Stress–flex orthogonality on spheres and a torus obstruction

Caius Folkerts

A stress balances forces along the edges of a framework; an infinitesimal flex moves its vertices without changing edge lengths to first order. The strong stress–flex conjecture asks whether these two kinds of data always satisfy a further orthogonality identity when a closed polyhedral surface is coned from a point. This note proves the identity for every polyhedral sphere in three dimensions, without convexity or embeddedness, and disproves the general conjecture with an explicit nine-vertex embedded torus. The distinction is topological: a closed vector-valued cochain has global face potentials on a sphere, whereas the torus supports a nonzero period.

Let S be a finite cell decomposition of a closed oriented surface with simple graph $G = (V, E)$. Realize its vertices as distinct points $p_i \in \mathbb{R}^3$, so that the vertices of each face lie in a plane and include three noncollinear points. The realization need not be embedded. Choose a point x outside every face plane, and join x to every vertex. An equilibrium stress assigns scalars ω_{ij} to the edges of G and ω_{*i} to the cone edges; at each vertex, the weighted sum of its incident edge vectors is zero. An infinitesimal motion with the cone point fixed assigns velocities u_i satisfying

$$(p_i - x) \cdot u_i = 0, \qquad (p_i - p_j) \cdot (u_i - u_j) = 0 \quad (ij \in E).$$

Theorem 1. *If S is a sphere, every equilibrium stress and every infinitesimal motion with the cone point fixed satisfy*

$$\sum_{i \in V} \omega_{*i} u_i = 0. \tag{1}$$

There is an embedded polyhedral torus with nine vertices, and a cone point outside all its face planes, for which this identity fails.

The torus disproves the strong stress–flex conjecture of Connelly, Gortler, Theran, and Winter [1], whose formulation allows closed surfaces with flat faces. The spherical result rests on the classical relation between stresses and polyhedral liftings [2]; the argument below gives the required correspondence directly. It identifies the topological obstruction.

1. Moving the cone point

Let R_x have rows $p_i - x$. A coned stress is encoded by a symmetric matrix M, supported off the diagonal on E, satisfying $MR_x = 0$. Its outer-edge stresses are

$\omega_{ij} = M_{ij}$, and its cone-edge stresses are $\omega_{*i} = -(M\mathbf{1})_i$. The equilibrium equation at the cone follows by summing the outer-vertex equations.

The following observation applies in every dimension. Moving the cone point in an arbitrary direction tests the corresponding component of the stress-weighted velocity.

Lemma 2. *Suppose an outer stress matrix M_0 of a framework coned from the origin extends continuously to outer stress matrices M_x for all cone points x near 0, while the outer vertices remain fixed. Then $(M_0\mathbf{1})^T U = 0$ for every infinitesimal motion U with the cone point fixed.*

Proof. Write $P = (p_i)_{i\in V}$ and $U = (u_i)_{i\in V}$, again with row vectors, and put $\alpha_x = M_x\mathbf{1}$. The infinitesimal edge equations give

$$\operatorname{tr}(P^T M_x U) = \sum_i (\alpha_x)_i p_i \cdot u_i - \sum_{ij\in E} (M_x)_{ij}(p_i - p_j)\cdot(u_i - u_j) = 0.$$

On the other hand, $M_x(P - \mathbf{1}x^T) = 0$, so the same trace equals $x \cdot (\alpha_x^T U)$. Set $x = tv$, divide by $t \neq 0$, and let t tend to zero. Continuity gives $v \cdot (\alpha_0^T U) = 0$ for every v. □

The required continuation comes from a vector space that does not depend on the cone point. Let

$$L(S) = \{a \in \mathbb{R}^V \mid a \text{ is the restriction of an affine function on each face}\}.$$

The affine functions may differ from face to face; their vertex values must agree along common edges.

Proposition 3. *If S is a sphere and x lies outside its face planes, the space of coned stresses is naturally isomorphic to*

$$L(S)\big/\{(b\cdot(p_i - x))_{i\in V} \mid b \in \mathbb{R}^3\}.$$

In particular, its dimension is $\dim L(S) - 3$*, and every stress extends real analytically under small motions of x.*

Proof. Fix x and abbreviate $r_i = p_i - x$. If an oriented edge $i \to j$ has face F on its left and face G on its right, assign to the dual edge $F \to G$ the vector

$$\beta_{FG} = M_{ij}(r_i \times r_j). \tag{2}$$

This assignment is antisymmetric. Around the dual cell corresponding to vertex i, its sum is, with a consistent orientation,

$$\sum_{j:ij\in E} M_{ij}(r_i \times r_j) = r_i \times \sum_{j:ij\in E} M_{ij} r_j = r_i \times (-M_{ii} r_i) = 0.$$

Thus β is closed. The first cohomology of the sphere vanishes, so there are vectors λ_F, unique up to a common additive vector, such that $\lambda_G - \lambda_F = \beta_{FG}$.

The difference across an edge annihilates both endpoint vectors. Consequently $a_i = \lambda_F \cdot r_i$ is independent of the incident face F, since those faces form a cycle around i. On F these values are given by the affine function $p \mapsto \lambda_F \cdot (p - x)$, and hence $a \in L(S)$.

Conversely, fix $a \in L(S)$. For each face there is a unique vector $\lambda_F(x)$ such that

$$\lambda_F(x) \cdot (p_i - x) = a_i \qquad (i \in V(F)). \tag{3}$$

Indeed, linear functions on $\mathbb{R}^3$ restrict to all affine functions on a plane missing the origin, and three noncollinear vertices of that plane give three linearly independent vectors $p_i - x$. Across an edge ij, the difference $\lambda_G - \lambda_F$ is perpendicular to r_i and r_j, so it has a unique expression $M_{ij}(r_i \times r_j)$. The cross product is nonzero because x is on no incident face plane. Telescoping these differences around i shows that $\sum_j M_{ij} r_j$ is parallel to r_i; there is therefore a unique diagonal entry M_{ii} making $MR_x = 0$.

This linear construction is onto by the first part of the proof. Its kernel consists of common face vectors $\lambda_F = b$, since the dual graph is connected, and thus has dimension three. Finally, keep a fixed while moving x. Solving (3) and recovering the entries of M uses rational functions with locally nonzero denominators, giving the asserted analytic continuation. □

After translating the cone point to the origin, Proposition 3 and Lemma 2 prove the spherical assertion of Theorem 1. More generally, the same argument proves (1) for any stress on a closed oriented surface whose cochain (2) has zero periods. Only the passage from a closed cochain to face potentials used the topology of the sphere.

2. **A torus with a nonzero period**

Let

$$\xi_1 = (1, 0), \qquad \xi_2 = (0, 1), \qquad \xi_3 = (-1, -1),$$

and, with indices read modulo three, set

$$A_k = (2\xi_k, -1), \qquad B_k = (4\xi_k, 0), \qquad C_k = (2\xi_k, 1).$$

Take the nine quadrilateral faces

$$A_k B_k B_{k+1} A_{k+1}, \qquad B_k C_k C_{k+1} B_{k+1}, \qquad C_k A_k A_{k+1} C_{k+1}.$$

They are the boundary of an embedded solid torus. To see this without a projection argument, put $D = \operatorname{conv}\{\xi_1, \xi_2, \xi_3\}$ and $T = \operatorname{conv}\{(2, -1), (4, 0), (2, 1)\}$. The map

$$\partial D \times T \longrightarrow \mathbb{R}^3, \qquad (\xi, r, z) \longmapsto (r\xi, z),$$

is a homeomorphism onto its image, with an inverse that recovers $r > 0$ from the gauge of D and then recovers ξ. Its boundary is precisely the stated nine-face surface. If ℓ ranges over the three side-defining linear forms of D, the face planes are $\ell = 2$ and $\ell \pm 2z = 4$. None contains the cone point $O = 0$.

Let $J(x, y) = (-y, x)$ and assign velocities

$$u_{A_k} = (J\xi_k, 0), \qquad u_{B_k} = u_{C_k} = 0, \qquad u_O = 0.$$

These define an infinitesimal flex of the coned framework. On the lower triangle, each relative velocity is perpendicular to its edge. Every other edge incident to A_k has horizontal component parallel to ξ_k, so its scalar product with u_{A_k} also vanishes; this includes the cone edge. All remaining endpoints are stationary.

The four vertices O, A_1, B_1, C_1 span a complete subgraph and satisfy

$$-O + A_1 - B_1 + C_1 = 0.$$

The affine dependence with coefficients $(-1, 1, -1, 1)$ gives an equilibrium stress by assigning $-\lambda_i\lambda_j$ to each of its six edges and zero elsewhere. Its cone-edge coefficients are $1, -1, 1$ at A_1, B_1, C_1, respectively. Therefore

$$\sum_i \omega_{*i} u_i = u_{A_1} = (0, 1, 0) \neq 0,$$

which proves the second assertion of Theorem 1.

The obstruction is visible on the dual surface. The cycle through the three lower trapezoids crosses the edges A_kB_k, of which only A_1B_1 is stressed. Its period is consequently, up to orientation,

$$A_1 \times B_1 = (0, -4, 0).$$

Thus the stress cannot be represented by global face potentials, precisely at the step that is automatic on a sphere.

References

[1] R. Connelly, S. J. Gortler, L. Theran, and M. Winter, *The Stress-Flex Conjecture*, preprint, 2024. arXiv 2404.15590.

[2] W. Whiteley, *Motions and stresses of projected polyhedra*, Structural Topology **7** (1982), 13–38. HDL 2099/989.