Undecidability of mixed-volume realization

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Abstract. Exact rational mixed-volume realizability is undecidable already for forty-four compact convex bodies in three dimensions, even when every prescribed coefficient is nonnegative. More precisely, rational membership in this fixed coefficient locus is co-recursively enumerable complete. The reduction encodes products of stochastic matrices in contractive systems of planar zonoids and forces those systems from complete cubic volume polynomials. Every coefficient, including repeated-index coefficients, is computed by finite rational matrix operations, while the reverse implication allows arbitrary compact convex realizations. The same construction gives a quadratic three-parameter section of the twelve-body locus whose boundary contains infinitely many distinct planar pieces, providing an explicit geometric obstruction to semialgebraicity.

1. Introduction

The volume polynomial of compact convex sets $K_1, \dots, K_n \subset \mathbb{R}^d$ is

$$F_K(t) = \mathrm{Vol}_d\left(\sum_{i=1}^{n} t_i K_i\right) = \sum_{|\alpha|=d} c_\alpha t^\alpha.$$

Let $\mathcal{V}_{n,d}$ denote the set of its complete coefficient arrays, with all K_i nonempty. Prescribing these coefficients asks for one simultaneous realization of all the mixed volumes, including those with repeated arguments. The Heine–Shephard problem seeks a characterization of this locus. An algorithmic version asks whether a finite array of rational numbers can be recognized as realizable.

Theorem 1.1. *Rational membership in $\mathcal{V}_{44,3}$ is co-recursively enumerable complete, also when inputs are restricted to nonnegative coefficients. In particular, it is undecidable, and there is no sound effective verifier accepting finite certificates for every realizable rational input.*

Here a set is *co-recursively enumerable* if an algorithm eventually recognizes every input outside it. Completeness is under computable many-one reductions. Thus nonrealizability has effective finite certificates, whereas no such complete system exists for realizability. The theorem fixes both the ambient dimension and the number of

bodies; padding by points gives the same conclusion for every $n \geq 44$ in dimension three.

The geometric construction starts from a rational probabilistic automaton. Its transition matrices act on states, while a second family of contracting planar maps assigns a distinct direction to each finite word. The acceptance value of that word is recorded in the length of an exposed edge of a convergent zonoid sum. The condition that every word remain below a cutpoint is then equivalent to the existence of one Minkowski summand.

To recover this infinite system from finitely many numbers, two ingredients are needed. A small collection of reference segments calibrates oblique projections between two coordinate planes, and equality in the planar Minkowski inequality forces the prescribed images and recurrences. At the same time, the geometry of the word directions makes every complete cubic coefficient rational and effectively computable. This exact summation is what permits a reduction whose input consists solely of finitely many rational coefficients.

The construction also retains real parameters. For fixed rational transition data it gives a rational quadratic map $\Psi(p, \theta)$ such that realizability is equivalent to

$$pP_w f \leq \theta \qquad \text{for every finite word } w.$$

A three-state example turns these inequalities into an intersection of halfspaces with infinitely many distinct exposed boundary planes. It yields an explicit polynomial section witnessing nonsemialgebraicity of $\mathcal{V}_{12,3}$ over $\mathbb{R}$. The sharper three-body threshold is established in [2]; the section here describes a different feature of the locus through the same encoding that produces the algorithmic obstruction.

2. **Encoding words by segment directions**

Fix an alphabet $\mathcal{A} = \{1, \dots, s\}$ with $s \geq 1$ and rational row-stochastic matrices $P_a \in \mathbb{Q}^{m \times m}$ sharing a rational stationary probability row λ. Thus $\lambda P_a = \lambda$ for every a. Let p be a rational probability row, $f \in [0,1]^m \cap \mathbb{Q}^m$ an output column, and $0 < \theta < 1$ a rational cutpoint. For $w = a_1 \cdots a_k$, write $P_w = P_{a_1} \cdots P_{a_k}$, including $P_\varnothing = I$. The condition to be encoded is

$$pP_w f \leq \theta \qquad (w \in \mathcal{A}^*). \tag{1}$$

Set

$$q = s + 1, \qquad \rho = \frac{1}{4s}, \qquad T_a = \rho \begin{pmatrix} q^{-1} & aq^{-1} \\ 0 & 1 \end{pmatrix}, \qquad E = [-e_2/2, e_2/2].$$

The geometric word vector and its tag are

$$g_w = T_w e_2 = \rho^{|w|}(t_w, 1), \qquad t_w = \sum_{i=1}^{|w|} a_i q^{-i}.$$

The tags lie in $[0,1)$ and are distinct. They are finite base-q expansions whose digits lie in $\{1,\dots,q-1\}$, so no expansion can be obtained from another by adding trailing zeros.

For $v \in \mathbb{R}^d$, write $S(v) = [-v/2, v/2]$. Define planar zonoids

$$K_0 = \sum_{w\in\mathcal{A}^*} S(g_w), \qquad K_j = \sum_{w\in\mathcal{A}^*} (1 + (P_w f)_j) S(g_w) \quad (1 \le j \le m). \tag{2}$$

The contribution of words of length k is $O((s\rho)^k)$, so the sums converge in Hausdorff distance. Every K_j has positive area, since the nonparallel segments E and $T_1 E$ occur with positive coefficients. Stationarity makes the unweighted body recoverable from the state bodies. With $\gamma = \lambda f$, the equality $\lambda P_w f = \gamma$ for every word gives

$$\sum_{j=1}^{m} \lambda_j K_j = (1+\gamma) K_0. \tag{3}$$

Put $b_j = 1 + f_j$ for $1 \le j \le m$. Splitting words by their first letter gives the state equations

$$K_j = b_j E + \sum_{a=1}^{s} \sum_{k=1}^{m} (P_a)_{jk} T_a K_k. \tag{4}$$

Their contraction is the reason these bodies can later be forced from their projected areas.

Lemma 2.1. *System* (4) *has a unique tuple of compact convex solutions. If a tuple satisfies the equations up to independent translations, it consists of translates of the state bodies in* (2).

Proof. Use Hausdorff distance induced by the planar infinity norm, and the maximum distance on tuples. Since each P_a is row-stochastic and $\|T_a\|_\infty = \rho$, the right side of (4) has Lipschitz constant $s\rho = 1/4$. The contraction theorem gives uniqueness and existence, the latter also supplied by (2).

If the equations contain translation errors v_j, their effect on translation vectors is the linear operator

$$(Lz)_j = \sum_{a,k} (P_a)_{jk} T_a z_k, \qquad \|L\| \le 1/4.$$

Solve $z = v + Lz$ and translate the jth body by $-z_j$. This removes all errors and reduces to the unique exact solution. □

Let $K_p = \sum_{j=1}^{m} p_j K_j$. The cutpoint condition can now be read from the lengths of exposed faces.

Lemma 2.2. *There is a compact convex set D satisfying*

$$D + K_p = (1 + \theta)K_0 \tag{5}$$

up to translation if and only if (1) *holds.*

Proof. If every deficit is nonnegative, the convergent sum $D = \sum_w (\theta - pP_w f)S(g_w)$ satisfies (5).

For the converse, write $F_u(K)$ for the exposed face of K in direction u. Absolute summability identifies each sum in (2) with the compact image of the product of its segments. A point lies in an exposed face exactly when every segment coordinate does, since its support deficit is the sum of nonnegative coordinate deficits. Thus exposed faces commute with these infinite sums.

At $n_w = (1, -t_w)$, only g_w is perpendicular to n_w; every other generator contributes a point to the exposed face. The faces of K_0 and K_p therefore have lengths $|g_w|$ and $(1 + pP_w f)|g_w|$, respectively. Taking lengths of the parallel exposed faces in (5) gives

$$0 \le \operatorname{length} F_{n_w}(D) = (\theta - pP_w f)|g_w|.$$

Every word consequently satisfies (1). □

3. Computing every coefficient

The bodies above contain infinitely many generators, but their mixed volumes can be summed by finite rational operations. The only comparisons needed are signs of affine expressions in word tags; no automaton acceptance inequality enters this calculation.

A rational recognizable weight has the form $c(w) = \ell M_w r$, where the matrices and vectors are rational and $\|M_a\|_\infty \le 1$. Initial and terminal vectors may be signed. This includes all state weights in (2), their finite rational linear combinations, and their prefix-filtered versions. For example, the weight that equals $c(v)$ when $w = av$ and zero otherwise is obtained by adding an initial state whose first transition on a enters the old representation. For the state weights, that entering row is a coordinate row and preserves the norm bound.

Lemma 3.1. *For recognizable weights c, d and rational A, B, C, the sums*

$$\sum_{u,v\in\mathcal{A}^*} c(u)d(v)\rho^{|u|+|v|}|A + Bt_u + Ct_v|$$

are rational and effectively computable. The same holds for one-word affine absolute-value sums and for polynomial moments in t_u, t_v of degree at most one in each variable over any regular relation on pairs of words.

Proof. Read the words synchronously over $\{0, 1, \dots, s\}^2 \setminus \{(0, 0)\}$, padding the shorter word by zeros and using two flags to forbid letters after padding begins. This gives a unique encoding for every pair, including the empty encoding for two empty words.

After clearing denominators, assume A, B, C are integers. A finite automaton determines the sign of the affine expression by tracking

$$R_k = q^k \left(A + B \sum_{i\le k} u_i q^{-i} + C \sum_{i\le k} v_i q^{-i} \right), \qquad R_{k+1} = qR_k + Bu_{k+1} + Cv_{k+1}.$$

The absolute contribution of every possible remaining tail, at this scale, is at most $T = |B| + |C|$. If $R_k > T$ or $R_k < -T$, the sign is settled and the state may be replaced by an absorbing positive or negative state. Otherwise it is an integer in the finite interval $[-T, T]$. At termination its sign is exactly the required sign, including zero.

To sum polynomial moments on a regular relation, tensor its deterministic finite automaton with the recognizable weight matrices and the geometric matrices T_a. For each nonzero letter of the first word, use $M_a \otimes T_a$, and use the identity for padding; do the same for the second word. The resulting finite rational transfer matrix W has absolute row sum at most

$$s^2\rho^2 + 2s\rho = \frac{9}{16} < 1.$$

Its resolvent $(I - W)^{-1} = \sum_{k\ge 0} W^k$ is therefore rational. Suitable initial and terminal vectors select the factors $c(u)\rho^{|u|}t_u^i$ and $d(v)\rho^{|v|}t_v^j$, since $T_w e_2 = \rho^{|w|}(t_w, 1)^T$. The resolvent sums their product over the accepted encodings, with its identity term including empty words. Split the affine absolute-value expression into its nonnegative and negative sign languages, expand it, and apply these moment sums. The one-word construction is identical with a smaller transfer bound. □

For nonnegative weights, let $Z_c = \sum_w c(w)S(g_w)$. The usual zonotope area formula gives

$$2V_2(Z_c, Z_d) = \sum_{u,v} c(u)d(v)\rho^{|u|+|v|}|t_v - t_u|.$$

Thus mixed areas, areas, and widths in rational directions are rational and computable.

To obtain cubic volume polynomials, place these planar bodies in

$$U = \{(x, y, 0)\}, \qquad V = \{(x, 0, z)\},$$

using coordinates (x, y) and (x, z), respectively. A tagged direction becomes $a(t) = (t, 1, 0)$ in U or $b(t) = (t, 0, 1)$ in V. Use the reference segments with vectors

$$e_1, e_2, e_3, \quad s_a = (-a, -q, 1) \quad (1 \le a \le s), \quad s_I = (0, -1, 1). \tag{6}$$

Proposition 3.2. *The complete cubic volume polynomial of these reference segments and any finite collection of the state zonoids K_j and image zonoids $T_a K_j$, placed in U or V, has effectively computable rational coefficients.*

Proof. For a finite zonotope, volume is the sum of the absolute determinants of unordered generator triples. Absolute convergence extends this formula to the countable sums. With repeated body labels, use independent copies of the same generating measure; the coefficient for multiplicity α is the ordered determinant sum with prescribed body labels, divided by $\prod_i \alpha_i!$.

All determinants reduce to the expressions

$$\begin{aligned} \det[a(x), a(y), r] &= r_3(x - y), \\ \det[b(x), b(y), r] &= r_2(y - x), \\ \det[a(x), b(y), r] &= r_1 - r_2 x - r_3 y, \\ \det[a(x), a(y), b(z)] &= x - y, \\ \det[a(x), b(y), b(z)] &= z - y. \end{aligned}$$

Three generators in one plane give zero; one planar generator and two reference vectors give an affine function of one tag, and three references give a rational determinant. In a three-word sum, the unused tag factors into an ordinary word-mass sum. Lemma 3.1 therefore computes every case. Finally $T_a g_w = g_{aw}$, so the image zonoids are represented by the prefix-filtered weights already included in that lemma. □

4. **Forcing the prescribed geometry**

A volume polynomial records both dimension and projected area. The following elementary facts explain how the reference segments turn those numerical records into fixed geometric maps.

Lemma 4.1. *If the total Minkowski sum of a tuple has interior, the degree of its volume polynomial in the variable belonging to K_i is* $\dim K_i$.

Proof. Translate K_i into its k-dimensional direction space W and let L be the sum of the other bodies. The projection of L to $\mathbb{R}^d/W$ has positive $(d-k)$-volume. Each of its nonempty fibers contributes a translate of tK_i to $tK_i + L$, so Fubini gives a lower bound ct^k. Boundedness in coordinates adapted to W gives an upper bound $C(1+t)^k$. The polynomial with all other variables equal to one therefore has degree k, and nonnegativity of mixed-volume coefficients identifies this with the degree in the original polynomial. □

Lemma 4.2. *Suppose the first three vectors of an ordered segment family form a basis and the fourth has every basis coordinate nonzero. Their absolute three-by-three minors determine the whole segment family up to a common linear transformation of determinant modulus one and independent translations.*

Proof. Normalize the first basis to the prescribed basis; its determinant modulus is fixed by its triple coefficient. Single-replacement minors give the absolute coordinates of every remaining vector. Coordinate reflections make the fourth vector equal to its prescribed value p. For another vector v, the minors involving p, v and one basis vector give $|p_i v_j - p_j v_i|$. Squaring determines $p_i p_j v_i v_j$ from the known coordinate squares. Since every p_i is nonzero, these products determine all relative nonzero coordinate signs. Only an overall sign remains, which does not change a centered segment. □

For $s = (s_1, s_2, 1)$, define $\pi_s(x, y, z) = (x - s_1 z, y - s_2 z)$. The determinant-one shear taking s to e_3 gives the exact extrusion identity

$$[\tau]\,\mathrm{Vol}_3(C + \tau S(s)) = \mathrm{Area}(\pi_s C). \tag{7}$$

For the reference vectors (6), the maps are

$$\pi_I(x, y, z) = (x, y + z), \qquad \pi_a(x, y, z) = (x + az, y + qz).$$

They restrict to the identity on U. On V, the first is the coordinate identification and the second is $L_a = \left(\begin{smallmatrix} 1 & a \\ 0 & q \end{smallmatrix}\right)$. Hence, with the fixed rational scalar $c = \rho/q$,

$$cL_a = T_a.$$

The final ingredient is the equality case of the planar Minkowski inequality [5]. If $\mathrm{Area}(xA + yB) = a(x + y)^2$ for $x, y \geq 0$ and $a > 0$, the two bodies are translates. Their areas and mixed area agree, so equality gives positive homothety and the equal areas fix its ratio at one. This conclusion holds for arbitrary compact convex planar sets.

Theorem 4.3. *From the rational automaton data one can compute a complete rational cubic coefficient array on*

$$N = (s+1)m + s + 5$$

labels which is realizable by compact convex sets in $\mathbb{R}^3$ *if and only if* (1) *holds. For fixed rational* P_a, f, λ*, the array is a rational polynomial map* $\Psi(p,\theta)$ *of degree at most two, and the equivalence holds for all real probability rows* p *and real* $0 < \theta < 1$*.*

Proof. Use the $s+4$ reference segments in (6), together with the canonical bodies

$$B_j = K_j \subset V, \qquad C_{a,j} = T_a K_j \subset U \quad (1 \le j \le m,\ 1 \le a \le s).$$

Proposition 3.2 computes their entire rational volume polynomial F_0. Adjoin a variable z, put $\eta = (1+\theta)/(1+\gamma)$, and make the formal substitutions

$$x_{B_j} \longmapsto x_{B_j} + (\eta\lambda_j - p_j)z \qquad (1 \le j \le m). \tag{8}$$

The resulting polynomial has N variables, and its full coefficient vector is the target. Since all affected B_j lie in one plane, F_0 has total degree at most two in those variables; this proves the polynomial degree bound in (p,θ).

If (1) holds, Lemma 2.2 and (3) give an actual $D \subset V$ with

$$D + K_p = \eta \sum_{j=1}^{m} \lambda_j K_j.$$

On the nonempty open cone of positive parameters satisfying $x_{B_j} > p_j z$, this identity makes the substituted sum equal to the genuine Minkowski sum including zD. Their volume polynomials agree on an open set and hence identically, proving realizability of every prescribed coefficient.

For the reverse implication, suppose arbitrary compact convex bodies realize the target. The coefficient on the first three reference variables is one, so the total sum has interior. Every reference variable has degree exactly one, so Lemma 4.1 makes its body a segment. Since $s_1 = (-1, -q, 1)$ has all three basis coordinates nonzero, it serves as the fourth vector in Lemma 4.2, which normalizes all reference bodies to (6). The coefficients of B_j and D with the segments e_1, e_3 are their y-widths, and are zero; thus these bodies can be translated into V. The analogous coefficients of $C_{a,j}$ with e_1, e_2 put them in U.

Write $a_j = \mathrm{Area}(K_j) > 0$. Read the coefficient of the reference segment s_a in the polynomial of $xC_{a,j} + ycB_j$. In the target it is

$$(\det T_a) a_j (x+y)^2.$$

In the competing realization, (7) identifies it with $\mathrm{Area}(xC_{a,j} + yT_aB_j)$. Planar Minkowski equality forces

$$C_{a,j} \text{ to be a translate of } T_aB_j. \tag{9}$$

Next form the actual body in U

$$H_j = b_jS(e_2) + \sum_{a,k}(P_a)_{jk}C_{a,k}.$$

In the canonical tuple the recurrence (4) makes it K_j, under the planar identification. The coefficient of s_I in the polynomial of $xH_j + yB_j$ is consequently $a_j(x+y)^2$. In the competing realization, (7) and Minkowski equality make B_j a translate of the planar shape of H_j. Together with (9), these are precisely the contractive equations (4), up to translation. Lemma 2.1 fixes every B_j to its canonical K_j, up to translation.

Finally, in V put $L = D + \sum_j p_jB_j$ and $M = \eta\sum_j \lambda_jB_j$. Both are actual convex bodies. On the parameters of $xL + yM$, substitution (8) cancels the coefficients p_j and leaves $\eta\lambda_j(x + y)$ for the state bodies. Extrusion along e_2 and (3) therefore give

$$\mathrm{Area}(xL + yM) = (1 + \theta)^2\,\mathrm{Area}(K_0)(x + y)^2.$$

Minkowski equality makes L, M translates, and the forced state shapes identify M with $(1+\theta)K_0$. Lemma 2.2 now forces (1). The argument uses only nonnegativity and unit sum of p, and $0 < \theta < 1$, proving the assertion for real parameters as well. □

An arbitrary automaton acquires a common stationary row by adjoining a zero-output absorbing state. Replace the transition matrices, output and initial row by

$$\widetilde{P}_a = \mathrm{diag}(1, P_a), \qquad \widetilde{f} = (0, f_1, \ldots, f_m)^T, \qquad \widetilde{p} = (0, p).$$

The acceptance values are unchanged, while the first coordinate row is stationary for every $\widetilde{P}_a$. The same construction therefore encodes arbitrary rational automata with $(s + 1)(m + 1) + s + 5$ labels.

5. **The algorithmic obstruction**

Strict cutpoint emptiness asks whether some finite word satisfies $pP_wf > \theta$. Rote [4, Theorem 1(a)] proves this problem undecidable for automata with four positive rational doubly stochastic 7×7 transition matrices, a rational initial distribution, output $f = e_1$ and cutpoint $1/7$. The uniform probability row is stationary, so these automata fall directly within Theorem 4.3. The complement of strict cutpoint emptiness is the universal property (1).

These automata already encode arbitrary Turing-machine halting. Neary [3, Lemma 9 and Theorem 11] maps an arbitrary Turing-machine halting instance to a binary tag system, and then to a five-pair Post correspondence instance, preserving existence of a halting computation. Rote's construction in Sections 2.8–2.9 transforms those instances into the four-letter seven-state automata, preserving existence of a word above the cutpoint. The end-pair argument is supplied in Rote's correction in Section 2.8. Therefore the universal cutpoint property is co-recursively enumerable complete for this fixed size. Its complement is directly enumerable by testing finite words with exact rational arithmetic.

Proof of Theorem 1.1. Apply Theorem 4.3 with $s = 4, m = 7$. It gives

$$N = 5 \cdot 7 + 4 + 5 = 44$$

and a computable reduction from the universal cutpoint property to rational membership in $\mathcal{V}_{44,3}$. Thus membership is co-recursively enumerable hard.

The upper bound follows from the finite rational-net criterion for volume polynomials [2, Theorem 8.2]. For each fixed n, d, that criterion gives an effective sequence of finite rational tests, all satisfied exactly by realizable coefficient arrays, such that every nonrealizable rational array fails a test. Running those tests enumerates the complement. This proves completeness.

To restrict the inputs to nonnegative coefficients, inspect the generated array. If an entry is negative, the source condition cannot hold, by the forward implication of Theorem 4.3. Replace that output by the nonnegative polynomial $x_1^3 + x_2^3$, padded by zero coefficients. It is not realizable, since Brunn–Minkowski forces the sum of two volume-one bodies in $\mathbb{R}^3$ to have volume at least eight, whereas this polynomial takes value two at $(1, 1)$. The modified reduction preserves the equivalence and has only nonnegative outputs.

A sound complete finite-certificate verifier for realizability would enumerate the realizable rational arrays. Together with the enumeration of nonrealizability, it would decide membership, contrary to the reduction. Equivalently, on a constructed array one could search for a certificate while enumerating violating words; one of the two searches would always terminate. □

The dependence on input coefficients can be confined to a fixed quadratic family.

Corollary 5.1. *There is a rational quadratic map from the probability simplex in $\mathbb{R}^{28}$ to the coefficient space of $\mathcal{V}_{91,3}$ for which rational membership of the image is undecidable.*

Proof. Use Rote's two fixed 28×28 matrices, fixed output and cutpoint $1/28$ [4, Theorem 1(d)]. Only the initial distribution varies. The construction in Sections 3.3–3.4 of [4] makes both row and column sums one, so the uniform probability row is stationary. Theorem 4.3 gives $N = 3 \cdot 28 + 7 = 91$, with the fixed quadratic dependence on the initial distribution. □

6. **An explicit section with infinitely many boundary planes**

The parameter form of the encoding also gives a concrete geometric obstruction over the real numbers. Take one letter and three states, so $s = 1$ and $m = 3$, and fix

$$P_1 = \begin{pmatrix} 5/8 & 1/8 & 1/4 \\ 1/8 & 5/8 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}, \qquad f = (1, 0, 0)^T.$$

Since P_1 is symmetric, its uniform probability row is stationary and

$$K_0 = \frac{K_1 + K_2 + K_3}{4}.$$

The encoding therefore has twelve labels, with membership condition $pP_1^k f \le \theta$ for every $k \ge 0$.

Theorem 6.1. *A rational quadratic three-parameter section of $\mathcal{V}_{12,3}$ is, near an interior parameter point, the cone*

$$\mathcal{S} = \bigcap_{k \ge 0} \{(A, B, C) \in \mathbb{R}^3 \mid A + 2^{-k} B + 4^{-k} C \le 0\}. \tag{10}$$

Its boundary has a relatively open piece in every plane $A + 2^{-k} B + 4^{-k} C = 0$. Consequently $\mathcal{V}_{12,3}$ is not semialgebraic over $\mathbb{R}$.

Proof. The vectors $\mathbf{1}$, $u = (1, -1, 0)^T$ and $v = (1, 1, -2)^T$ are eigenvectors of P_1 with eigenvalues $1, 1/2, 1/4$. Hence

$$P_1^k f = \tfrac{1}{3}\mathbf{1} + \tfrac{1}{2}\, 2^{-k} u + \tfrac{1}{6}\, 4^{-k} v.$$

On the affine parameter space $\sum p_i = 1$, use coordinates

$$A = \tfrac{1}{3} - \theta, \qquad B = \frac{p_1 - p_2}{2}, \qquad C = \frac{p_1 + p_2 - 2p_3}{6}.$$

Their inverse is

$$p_1 = \tfrac{1}{3} + C + B, \qquad p_2 = \tfrac{1}{3} + C - B, \qquad p_3 = \tfrac{1}{3} - 2C, \qquad \theta = \tfrac{1}{3} - A.$$

A sufficiently small ball Ω about the origin lies entirely in the domain $p_i > 0$, $0 < \theta < 1$. Theorem 4.3 identifies the preimage of $\mathcal{V}_{12,3}$ on this ball with $\mathcal{S} \cap \Omega$.

Put $t_k = 2^{-k}$ and $H_k = \{A + Bt_k + Ct_k^2 = 0\}$. For sufficiently small $\varepsilon > 0$, the point

$$q_k = \varepsilon(-t_k^2, 2t_k, -1)$$

lies in Ω. The jth defining expression of (10) takes value $-\varepsilon(t_j - t_k)^2$ there. It vanishes for $j = k$ and is at most $-\varepsilon t_k^2/4$ for every $j \neq k$. Since all coefficient vectors $(1, t_j, t_j^2)$ have norm at most $\sqrt{3}$, a relatively open neighborhood of q_k in H_k preserves every other inequality strictly. Each such point is on the boundary, as a small displacement across H_k changes membership.

If $\mathcal{S} \cap \Omega$ were semialgebraic, its boundary would lie in the union of the zero sets of finitely many nonzero defining polynomials. Their product would vanish on a relatively open subset of every H_k, hence on every entire plane H_k. Every distinct linear form $A + Bt_k + Ct_k^2$ would divide that nonzero product, which is impossible. Polynomial preimages of semialgebraic sets are semialgebraic, so the encoding map proves the final assertion. □

Padding by points carries this explicit section into every $\mathcal{V}_{n,3}$ with $n \geq 12$. The example gives a direct negative answer to the semialgebraicity question of Averkov, Borger and Soprunov [1] in these dimensions. On the relatively open boundary piece in H_k, the deficit for the word of length k vanishes while all other deficits remain positive. That plane therefore records the disappearance of the corresponding exposed edge of the summand D. The undecidability theorem shows that exact rational feasibility of the word inequalities cannot be replaced by a terminating recognition procedure.

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