Quadratic generation for smooth polytopes with long edges

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Abstract. Every smooth lattice polytope of dimension n whose edges have lattice length at least n is normal and has a toric ideal generated by quadrics. Thus a single edge bound gives both projective normality and quadratic equations in every dimension. We prove two mixed decomposition theorems; an interior point of a sum admits every prescribed parity in its first summand at edge bound n, and multiplication by an adjoint factor is surjective at edge bound $n+1$. The common mechanism is a lattice-covering theorem for averages of basis boxes. Large projective-space fibers account for the failure of the requisite boxes; their disjointness permits a simultaneous correction. A general exchange argument then reduces quadratic generation in all degrees to the creation of a repeated summand in degree three.

1. Introduction

A smooth lattice polytope determines a projective embedding of a smooth toric variety, but smoothness alone does not yet provide a general description of the generators and relations of its homogeneous coordinate ring. The normality question associated with Oda and the quadratic-generation question associated with Bøgvad ask whether these generators always have degree one and these relations always have degree two; see, for example, [2]. We establish both conclusions under an edge bound depending only on the dimension.

Let $M \cong \mathbb{Z}^n$, let $P \subset M_{\mathbb{R}}$ be a lattice polytope, and write $P_M = P \cap M$. The polytope is *smooth* if the primitive directions of its edges at each vertex form a basis of M. It has the *integer decomposition property* if every point of $kP \cap M$ is a sum of k points of P_M, for every integer $k \geq 1$. Over an arbitrary field $\Bbbk$, its toric ideal is

$$I_P = \ker\bigl(\Bbbk[x_a : a \in P_M] \longrightarrow \Bbbk[M][t], \quad x_a \longmapsto \chi^a t\bigr).$$

Theorem 1.1. *Let P be a smooth n-dimensional lattice polytope, where $n \geq 1$. If every edge has lattice length at least n, then P has the integer decomposition property and*

$$I_P = \bigl(x_a x_b - x_c x_d : a, b, c, d \in P_M,\ a + b = c + d\bigr).$$

The assertion is independent of the ground field. In the language of projective embeddings it is property N_1; the embedding is projectively normal and its homogeneous ideal is generated by quadrics. Bounds on powers of line bundles and bounds on the individual edges are different kinds of positivity; the theorem uses only the latter. Recent work of Song and Wen [3] gives the bound $n-1+p$ for property N_p under additional conditions on the fan. Theorem 1.1 gives the case $p=1$ on every smooth projective fan.

The proof requires more than the existence of integral decompositions. To change one decomposition into another, we must control the residue class of a summand, including when the target approaches a face. The following mixed statements provide that control. If A is an ample integral divisor on a smooth projective toric variety, write P_A for its lattice polytope; the degree of A on an invariant curve is the lattice length of the corresponding edge.

Theorem 1.2 (Prescribed parity)**.** *Let A, B be ample integral divisors on a smooth projective toric n-fold, each having degree at least n on every invariant curve. For every $m \in \operatorname{int}(P_A + P_B) \cap M$ and $\alpha \in M/2M$, there are $a \in P_A \cap M$ and $b \in P_B \cap M$ such that $a + b = m$ and $a \equiv \alpha \pmod{2M}$.*

Theorem 1.3 (Adjoint multiplication)**.** *Let A, B be ample integral divisors on a smooth projective toric n-fold, each having degree at least $n+1$ on every invariant curve. Then*

$$(P_A \cap M) + (\operatorname{int} P_B \cap M) = \operatorname{int}(P_A + P_B) \cap M.$$

Equivalently, multiplication of sections of A and $B + K_X$ onto those of $A + B + K_X$ is surjective.

The distinction between the two bounds already appears for a simplex. The polytope $n\Delta_n$ contains the unit boxes at its vertices, but an interior lattice point first appears in $(n+1)\Delta_n$. In the proof, boxes of side two enforce parity, while a unit box inside an adjoint polytope supplies an interior summand. A simplex factor that is too short to contain the desired box is treated by distributing its homogeneous coordinates directly.

Section 2 explains why these are the right decomposition statements by deriving quadratic generation from an exchange criterion for arbitrary lattice polytopes. Sections 3 and 4 develop the toric geometry and lattice-covering lemma. A corner box can fail along a contractible projective-space fiber of dimension greater than $n/2$, but distinct such loci are disjoint. Section 5 uses this fact to construct corrected boxes, and Sections 6 and 7 prove the mixed theorems by induction on dimension.

2. From parity to quadratic generation

A degree-k decomposition of $z \in M$ is a multiset of k points of P_M with sum z. A *quadratic exchange* replaces two entries by two other points of P_M having the same sum. The following criterion isolates the algebraic part of the argument.

Proposition 2.1. *Suppose P has the integer decomposition property and the following properties, in the saturated affine lattice of each face*

(i) *for every face F, every lattice point of* $\operatorname{relint}(2F)$ *admits a decomposition in F with any prescribed parity of the first summand;*

(ii) *for every proper face G, every lattice point of* $\operatorname{relint}(2G)$ *is the sum of a lattice point of G and a lattice point of* $\operatorname{relint} G$.

Then every decomposition fiber is connected by quadratic exchanges, and I_P is generated by quadrics over every field.

Proof. We first change any triple into one with a repeated entry. Let (a, b, c) be a triple and F its smallest containing face. If some pair, say a, b, already has smallest containing face F, then $a + b \in \operatorname{relint}(2F)$. By (i), write $a + b = p + q$ with $p \equiv c$ modulo twice the face lattice. The midpoint $(p + c)/2$ is integral and belongs to F, so

$$(a, b, c) \longrightarrow (p, q, c) \longrightarrow \big((p + c)/2, (p + c)/2, q\big)$$

are quadratic exchanges.

Otherwise choose a pair with smallest containing face $G \subsetneq F$. If the pair repeats, there is nothing to prove; if it does not, (ii) changes it into u, v with $u \in \operatorname{relint} G$. Every face containing u contains G, so u and the third entry have smallest containing face F. The preceding argument applies. Here G is a proper face of P, even when $F = P$, which is precisely where condition (ii) is needed.

To connect two repeated triples with common sum, write

$$z = a + 2b = c + 2d, \qquad e = z - b - d = (a + c)/2.$$

Then $e \in P \cap M$, so the two exchanges

$$(a, b, b) \longrightarrow (b, d, e) \longrightarrow (c, d, d)$$

connect the triples. Thus degree-three fibers are connected. For $k \geq 4$, induct on k and create a repeated entry in each of two decompositions

$$z = 2b + \sum_{i=1}^{k-2} a_i = 2d + \sum_{i=1}^{k-2} c_i.$$

The point

$$q = z - b - d = \tfrac{1}{2}\left(\sum_i a_i + \sum_i c_i\right) \in (k-2)P \cap M$$

has an integral decomposition $q = g_1 + \cdots + g_{k-2}$. Holding one b fixed, induction connects the first state to $(b, d, g_1, \ldots, g_{k-2})$; holding d fixed connects this state to the second. Finally, distinct target characters are linearly independent, so the kernel is spanned by differences of monomials in common fibers. Along an exchange path each such difference telescopes into monomial multiples of quadratic binomials. □

Theorem 1.2 supplies (i) for every face of the polytope in Theorem 1.1. A proper face of dimension d inherits the edge bound $n \geq d+1$, so Theorem 1.3, applied to that face twice, supplies (ii). It remains to prove the mixed statements and the integer decomposition property.

3. **Contractible curves and local boxes**

Throughout the geometric proof, X is a smooth projective toric n-fold with character lattice M and cocharacter lattice N. Write u_ρ for its primitive ray generators and D_ρ for the invariant divisors. For a real invariant divisor $A = \sum_\rho a_\rho D_\rho$, set

$$P_A = \{x \in M_{\mathbb{R}} : \langle x, u_\rho \rangle + a_\rho \geq 0 \text{ for every } \rho\}.$$

The Cartier point $v_\sigma(A)$ at a maximal cone σ has equality on its rays. Nefness means that all these points belong to P_A. For nef real divisors,

$$P_A = \operatorname{conv}\{v_\sigma(A)\}_\sigma, \qquad P_{A+B} = P_A + P_B = \operatorname{conv}\{v_\sigma(A) + v_\sigma(B)\}_\sigma. \tag{1}$$

These formulas include lower-dimensional nef polytopes. Integral nef divisors have integral Cartier points, and $K_X = -\sum_\rho D_\rho$.

3.1. **The contraction data**

We use Casagrande's description of contractible classes and their behavior under smooth invariant blowups [1]. A normalized extremal class is the class of an actual invariant curve. It is contractible, and every contractible class is represented by an invariant curve and has a primitive relation

$$x_0 + \cdots + x_r = \sum_{j=1}^{q} a_j y_j, \qquad r \geq 1, \quad a_j \in \mathbb{Z}_{>0}, \quad r + q \leq n. \tag{2}$$

For $q = 0$ the right side is zero. The positive rays form a primitive collection; every proper subset spans a cone, whereas the full set does not. Intersections with invariant divisors are the displayed coefficients, so

$$K_X \cdot \gamma = -(r+1) + \sum_j a_j.$$

An edge bound e therefore gives $A \cdot \gamma \geq e$ for every contractible class. Nefness, on the other hand, is tested on normalized extremal classes.

If $q = 0$, the contraction is a $\mathbb{P}^r$ bundle over a smooth projective toric base of dimension $n - r$. If $q > 0$, its exceptional locus is

$$Y = V(\langle y_1, \ldots, y_q \rangle) \longrightarrow T, \qquad b := \dim T = n - r - q, \tag{3}$$

a $\mathbb{P}^r$ bundle. The normal line bundles supplied by the D_{y_j} restrict to $\mathcal{O}_{\mathbb{P}^r}(-a_j)$ on each fiber. The base is projective, because the fiber class on the projective variety Y is numerically effective [1, Corollary 3.1].

For a smooth invariant blowup $\pi : X' \to X$ with exceptional divisor E, every contractible class other than a blowup-fiber line pushes to a primitive relation, and pushforward is injective on these classes. If the pushed class is not contractible, the original class has E-degree one [1, Theorem 5.1]. For center (3), the unique contractible lift of its fiber class has relation

$$x_0 + \cdots + x_r = a_{\min} e + \sum_j (a_j - a_{\min}) y_j, \quad e = \sum_j y_j, \quad a_{\min} = \min_j a_j, \tag{4}$$

with zero coefficients omitted. Its E-degree is $-a_{\min}$, as follows from the type-a case of [1, Lemma 5.4]. For disjoint centers these statements apply successively, with intersections against the other exceptional divisors unchanged.

3.2. **Coordinates on the projective bundles**

The bundles above split equivariantly. Any r fiber rays extend to a smooth maximal cone, hence span a saturated sublattice. Smoothness of the bundle gives one primitive lift of each base ray, and the contraction fan contains every proper fiber cone over every base cone. After an integral splitting of the lattice sequence, the fiber coordinates of the lifted rays are integral piecewise-linear functions on the base fan. They define line bundles $E_0, \ldots, E_r$, one normalized to zero, with bundle

$$\mathbb{P}_T \left(\bigoplus_{i=0}^{r} \mathcal{O}(E_i) \right).$$

Use the convention that an integral homogeneous fiber weight $\alpha \geq 0$, $|\alpha| = k$, of a divisor C of fiber degree k has base divisor

$$C_\alpha = C_{\text{base}} + \sum_i \alpha_i E_i.$$

The endpoint divisors C_{ke_i} restrict to the coordinate sections, so their polytopes are faces of P_C. For $k > 0$,

$$C_\alpha = \sum_i \frac{\alpha_i}{k} C_{ke_i}. \tag{5}$$

Thus edge bounds and primitive-relation inequalities verified at the endpoints hold at every actual integral weight. Adjunction takes the useful form

$$(C + K_X)_\beta = K_T + C_{\beta+\mathbf{1}}, \qquad |\beta| = k - r - 1. \tag{6}$$

In a multiplication problem, the sum of the base-weight divisors depends only on the total fiber weight.

The normal-height, fiber, and base coordinates are integral and saturated. The normal rays of a smooth face form part of a lattice basis, so the height map onto $\mathbb{Z}^q$ is surjective and its kernel is the face lattice. The bundle lattices form a split exact sequence. Prescribing parity in these successive coordinates therefore prescribes parity in M.

3.3. **Two local criteria**

At $v_\sigma(A)$ let $e_{\sigma,1}, \dots, e_{\sigma,n}$ be the primitive outgoing edge basis. Its corner box of side s is

$$v_\sigma(A) + \sum_i [0, s] e_{\sigma,i}.$$

Lemma 3.1 (Corner criterion)**.** *An ample real divisor A contains all its corner boxes of side s if $A \cdot \gamma \geq sr$ for every extremal relation* (2)*. Conversely, containment implies this inequality for every contractible class.*

Proof. For $S \subseteq \sigma(1)$ put $D_S = \sum_{\rho \in S} D_\rho$. A cone contains at most r positive rays of a primitive collection, and negative coefficients can only decrease $D_S \cdot \gamma$. Hence $A - sD_S$ is nef. Its Cartier point on σ is the box corner $v_\sigma(A) + s \sum_{\rho_i \in S} e_{\sigma,i}$ and belongs to $P_{A - sD_S} \subseteq P_A$.

Conversely, the contraction fan has a maximal cone containing all negative rays and all but one positive ray of a contractible relation. At the associated vertex the slack of the missing-positive-ray inequality is $A \cdot \gamma$; at the box corner selecting the other r positive rays it is $A \cdot \gamma - sr$. Containment makes this nonnegative. □

Faces are smooth in their saturated lattices and inherit contained corner boxes. In particular, edge bound n gives unit boxes in dimension n.

Lemma 3.2 (Displaced slices)**.** *Suppose every invariant degree of an ample real divisor C is at least e. If $h_j \geq 0$ and $|h| := \sum_j h_j \leq e$, then $C_h = C - \sum_j h_j D_{y_j}$ is nef, is ample when $|h| < e$, and has every invariant degree at least $e - |h|$. Its restriction to Y in* (3) *is the exact slice of P_C at normal heights h_j, with fiber degree $C \cdot \gamma + \sum_j a_j h_j$.*

Moreover, if a smooth real polytope has all edges of length at least λ, every vertex has height at least λ above each nonincident primitive facet.

Proof. A positive prime-divisor intersection with an extremal class is one, so $C - eD_\rho$ is nef. The expression

$$C_h = \left(1 - \frac{|h|}{e}\right) C + \sum_j \frac{h_j}{e}(C - eD_{y_j})$$

proves all positivity and degree assertions, including for nonextremal invariant curves. The face defined by the normal equalities in P_{C_h} is the restriction polytope, proving exactness; the fiber degree follows from (2).

At a vertex outside a facet, an outgoing edge decreases the facet height, since otherwise the tangent cone would make that vertex a global minimum. Its primitive height slope is a negative integer and its length is at least λ, giving the last assertion. □

4. **Averages of lattice basis boxes**

The geometric arguments will place a target in a convex hull of matched box centers. The next lemma converts this real-valued information into a lattice decomposition; the bases of the different boxes may be unrelated.

Lemma 4.1 (Lattice covering)**.** *Let $v_1, \ldots, v_N$ be vectors of a full lattice $\Gamma \subset \mathbb{R}^d$ spanning $\mathbb{R}^d$. If c is a convex combination of incidence vectors of subsets that are real bases, then*

$$Z(c) = \sum_j [-c_j/2, c_j/2] v_j$$

meets every translate of Γ. In particular, every convex Minkowski average of centered integral basis boxes is lattice covering.

Proof. We choose an ellipsoidal norm so that comparison with neighboring lattice points places every nearest-point error inside $Z(c)$. The required balance is supplied by a determinant minimization.

Let $\mathcal{B}$ be the bases of the configuration and $\mathcal{P} = \operatorname{conv}\{\mathbf{1}_B : B \in \mathcal{B}\}$. First take $c \in \operatorname{relint} \mathcal{P}$, put $W = \operatorname{span}\{\mathbf{1}_B - \mathbf{1}_{B'}\}$, and minimize on W the function

$$f(t) = \log\det\left(\sum_j e^{t_j} v_j v_j^{\mathsf{T}}\right) - c \cdot t.$$

Cauchy–Binet expands the determinant as $\sum_B \det(V_B)^2 e^{\mathbf{1}_B \cdot t}$. Relative interior gives $\varepsilon > 0$ with

$$\max_B (\mathbf{1}_B - c) \cdot t \geq \varepsilon \|t\| \qquad (t \in W).$$

Thus f is coercive and attains its minimum. The logarithmic gradient lies in $\operatorname{aff} \mathcal{P}$, so vanishing derivatives on W give

$$c_j = a_j v_j^{\mathsf{T}} Q v_j, \qquad a_j = e^{t_j}, \quad Q = \left(\sum_j a_j v_j v_j^{\mathsf{T}}\right)^{-1}.$$

For any $x \in \mathbb{R}^d$, choose a nearest lattice point $z \in \Gamma$ in the Q-norm. Comparison with $z \pm v_j$ yields

$$|v_j^{\mathsf{T}} Q(x - z)| \leq \tfrac{1}{2} v_j^{\mathsf{T}} Q v_j.$$

Since $\sum_j a_j v_j v_j^{\mathsf{T}} Q = I$,

$$x - z = \sum_j a_j \big(v_j^{\mathsf{T}} Q(x - z)\big) v_j \in Z(c).$$

For boundary c, approximate by relative-interior points. The zonotopes converge and remain bounded, so the corresponding lattice points have a constant subsequence; passage to the limit finishes the proof. □

Suppose P_A, P_B contain boxes with the same basis at each maximal cone and centers c_σ, d_σ. If $m = \sum_\sigma \lambda_\sigma (c_\sigma + d_\sigma)$, the common centered average Z satisfies

$$\sum_\sigma \lambda_\sigma c_\sigma + Z \subseteq P_A \cap (m - P_B). \tag{7}$$

Unit boxes and $\Gamma = M$ give a lattice decomposition. Side-two boxes and $\Gamma = 2M$ give every parity class. This explains why the centers, as well as the boxes, must fit into a common nef-divisor construction.

Lemma 4.2. *If two ample integral polytopes on the same smooth fan contain their unit corner boxes, then*

$$(P_A \cap M) + (P_B \cap M) = P_{A+B} \cap M.$$

Proof. The center at σ is $c_\sigma = v_\sigma(A) + \frac{1}{2}\sum_i e_{\sigma,i}$. For a primitive facet normal u, its slack is at least $\frac{1}{2}\sum_i |\langle u, e_{\sigma,i}\rangle| \geq \frac{1}{2}$, because the box is contained and the final sum is a positive integer. Thus these centers are feasible Cartier points for the nef real divisor $A + K_X/2$; likewise for B.

An interior lattice target has every facet slack at least one and therefore lies in

$$P_{A+B+K_X} = P_{A+K_X/2} + P_{B+K_X/2}.$$

Equation (1) expresses it as an average of matched centers, so (7) and Lemma 4.1 apply. A boundary target reduces to matched proper faces, which inherit the smooth fan and unit boxes; induction on dimension completes the proof. □

Corollary 4.3. *Every smooth n-polytope with edges of length at least n has the integer decomposition property.*

Proof. The corner criterion gives unit boxes. Apply Lemma 4.2 to P and $(k-1)P$, and induct on k. □

The next allocation replaces a missing box on a simplex fiber.

Lemma 4.4 (Simplex parity)**.** *For $d \geq 1$ and $k, l \geq d$, every interior lattice target in $k\Delta_d + l\Delta_d$ has a decomposition in each prescribed parity of its first factor.*

Proof. Write the homogeneous target as $m_i > 0$, $\sum_{i=0}^{d} m_i = k+l$. Let $\varepsilon_i, \eta_i \in \{0,1\}$ be the requested and complementary parities, including the zeroth coordinates determined by the degrees. Then $\varepsilon_i + \eta_i \leq m_i$, and the parity-vector sums agree with k, l modulo two. Also $\sum_i \varepsilon_i \leq k$; when $k = d$, parity excludes $d+1$ ones, and when $k \geq d+1$ the bound is automatic. The same holds for η, l. The intervals

$$\varepsilon_i \leq a_i \leq m_i - \eta_i, \qquad a_i \equiv \varepsilon_i \pmod 2$$

therefore admit total k; subtract their lower bounds and divide by two, then distribute the resulting integral total among the integral capacities. □

5. **Large contraction loci and corrected boxes**

The corner criterion shows that a side-two box can fail only along a contractible fiber of dimension greater than $n/2$. Two such fibers cannot meet if their numerical classes

differ; an intersection would have positive dimension and contain a curve whose class is a positive multiple of both classes. Thus the loci

$$Y_i \longrightarrow T_i, \quad r_i > n/2, \quad q_i \geq 1, \quad b_i = n - r_i - q_i$$

are pairwise disjoint. The same argument shows that a class with $q_i = 0$ is the only class in any collection of this kind.

Blow up any subcollection with $q_i \geq 2$. For $q_i = 1$ make no modification and write $E_i = Y_i$. Over a point of T_i, the divisor E_i is

$$\mathbb{P}_{\mathbb{P}^{r_i}} \left(\bigoplus_{j=1}^{q_i} \mathcal{O}(-a_{ij}) \right).$$

With the convention of lines, $\mathcal{O}_{E_i}(E_i)$ is the negative tautological line bundle; its dual is ample on this fiber. Thus E_i has negative degree on every curve contracted by $E_i \to T_i$, and the E_i remain disjoint.

Lemma 5.1. *Let η be a normalized extremal class on the resulting variety, with s positive rays and q' negative rays.*

(i) *If $E_i \cdot \eta = 1$, then $s \leq b_i + 2$.*

(ii) *If E_i, E_j both have positive degree, their two rays form the entire positive collection, and $s = 2$, $q' \geq n - 1 - b_i - b_j$.*

(iii) *Every maximal cone containing E_i contains exactly r_i rays of the original positive collection $U_i = \{x_{i,0}, \ldots, x_{i,r_i}\}$.*

Proof. For (i), a $\mathbb{P}^{s-1}$ contraction fiber meets E_i in a coordinate hyperplane $\mathbb{P}^{s-2}$ on which E_i has degree one. Its map to T_i has no positive-dimensional fiber, since a curve in one would have both positive and negative E_i-degree. Hence $s - 2 \leq b_i$.

For (ii), disjoint divisor rays are a noncone pair, so minimality makes this pair the entire positive collection. The contraction locus is a $\mathbb{P}^1$ bundle over Z of dimension $n-1-q'$; for $q' = 0$ use the whole variety. The two divisors give disjoint sections with dual normal line bundles. The joint map $Z \to T_i \times T_j$ has no positive-dimensional fiber, since both dual bundles would be negative on a curve in it. Thus $\dim Z \leq b_i + b_j$.

For (iii), suppose an old maximal cone containing the q_i negative rays contains k rays of U_i. Primitivity gives $k \leq r_i$. Remove these rays and the negative rays, leaving ν on $n - k - q_i$ rays. The contraction fan supplies a cone containing ν, all negative rays, and any r_i positive rays; its $n + r_i - k$ rays force $k \geq r_i$. Star subdivision preserves these r_i positive rays. □

5.1. **Original boxes away from the loci**

We must select all obstructing *contractible* classes here, because an extremal class after blowup may push to a nonextremal contractible class.

Proposition 5.2. *Let all invariant degrees of an ample divisor L be at least n. Select every contractible class with $L \cdot \gamma_i < 2r_i$, and suppose none has $q_i = 0$. At every original vertex outside their loci, the corner box of side two is contained in P_L.*

Proof. Fix its maximal cone σ and $S \subseteq \sigma(1)$. The desired box corner is the Cartier point of $L - 2D_S$, where $D_S = \sum_{\rho \in S} D_\rho$. We will make this divisor nef by subtracting exceptional divisors over the selected loci, leaving its Cartier point at σ unchanged. The correction must repair each negative fiber degree without creating a negative degree on a different curve.

Activate the selected indices with $\delta_i = 2D_S \cdot \gamma_i - L \cdot \gamma_i > 0$, and write

$$A_i = |S \cap U_i|, \quad k_i = |S \cap \{y_{i,1}, \dots, y_{i,q_i}\}|, \quad \tau_i = \lceil \delta_i / a_{i,\min} \rceil .$$

Since σ is outside Y_i,

$$A_i \le r_i, \quad k_i \le q_i - 1, \quad 1 \le \tau_i \le \delta_i \le 2A_i - 2k_i - n, \quad \tau_i \le 2r_i - n. \tag{8}$$

Blow up the activated loci of codimension at least two, retaining divisors in codimension one. We claim that

$$B_S = \pi^*(L - 2D_S) - \sum_i \tau_i E_i \tag{9}$$

is nef. A blowup-fiber line has degree $\tau_i > 0$. Consider any other normalized extremal class η.

If no E_i is positive, successive pushdown gives an old contractible class γ. For an inactive class, $(L - 2D_S) \cdot \gamma \ge 0$; use $D_S \cdot \gamma \le r$ for an unselected class and inactivity otherwise. All exceptional corrections help. For an activated class, injectivity identifies the unique lift (4); its degree is $-\delta_i + \tau_i a_{i,\min} \ge 0$, with zero intersections against other E_j.

If exactly one E_i is positive, the image of an invariant representative is a nonzero effective multiple of an old invariant curve, so $\pi^* L \cdot \eta \ge n$. Since $\pi^* D_S = \sum_{\rho \in S} \widetilde{D}_\rho + \sum_i k_i E_i$, Lemma 5.1 gives

$$\begin{aligned} B_S \cdot \eta &\ge n - 2(s-1) - 2k_i - \tau_i \\ &\ge n - 2(b_i + 1) - 2(q_i - 1) - \tau_i = 2r_i - n - \tau_i \ge 0. \end{aligned}$$

If two E_i, E_j are positive, then $s = 2$ and neither positive ray is in S. Put $b = b_i + b_j$, $k = k_i + k_j$, $A = A_i + A_j$, and $\tau = \tau_i + \tau_j$. Let N_S be the sum of negative wall-coefficient magnitudes on old rays in S. Discarding other helpful terms gives

$$B_S \cdot \eta \geq n - 2k + 2N_S - \tau.$$

The bounds (8) first give $B_S \cdot \eta \geq 2b + 4 - n$. For a complementary estimate, each U_i, U_j has all but one ray in an invariant wall representing η, so their intersection has at most one ray outside the wall. By Lemma 5.1(ii), at most b wall rays have zero coefficient. Consequently

$$N_S \geq |S \cap U_i \cap U_j| - b - 1 \geq A - n - b - 1.$$

Together with $\tau \leq 2A - 2k - 2n$, this gives $B_S \cdot \eta \geq n - 2b - 2$. The two lower bounds sum to two, so one is positive. Three positive exceptional rays are impossible, because any two already form a primitive noncone pair. This proves nefness.

The unchanged cone σ contains no exceptional ray, so its Cartier point for B_S is $v_\sigma(L) + 2\sum_{\rho_i \in S} e_{\sigma,i}$. It belongs to $P_{B_S} \subseteq P_L$, since (9) subtracts effective divisors. Varying S supplies every box corner. □

6. **Prescribing the parity of a summand**

We prove Theorem 1.2 by induction on n. Dimension zero is immediate. For intervals, an interior target gives a feasible first-summand interval with integral endpoints and length at least one, so both parities occur. For $X = \mathbb{P}^n$, Lemma 4.4 applies.

Select the union of all contractible classes on which at least one of A, B has degree less than $2r$. Every selected r exceeds $n/2$, so the loci are disjoint. We first treat a possible fiber-type class, then build boxes and their centers on a common blowup. Targets outside the sum of the center polytopes will lie close to an old facet or a selected locus, where induction applies.

6.1. **A fiber-type class**

If a selected class has $q = 0$, it is the sole selected class and X is a split $\mathbb{P}^r$ bundle over a base of dimension $b = n - r < n/2$. Both fiber degrees are at least $n \geq r$. Lemma 4.4 splits the target's homogeneous fiber coordinates in the prescribed parity. Each resulting base-weight divisor has edge degrees at least n, inherited from the coordinate-section faces through (5). Their sum depends only on the total weight, and the base target is interior because the original target satisfies the corresponding facet

inequalities strictly. Induction in dimension b gives the base parity, which combines with the fiber parity to give the prescribed class in M.

Henceforth every selected $q_i \geq 1$. Blow up their disjoint loci as in Section 5.

6.2. **Shifted boxes and their centers**

For $L = A, B$, set $d_{L,i} = L \cdot \gamma_i$ and

$$t_{L,i} = \max\left\{2(q_i - 1), \left\lceil \frac{2r_i - d_{L,i}}{a_{i,\min}} \right\rceil, 0\right\}.$$

Both factors use the same selected loci, even if one had no obstruction there. Since $n = r_i + q_i + b_i$ and $r_i > q_i + b_i$,

$$t_{L,i} \leq \max(2q_i - 2, 2r_i - n) \leq n - 2b_i - 2,$$
$$t_{L,i} \leq r_i + q_i - 2.$$

Define the real center divisor

$$\begin{aligned} H_L &= \pi^* L + K_{X'} - \sum_i t_{L,i} E_i \\ &= \pi^*(L + K_X) - \sum_i (t_{L,i} - q_i + 1) E_i. \end{aligned}$$

We check nefness first, since it will allow all the boxes to be averaged. A blowup-fiber line has degree $t_{L,i} - q_i + 1 \geq 0$. Every other invariant representative has $\pi^* L$-degree at least n. If no selected divisor is positive, the degree is at least $n - s + \sum c_j \geq 0$, where c_j are negative-coefficient magnitudes; here $s \leq n$, since $X' \neq \mathbb{P}^n$. With one positive E_i, Lemma 5.1 gives

$$H_L \cdot \eta \geq n - s - t_{L,i} \geq r_i + q_i - 2 - t_{L,i} \geq 0.$$

With two positive divisors, $s = 2$ and

$$\begin{aligned} H_L \cdot \eta &\geq n - 2 + q' - t_{L,i} - t_{L,j} \\ &\geq r_i + q_i + r_j + q_j - 3 - t_{L,i} - t_{L,j} \geq 1. \end{aligned}$$

Thus H_L is nef.

Outside the selected loci, Proposition 5.2 supplies the original side-two boxes, centered at the Cartier points of H_L. At a vertex of Y_i, let $e_1, \ldots, e_{q_i}$ be the old primitive normal directions and complete them by tangent directions. On the new cone indexed by j, the normal basis is

$$e_j, \qquad e_l - e_j \quad (l \neq j). \tag{10}$$

Take the side-two box based at $v + t_{L,i}e_j$ in this basis, with the unchanged tangent directions. Its normal heights satisfy

$$h_l \geq 0, \qquad t_{L,i} \leq |h| \leq t_{L,i} + 2 \leq n - 2b_i;$$

the lower bounds use $t_{L,i} \geq 2(q_i - 1)$.

By Lemma 3.2, the tangent slice is the restriction of $L - \sum_l h_l D_{y_l}$ to Y_i. Its fiber degree is at least $d_{L,i} + a_{i,\min} t_{L,i} \geq 2r_i$, while every other edge degree is at least $n - |h| \geq 2b_i$. In a split $\mathbb{P}^{r_i}$ bundle, every nonfiber primitive collection lifts a base primitive collection and has at most $b_i + 1$ rays. The corner criterion therefore supplies all tangent side-two boxes, proving containment of the full shifted box. For $b_i = 0$, only the fiber inequality is needed, even at the nef endpoint $|h| = n$.

Its base point is the Cartier point of $\pi^* L - \sum_i t_{L,i} E_i$, so its center is exactly the Cartier point of H_L. When $q_i = 1$, this is simply a normal shift in the original basis.

6.3. **Targets covered by the centers**

For an interior lattice target $m \in P_A + P_B$, let $d_\rho(m) > 0$ be its primitive facet slacks. We have $m \in P_{H_A + H_B}$ whenever

$$d_\rho(m) \geq 2 \quad \text{for every old ray,}$$

$$\sum_{j=1}^{q_i} d_{y_{ij}}(m) \geq t_{A,i} + t_{B,i} + 2 \quad \text{for every selected locus.}$$

For $q_i \geq 2$, the second condition is the exceptional-ray inequality; for $q_i = 1$, it strengthens the selected old-facet inequality. Nefness expresses m as an average of matched center sums. The contained side-two boxes and Lemma 4.1, with $\Gamma = 2M$, give every required parity.

6.4. **Targets near an old facet**

If $d_\rho(m) = 1$ at an old facet D, the desired normal parity chooses heights $0 + 1$ or $1 + 0$. The two slices are ample integral polytopes on the smooth facet fan, with edge bounds at least n and $n - 1$ by Lemma 3.2. Their sum is the exact first slice of $P_A + P_B$, and the target is interior in it. Induction in dimension $n - 1$ supplies the tangential parity, which combines with the chosen normal parity by saturation of the face lattice.

6.5. **Targets near a selected locus**

The remaining case is a selected locus with total normal height

$$D = \sum_{j=1}^{q} d_j \le t_A + t_B + 1, \qquad d_j = d_{y_j}(m) > 0.$$

Put $R = n - 2b$. The shift bounds give

$$D \le 2R - 3. \tag{11}$$

We need a split $d = h+k$ in the prescribed normal parity with $h, k \ge 0$ and $|h|, |k| \le R$. Let $\varepsilon_j, \eta_j \in \{0, 1\}$ be the desired and complementary parities. Positivity of d_j gives $\varepsilon_j + \eta_j \le d_j$. Set

$$c_j = \frac{d_j - \varepsilon_j - \eta_j}{2}, \quad E = \sum_j \varepsilon_j, \quad F = \sum_j \eta_j, \quad C = \sum_j c_j.$$

The capacities for $h = \varepsilon + 2u$ and $k = \eta + 2v$ are

$$U = \lfloor (R - E)/2 \rfloor, \qquad V = \lfloor (R - F)/2 \rfloor.$$

They are nonnegative because $E, F \le q \le R$, and (11) implies $U + V \ge C$. Allocate the C integral tokens among the coordinate capacities c_j so that $u_j + v_j = c_j$, $\sum u_j \le U$, and $\sum v_j \le V$. This gives the required normal split.

The exact slices at h, k are nef restrictions to Y, ample there when $b > 0$, and are split $\mathbb{P}^r$-bundle polytopes. Their fiber degrees are at least $n \ge r$. The target is interior in their sum; the sum divisor is the restriction of $A + B - \sum_j d_j D_{y_j}$, which is ample because $D < 2n$, and its incident facet slacks are the original positive slacks. Split the homogeneous fiber coordinates in the desired parity by Lemma 4.4. Each resulting base weight has edge bound at least $n - |h| \ge 2b$ or $n - |k| \ge 2b$, and their sum depends only on the fixed total coordinates. Induction in dimension b supplies the base parity, with no further step when $b = 0$.

These cases exhaust the targets, proving Theorem 1.2.

7. **An interior summand and adjoint multiplication**

We prove Theorem 1.3 by a separate induction on dimension. The geometry follows the parity argument, but the second factor must remain inside P_B, changing the displacement and the center divisor. Dimension zero is immediate. On $\mathbb{P}^n$, homogeneous-coordinate allocation applies to simplices of degrees a and $b - n - 1$, including a point as the second simplex.

Assume $X \neq \mathbb{P}^n$. Then $B + K_X$ is ample; every extremal positive collection has at most n rays, so its degree is at least $n + 1 - (r + 1) + \sum_j a_j \geq 1$.

7.1. **Selected loci and inner slices**

Select all contractible classes with

$$\delta_i = 2r_i + 1 - B \cdot \gamma_i - \sum_j a_{ij} > 0. \tag{12}$$

The bound $B \cdot \gamma_i \geq n + 1$ gives $r_i > n/2$, so their loci are disjoint.

We need the following exact inner-slice description. For $C = A, B$, $h \geq 0$, and $|h| \leq n$, the divisor $C_h = C - \sum_j h_j D_{y_j}$ is ample with edge bound $n + 1 - |h|$. If every $h_j \geq 1$, the slice of P_{B+K_X} at these original normal heights is

$$P_{B_h|_Y + K_Y}. \tag{13}$$

For incident facets this is adjunction, $K_Y = (K_X + \sum_j D_{y_j})|_Y$. A nonincident facet has slack at least $n + 1 - |h| \geq 1$ at every vertex of the displaced face, by Lemma 3.2; hence the canonical inequalities make no further cut. This also proves exactness at the real heights used below.

If a selected class has $q = 0$, then X is a split $\mathbb{P}^r$ bundle over a base of dimension $b = n - r < n/2$. Its two original fiber degrees are at least $n + 1$, and the adjoint second degree is $B \cdot \gamma - r - 1 \geq b \geq 0$. Allocate the integral fiber coordinates between an A weight α and an adjoint B weight β. By (6), the base target has divisor $A_\alpha + B_{\beta+\mathbf{1}} + K_T$. The two parent base divisors have edge bound $n + 1 \geq b + 1$, inherited from coordinate-section faces, so induction in dimension b supplies the base decomposition. We may now assume every selected $q_i \geq 1$.

7.2. **Inner unit boxes away from the loci**

Fix a maximal cone σ outside all selected loci and $S \subseteq \sigma(1)$. Set

$$D_S = \sum_{\rho \in S} D_\rho, \qquad D = B + K_X - D_S.$$

As $D_S \cdot \gamma \leq r$, every class on which D is negative is selected. Activate those with $\delta_i^S = -D \cdot \gamma_i > 0$ and set

$$\tau_i = \left\lceil \delta_i^S / a_{i,\min} \right\rceil, \qquad k_i = |S \cap \{y_{i,1}, \ldots, y_{i,q_i}\}|.$$

Then

$$k_i \leq q_i - 1, \qquad 1 \leq \tau_i \leq \delta_i^S \leq 2r_i - n - q_i.$$

Blow up the activated centers and put $D' = \pi^* D - \sum_i \tau_i E_i$. A blowup-fiber line has degree τ_i. With no positive exceptional divisor, the pushdown and unique-lift argument of Proposition 5.2 gives nonnegative degree.

For the other cases, use

$$D' = \pi^* B + K_{X'} - \sum_{\rho \in S} \widetilde{D}_\rho - \sum_i (q_i - 1 + k_i + \tau_i) E_i.$$

With one positive E_i, Lemma 5.1 yields

$$\begin{aligned} D' \cdot \eta &\geq n + 1 - s - (s-1) - (q_i - 1 + k_i + \tau_i) \\ &\geq 2r_i - n - \tau_i \geq q_i > 0. \end{aligned}$$

With two positive divisors, $s = 2$ and

$$\begin{aligned} D' \cdot \eta &\geq n - 1 + q' - (q_i - 1 + k_i + \tau_i) - (q_j - 1 + k_j + \tau_j) \\ &\geq 2n + 2 - r_i - r_j > 0. \end{aligned}$$

Thus D' is nef. Its Cartier point on the unchanged cone is

$$v_\sigma(B) + \sum_{l=1}^{n} e_{\sigma,l} + \sum_{\rho_l \in S} e_{\sigma,l},$$

and belongs to P_{B+K_X} because the correction subtracts effective divisors. Varying S supplies the unit box based at the Cartier point of $B + K_X$. The ordinary A unit boxes are contained everywhere by the corner criterion.

7.3. **Shifted unit boxes**

On the common blowup of all selected loci, put

$$t = \max\{2(q-1),\ q - 1 + \lceil \delta / a_{\min} \rceil\} \tag{14}$$

at a locus with parameters r, q, b. Then

$$t \leq \max(2q - 2, 2r - n - 1) \leq n - 2b - 2.$$

For the first factor, use the new basis (10) and the unit box based at the Cartier point of $Q_A = \pi^* A - \sum_i (q_i - 1) E_i$. Its heights satisfy $h_l \geq 0$ and $q - 1 \leq |h| \leq q$. In the exact tangent slice $A_h|_Y$, the fiber degree is at least $n + 1 \geq r$, and every other edge degree is at least $n + 1 - q \geq b$. The corner criterion supplies every tangent unit box, so the full box lies in P_A.

For the second factor, take the unit box based at the Cartier point of $Q_B = \pi^*B + K_{X'} - \sum_i t_i E_i$. On the cone indexed by j, its initial normal heights are $h_l = 1$ for $l \neq j$ and $h_j = t - q + 2$. Throughout the box,

$$h_l \geq 1, \qquad t + 1 \leq |h| \leq t + 2 \leq n - 2b.$$

The tangent slice is $B_h|_Y + K_Y$ by (13). Before adjunction, its fiber degree is at least

$$B \cdot \gamma + \sum_l a_l + a_{\min}(t + 1 - q) \geq 2r + 1,$$

so the inner fiber degree is at least r. Every other parent edge degree is at least $n + 1 - |h| \geq 2b + 1$. For a nonfiber extremal collection of size $s \leq b + 1$, adjunction leaves degree at least $2b + 1 - s \geq s - 1$. Thus the corner criterion supplies the tangent boxes, and the full second box lies in P_{B+K_X}.

7.4. **Nefness of the centers**

The centers of the matched boxes are Cartier points of

$$H_A = \pi^*A + \tfrac{1}{2}K_{X'} - \sum_i (q_i - 1)E_i,$$
$$H_B = \pi^*B + \tfrac{3}{2}K_{X'} - \sum_i t_i E_i.$$

At unchanged cones these are also the centers of the preceding boxes. We verify nefness in the same three cases of Lemma 5.1.

For H_A, a blowup-fiber line has degree $(q_i - 1)/2$. Every noncontracted extremal representative has π^*A-degree at least $n + 1$. With no positive selected divisor, the bound $n + 1 - s/2 \geq 0$ suffices. With one positive E_i,

$$H_A \cdot \eta \geq n + 1 - \tfrac{1}{2}s - (q_i - 1) \geq r_i + \tfrac{1}{2}b_i + 1 > 0.$$

With two it is at least $n - (q_i - 1) - (q_j - 1) > 0$, because $q_i + q_j < n$.

For H_B, a blowup-fiber line has degree $t_i - \frac{3}{2}(q_i - 1) \geq \frac{1}{2}(q_i - 1)$. If no selected divisor is positive, rewrite

$$H_B = \pi^*(B + \tfrac{3}{2}K_X) - \sum_i \big(t_i - \tfrac{3}{2}(q_i - 1)\big)E_i.$$

The correction coefficients are nonnegative, and the class pushes to an old contractible class. For a nonselected class, failure of (12) gives

$$(B + \tfrac{3}{2}K_X) \cdot \gamma \geq \tfrac{1}{2}\left(r - 1 + \sum_j a_j\right) \geq 0.$$

For a selected class, its unique lift and (14) give

$$H_B \cdot \eta \geq \tfrac{1}{2}\left(r - 1 + \sum_j a_j - a_{\min}(q-1)\right) \geq 0.$$

With one positive E_i,

$$H_B \cdot \eta \geq n + 1 - \tfrac{3}{2}(b_i + 2) - (n - 2b_i - 2) = \tfrac{1}{2}b_i \geq 0.$$

With two, the bounds on q' and t_i yield

$$H_B \cdot \eta \geq n - 2 + \tfrac{3}{2}q' - t_i - t_j \geq \tfrac{1}{2}(n + b_i + b_j + 1) > 0.$$

Both center divisors are therefore nef.

7.5. **Decomposing the targets**

Take $m \in P_{A+B+K_X} \cap M$ and write d_ρ for its primitive heights relative to $A + B$. If

$$\begin{aligned} d_\rho &\geq 2 \quad \text{for every old ray,} \\ \sum_{j=1}^{q_i} d_{y_{ij}} &\geq q_i + t_i + 1 \quad \text{for every selected locus,} \end{aligned} \tag{15}$$

then $m \in P_{H_A+H_B}$. At an exceptional ray the two center heights are $q_i - \frac{1}{2}$ and $t_i + \frac{3}{2}$, explaining the second threshold. Averaging matched centers and applying Lemma 4.1 to their unit boxes produces a lattice point in $P_A \cap (m - P_{B+K_X})$.

If $d_\rho = 1$ at an original facet D, the first factor has height zero and the inner second factor height one. The parent slices $A_0 = A|_D$ and $B_1 = (B - D)|_D$ have edge bounds $n + 1$ and n. Adjunction and the nonincident-facet reserve identify the inner second slice with $B_1 + K_D$, so dimension-$(n-1)$ induction decomposes the tangential target.

Finally, suppose all old heights are at least two but (15) fails at a selected locus. Then $D = \sum_{j=1}^{q} d_j$ satisfies $2q \leq D \leq q + t$. Give the second parent heights $k = (1, \ldots, 1)$ and the first heights $h = d - k$. Thus $h \geq 0$, $|h| = D - q \leq t$, and $|k| = q$, and both displaced parent slices are ample and exact. Put $U = A_h|_Y$, $V = B_k|_Y$. The remaining target belongs to P_{U+V+K_Y}. The second adjoint divisor is nef; its fiber degree is

$$B \cdot \gamma + \sum_j a_j - r - 1 \geq b + 2q \geq 0,$$

and every other extremal collection has size $s \leq b + 1$, giving degree at least $(n + 1 - q) - s \geq r > 0$.

Allocate the target's homogeneous fiber vector $w \geq 0$ as $w = \alpha + \beta$, with $|\alpha|$ equal to the fiber degree of U. Integral allocation is possible by distributing this total among the capacities w_i. The base target has divisor

$$U_\alpha + V_{\beta+\mathbf{1}} + K_T.$$

Its two parent base weights are ample on the same fan, with edge bounds at least $n+1-t \geq 2b+3$ and $n+1-q = r+b+1 \geq b+1$. Induction in dimension b supplies the base decomposition and completes the allocation in all coordinates. This proves Theorem 1.3.

Proof of Theorem 1.1. Corollary 4.3 gives the integer decomposition property. Every face inherits smoothness and the edge bound, so Theorem 1.2 gives condition (i) of Proposition 2.1. A proper face of dimension d has edges at least $n \geq d+1$; Theorem 1.3, applied twice to that face, gives condition (ii). The exchange criterion completes the proof. □

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