose a complete smooth fan satisfies the positive-coordinate criterion. Every linear subspace generated by some of its rays is the span of a cone of the fan. Quotienting by this subspace and discarding zero vectors and repeated positive ray directions gives the exact ray system of another complete smooth fan satisfying the same criterion; antipodal directions are retained. In particular, the simplification of any contraction of its unoriented ray matroid is the simplified unoriented ray matroid of another deeply monotone fan.*

Proof. Let L have dimension k, and choose k independent rays spanning it. Choose x in the relative interior of their positive cone, avoiding every intersection $L \cap \operatorname{span}\gamma$ of dimension less than k, for $\gamma \in \Sigma$. These are finitely many proper linear subspaces of L. In a maximal-cone basis containing x, write $x = T\lambda$ for the chosen k ray columns, with $\lambda > 0$. Each column of T has at most one positive entry. Thus x has at most k positive coordinates in this basis. Its minimal fan face γ has dimension at most k. The choice of x forces $\dim\gamma = k$ and $\operatorname{span}\gamma = L$.

The quotient star $\operatorname{Star}(\gamma)$ is a complete smooth fan in $N/(N \cap L)$. Its lattice is the indicated quotient because the generators of γ extend to a lattice basis. Let u be any original ray whose image $\bar{u}$ is nonzero. Choose a maximal quotient cone containing $\bar{u}$ and lift it to a maximal cone containing γ. In the lifted basis, the coordinates of u outside γ are nonnegative and not all zero. Their sum is at most one, so exactly one is one and all the others are zero. Hence $\bar{u}$ is exactly a primitive ray of the quotient fan. Conversely, every quotient-fan ray is an image of an original ray.

Finally, relative to any quotient maximal-cone basis, the coordinates of a projected ray are a subset of its coordinates in the lifted basis. Their positive sum is still at most one. Thus the quotient fan satisfies the criterion and is deeply monotone by Proposition 6.3. This proves the assertion for any ray-generated subspace, hence for every matroid contraction. □

Lemma 6.5 (The rank-three obstruction)**.** *A uniformly unimodular smooth Fano ray system of rank three cannot contain a restriction isomorphic to the cycle matroid of K_4.*

Proof. Regular orientation uniqueness and integral rigidity put the six lines of the restriction in the form

$$e_1, e_2, e_3, \quad e_1 - e_2, e_1 - e_3, e_2 - e_3.$$

Any further unimodular column has coordinates in $\{0, 1, -1\}$. If two nonzero coordinates i, j have the same sign, its determinant with $e_i - e_j$ and the remaining coordinate vector has absolute value two. Hence it has at most two nonzero coordinates, which

must be opposite when there are two. It lies on one of the six existing lines. Thus the whole line system consists of those of A_3. Use the incidence convention $i \to j \mapsto e_i - e_j$. Thus its rays are directed incidence vectors on four graph vertices, with at least one direction present on every pair. Positive spanning makes this semicomplete directed graph strongly connected.

A strongly connected semicomplete directed graph has a directed Hamilton cycle. For completeness, take a longest directed cycle. An outside vertex with both an incoming and an outgoing adjacency to the cycle can be inserted between two consecutive cycle vertices. If none can be inserted, every outside vertex receives from all cycle vertices or sends to all of them. Strong connectivity supplies an edge from a receiving vertex to a sending vertex; inserting these two vertices also lengthens the cycle. Thus no vertex is outside.

Label the four-cycle $1 \to 2 \to 3 \to 4 \to 1$. Choose a direction on each diagonal. Rotating the cycle labels puts the chosen diagonals in the form $1 \to 3$ and $2 \to 4$. The four edges

$$1 \to 2, \quad 2 \to 4, \quad 1 \to 3, \quad 3 \to 4$$

form a directed diamond. Assign vertex potentials $(2, 1, 1, 0)$. If $1 \to 4$ is absent, every available root has value at most one, and the roots of this diamond are exactly the four with value one. They form a quadrilateral facet of the ray hull.

If $1 \to 4$ is present, use instead potentials $(1, 1, 0, 0)$. The four available roots

$$1 \to 3, \quad 1 \to 4, \quad 2 \to 3, \quad 2 \to 4$$

are precisely the roots at value one. They again form a quadrilateral facet; no additional A_3 root can have larger value. In either case the ray hull is not simplicial. This contradicts smoothness of its Fano fan. □

Theorem 6.6 (Series-parallel structure of deeply monotone fans)**.** *The ray matroid of every deeply monotone toric variety is a direct sum of series-parallel networks. Equivalently, it is a graphic matroid with no $M(K_4)$ minor. Its actual primitive ray configuration is integrally isomorphic to directed incidence vectors of a graph with no K_4 minor, with antipodal directions allowed.*

Proof. Uniform unimodularity follows from Corollary 6.2, so the ray matroid is regular. Suppose it has an $M(K_4)$ minor, expressed as $M/C \setminus D$. In a vector representation of M/C, let H be the rank-three span of the surviving K_4 restriction. Choose elements of D whose images form a basis modulo H, and contract them too. Their span meets H trivially, so the K_4 restriction is preserved, while the whole remaining matroid now has rank three. By Lemma 6.4, a deeply monotone three-dimensional fan has the same simplified unoriented ray matroid. It therefore still contains the six-line K_4 restriction, contradicting Lemma 6.5.

A matroid with neither $U_{2,4}$ nor $M(K_4)$ as a minor is a direct sum of series-parallel networks; this is the standard excluded-minor characterization of that class [11]. Regularity already excludes $U_{2,4}$, so it applies here. Series-parallel networks have graph realizations without a K_4 minor. Finally, regular orientation uniqueness and integral rigidity identify the actual signed primitive vectors with a directed incidence realization, including any antipodal pairs. □

Proposition 6.7 (The exact minor class)**.** *The matroids occurring as minors of ray matroids of deeply monotone toric varieties are exactly the series-parallel matroids. More constructively, every series-parallel matroid on $m \geq 1$ elements is a minor of the ray matroid of a deeply monotone smooth Fano fan of dimension at most m, with $2n$ rays and 2^n maximal cones in dimension n. Its anticanonical polytope is combinatorially a cube.*

Proof. Necessity follows from Theorem 6.6 and minor closure of the series-parallel class. For the converse we give one elementary fan operation. Let Σ be deeply monotone and choose one of its rays $u \in N$. In $N \oplus \mathbb{Z}$ retain all $(v, 0)$, and add

$$a = (u, 1), \qquad b = (0, -1).$$

For each old maximal cone γ, take the two cones generated by γ together with a or with b. They form a complete smooth fan. In the upper halfspace a point (x, t) is represented using the old cone containing $x - tu$, while in the lower halfspace one uses the cone containing x. The two subdivisions agree on $t = 0$; they are integral linear shears of product cones.

In a new maximal-cone basis, old rays have their old coordinates and a final zero coordinate. The opposite new ray has the old coordinates of u and a final coordinate

-1, since $a + b = (u, 0)$. Every positive-coordinate sum is therefore still at most one. The new fan is deeply monotone and smooth Fano by Proposition 6.3.

On matroids this adjoins a triangle $\{e, a, b\}$ along the selected element e, retaining e. Deleting e gives a series extension; contracting b gives a parallel extension of e. The same description remains true after old minor operations whenever the surviving element is eligible for the corresponding extension, since the new coordinate is independent of the old ambient space.

Start from the fan of $\mathbb{P}^1$. Deleting one of its antipodal rays gives a one-element nonloop matroid, and contracting one gives a one-element loop matroid. Apply the triangle operation to implement the defining series and parallel constructions, and use products for direct sums. This realizes every series-parallel matroid as a minor. Each extension adds one target element and one ambient dimension; products add both quantities. Hence the dimension is at most m. The initial fan has two rays and two maximal cones; each operation adds two rays and doubles the number of maximal cones. Its cone complex is combinatorially the join with a zero-sphere, so induction also gives the asserted cubical anticanonical polytope. □

6.1. **Intersections along series-parallel cycles**

The graph structure of a unimodular system can force a stronger bound than the ambient dimension alone. In the following statement, parallel edges are allowed; "no K_4 minor" refers to the underlying undirected multigraph.

Proposition 6.8. *Suppose the primitive rays of a smooth complete fan are integrally isomorphic to a collection of directed incidence vectors of a graph with no K_4 minor. Then every Thomsen stratum σ and every invariant curve C satisfy*

$$d_\sigma \cdot C \leq \dim \sigma.$$

No projectivity assumption is needed.

Proof. The incidence configuration is uniformly unimodular. Every vertex of a lifted arrangement cell is integral, since its independent active ray normals form a lattice basis. Translate one such vertex to the origin. A representative θ now pairs in $(-1, 1)$ with every ray, and its integral pairings are precisely its zero pairings. Represent θ by a potential h on the vertices of the graph. The active edges are exactly those joining equal potentials. In a connected component of the original graph, if the active-edge subgraph has c components, its contribution to the stratum dimension is $c - 1$.

The nonzero support of a normalized wall relation is a matroid circuit, hence a simple graph cycle, including the possibility of a cycle of two parallel edges. Denote

this cycle by Z. Orient its traversal so that its circuit coefficients $\varepsilon_e \in \{1, -1\}$ are 1 on edges directed with the traversal and -1 on edges directed against it. The active components partition the vertices of Z.

We first show that this partition is noncrossing in their cyclic order. Suppose instead that two disjoint connected active components A, B have vertices a_1, b_1, a_2, b_2 in that order on Z. An A-path from a_1 to a_2 avoids b_1, b_2, so one of its segments between consecutive intersections with Z joins opposite arcs of $Z \setminus \{b_1, b_2\}$. Call this segment P_A and its endpoints a, a'. Its interior is disjoint from Z, and a, a' separate b_1, b_2 on Z. Applying the same argument to a B-path from b_1 to b_2 gives a segment P_B with interior disjoint from Z and endpoints on opposite arcs of $Z \setminus \{a, a'\}$. The paths P_A, P_B are disjoint because A, B are disjoint. Their four endpoints alternate on Z. The cycle together with these two paths is a subdivision of K_4, a contradiction.

Contract each part of this noncrossing partition, retaining edge multiplicities, and discard loops. The resulting connected multigraph is a cactus, in which each edge lies in exactly one cycle, and its cycles meet only at cut vertices. Here is an elementary induction. A noncrossing partition has an interval part; after compressing consecutive equal labels, this gives a label occurring once. Remove that label from the cyclic word. If its two neighboring labels agree, restoring it adds a pendant two-edge cycle. If they are distinct, restoring it subdivides the edge joining them. Both operations preserve the cactus property. This proves the claim, starting from the one-label case.

Let q be the number of parts visited by Z. Its traversal is an Euler traversal of this cactus and traverses each of its constituent cycles coherently. Along a constituent cycle of length l, a potential can strictly decrease at most $l - 1$ times. The total number of decreases is consequently at most

$$\sum_{\text{constituent cycles}} (l - 1) = q - 1.$$

The case $q = 1$ has no nonloop edges and contributes zero. If the original connected graph component containing Z has c active components, then $q - 1 \le c - 1 \le \dim \sigma$.

Finally, because all nonzero ray pairings are in $(-1, 1)$, the wall intersection is

$$d_\sigma \cdot C = \#\{\text{decreases along } Z\} - \#\{e \in Z \mid \ \varepsilon_e = -1,\ \langle \theta, u_e \rangle \neq 0\}.$$

Here the incidence convention is $v \to w \mapsto E_v - E_w$. Thus a forward edge contributes 1 exactly at a decrease, while an oppositely directed nonactive edge contributes -1 exactly at an increase. Active edges contribute zero. The second term is nonnegative, so the required bound follows. □

Corollary 6.9. *For a smooth projective toric variety whose anticanonical polytope is deeply monotone, every Thomsen stratum and invariant curve satisfy*

$$d_\sigma \cdot C \leq \dim \sigma.$$

Proof. Apply Theorem 6.6 and Proposition 6.8. □

6.2. **Section rings and syzygies**

Proposition 6.10 (Graphic normals and quadratic initial ideals)**.** *Let P be a lattice polytope whose defining integral inequalities, after an integral change of coordinates, have normals among e_i and $e_i - e_j$, with either sign allowed. Its lattice-point toric ideal has a squarefree quadratic initial ideal. In particular, its semigroup algebra is normally generated and Koszul. Consequently, every nef line bundle on a smooth complete toric variety with an integral graphic ray configuration has a Koszul section ring with a squarefree quadratic initial ideal.*

Proof. This is the type-A alcoved-polytope construction [9]; we give its elementary triangulation proof. Triangulate each unit cube in $\mathbb{R}^n$ by the simplices whose vertices form coordinatewise chains of its $0/1$ vertices. The triangulations agree on common faces. Their edges join comparable lattice points whose coordinate differences have absolute value at most one. Thus a clique is a chain in a single unit cube and is a simplex, proving that the global triangulation is flag. Each maximal simplex is unimodular.

The triangulation is regular with the exact height function

$$H(z) = (n+1)\sum_i z_i^2 - \Big(\sum_i z_i\Big)^2 \qquad (z \in \mathbb{Z}^n).$$

For the simplex with successive prefix vertices in $[0,1]^n$, its supporting linear form is $\ell(z) = \sum_i (n+2-2i) z_i$. Indeed,

$$H(z) - \ell(z) = \sum_i z_i(z_i - 1) + \sum_{i<j} (z_i - z_j)(z_i - z_j - 1) \geq 0$$

for every integral z. Equality holds exactly at the $0/1$ prefix vertices. Translation and permutation give every simplex of the triangulation.

The integer hyperplanes $x_i = c$ and $x_i - x_j = c$ are unions of faces of this triangulation. Therefore P is a union of simplices. Convexity makes this the induced subcomplex on $P \cap \mathbb{Z}^n$, so the restricted triangulation remains regular, unimodular and flag. This also applies when P is lower-dimensional; its simplices are unimodular in their induced affine lattices.

For completeness, unimodularity gives a unique minimum-height monomial in each toric fiber. The lattice point of degree k has integral barycentric coefficients in the smallest simplex containing its quotient by k. The other monomials are exactly those whose supports are nonfaces of the triangulation. Hence the initial ideal is its Stanley–Reisner ideal. Flagness makes that ideal squarefree and generated in degree two. A quadratic Gröbner basis gives Koszulness. Finally, the section polytope of a nef line bundle on an integral graphic fan has exactly the displayed difference inequalities, and its section ring is the corresponding semigroup algebra. □

Corollary 6.11 (Syzygies on deeply monotone toric varieties)**.** *Let X be a smooth projective complex toric n-fold, $n \geq 2$, whose anticanonical polytope is deeply monotone. Every nef line bundle has a Koszul section ring with a squarefree quadratic initial ideal. Every ample line bundle is very ample and gives a projectively normal embedding with ideal generated by quadrics.*

For $p \geq 2$, a line bundle L satisfying

$$L \cdot C \geq \left\lfloor \frac{p(n+p)}{p+1} \right\rfloor \qquad \text{for every invariant curve } C$$

is very ample and satisfies Property N_p.

Proof. Theorem 6.6 gives an integral graphic representation of the ray configuration, so Proposition 6.10 proves the first assertions. Theorem 6.6 and Proposition 6.8 give $d_\sigma \cdot C \leq \dim \sigma$ for every stratum and curve. The diagonal-resolution criterion of [6] therefore applies with its degree bound replaced by k. For $q \geq 1$, the maximum of the sum of the q smallest entries of an ordered $(q+1)$-tuple of positive integers with total at most $n+q$ is

$$n + q - \left\lceil \frac{n+q}{q+1} \right\rceil = \left\lfloor \frac{q(n+q)}{q+1} \right\rfloor .$$

At any fixed total, the largest entry is at least the ceiling of the average, and the resulting upper bound increases with the total. Equality is attained by distributing $n+q$ as evenly as possible among the entries. These entries are at most n. The displayed maximum increases with q, so the largest required threshold occurs at $q = p$. It is positive for $n \geq 2$, $p \geq 1$, making L ample and hence very ample. Normal generation has already followed from the unimodular triangulation. □

7. The largest ray configurations

The largest possible ray systems admit a more concrete description. Heller's extremal theorem for unimodular systems provides the sharp bound; the additional statement

below determines all compatible fans on the extremal system.

Theorem 7.1 (Extremal ray systems and their fans)**.** *A complete uniformly unimodular fan of rank n has at most $n(n+1)$ rays. Equality holds precisely when its ray system is integrally isomorphic to*

$$U_{A_n} = \{e_i - e_j \mid 0 \le i, j \le n,\ i \ne j\} \subset \{(x_0, \dots, x_n) \in \mathbb{Z}^{n+1} \mid \textstyle\sum_i x_i = 0\}.$$

For this ray system, obtain all compatible complete fans by choosing, for every nonempty proper subset $S \subset \{0, \dots, n\}$, a subdivision of

$$F_S = \operatorname{conv}\{e_i - e_j \mid i \in S,\ j \notin S\} \cong \Delta_{|S|-1} \times \Delta_{n-|S|}$$

into polytopes using only its vertices, with the subdivisions agreeing on each common face $F_S \cap F_T$, and then taking cones from the origin over all cells and their faces. Every compatible complete fan occurs in this way, uniquely.

The fan is smooth if and only if all these subdivisions are triangulations. For every smooth fan on this extremal ray system, the anticanonical divisor is nef and big. This conclusion also holds for the nonprojective smooth fans; the projective ones are weak Fano.

Proof. Discard one vector from every antipodal pair, retaining one representative of each line. The resulting simple regular matroid has rank n. Heller's theorem states that it has at most $\binom{n+1}{2}$ elements, with equality exactly for the cycle matroid of K_{n+1} [8]. Each line supports at most two distinct primitive rays, proving the bound. Equality forces both orientations of every line and the complete-graph matroid. Uniqueness of regular representations, together with the lattice-basis normalization already proved, identifies this signed ray system integrally with U_{A_n}. Conversely, incidence matrices are totally unimodular, and U_{A_n} positively spans its root lattice, so it is an allowed ray system.

Let $P = \operatorname{conv}(U_{A_n})$. A linear functional on the roots has the form $e_i - e_j \mapsto h_i - h_j$. Its maximizing roots are precisely those directed from the set of maximum values of h_i to the set of minimum values. Such a face has dimension $n-1$ exactly when these two nonempty sets together exhaust all labels. Thus the facets of P are exactly the F_S. The displayed product description follows by recording the source and target barycentric coordinates separately. In particular, the facet cones form the canonical seed fan.

Now let K be a cone of any fan with exact ray system U_{A_n}, and draw its generators as directed edges. A vertex cannot have both an incoming and an outgoing edge, because edges $i \to j$ and $j \to k$ with $i \ne k$ would put the required ray $e_i - e_k =$

$(e_i - e_j) + (e_j - e_k)$ inside K without making it an extreme ray, contrary to the fan intersection axiom. If $i = k$, the two rays are antipodal, contrary to pointedness. Therefore the generators of K form a directed bipartite graph, with every edge going from a source to a sink.

If K is full-dimensional, the incidence rank formula shows that this graph is connected and uses all $n+1$ labels. Its source set S therefore determines a unique cut, and $K \subseteq \mathrm{cone}(F_S)$. It follows that every complete fan on U_{A_n} subdivides the canonical seed. Slicing each cut cone by its facet hyperplane gives the stated subdivision of F_S. The fan intersection axiom gives compatibility on common faces. Conversely, compatible face-to-face subdivisions of the facets glue to a subdivision of the boundary sphere of P; coning from its interior point 0 gives a complete face-to-face fan with exactly the prescribed rays.

A full-dimensional simplicial cone has a ray basis, which is unimodular; hence simpliciality and smoothness coincide. Finally, the seed's convex anticanonical support function is linear on every cone of any such subdivision. On a smooth subdivision it gives a nef anticanonical divisor, and its full-dimensional section polytope gives bigness. □

7.1. **All fans on the twelve roots of type** A_3

In rank three, the root polytope has eight triangular facets and six square facets. The triangles are fixed in every compatible fan, while each square is either left whole or split along one of its two diagonals. These choices are independent because the facet boundaries are simplicial. Hence there are $3^6 = 729$ labelled complete fans, of which $2^6 = 64$ are smooth.

The projective ones are determined by one relation. Write $u_{ij} = e_i - e_j$ for $0 \le i, j \le 3$, and assign a support value h_{ij} to each ray. For the square with source set $\{a, b\}$ and complement $\{c, d\}$, ordered by $a < b$ and $c < d$, put

$$L_{ab} = h_{ac} + h_{bd} - h_{ad} - h_{bc}.$$

The square relation is $u_{ac} + u_{bd} = u_{ad} + u_{bc}$, so an unsplit square requires $L_{ab} = 0$, while its two possible diagonals require opposite strict signs. In source-set order $(01, 02, 03, 12, 13, 23)$, the six forms satisfy exactly one linear relation,

$$L_{01} - L_{02} + L_{03} + L_{12} - L_{13} + L_{23} = 0. \tag{7.1}$$

Comparing the coefficients of the twelve h_{ij} shows that every relation is a scalar multiple of this one.

For example, split only the square with source set 01 and leave the other five squares whole. Their five vanishing forms force $L_{01} = 0$, whereas strict convexity at the new diagonal requires it to be nonzero. This is a nonprojective fan obtained from the canonical projective fan by one primitive replacement. The replacement removes the square and inserts its two triangles. Its boundary and degree agree, and no proper conformal subreplacement is possible, because the outer facets force both triangles as soon as the square is removed.

More generally, the rank-five image of the support-value map is exactly the hyperplane (7.1). A small perturbation of the canonical support values preserves strict convexity across all the old walls, so projectivity is equivalent to realizing the prescribed zero and sign pattern in this hyperplane. A nonzero pattern fails exactly when its nonzero terms, with the signs displayed in (7.1), all have the same sign. There are $2(2^6 - 1) = 126$ such patterns. Thus exactly 603 of the 729 labelled fans are projective, and exactly two of the 64 smooth fans are nonprojective. The two smooth exceptions choose all six diagonals so that the six signed terms in (7.1) are either all positive or all negative.

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