Rank-three obstructions to coordinate projection volumes

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Abstract. For every $n \geq 5$, a positive hollow matrix of rank three cannot be approximated by the volumes of the coordinate $(n-2)$-dimensional projections of a convex body in $\mathbb{R}^n$. These matrices have one positive eigenvalue, and their small perturbations yield open sets of counterexamples with strict Lorentzian signature. This answers the closure version of Huh's projection-volume question in the negative. The obstruction comes from the polygonal cone determined by a rank-three matrix and the equality cases of Minkowski's quadratic inequality. A compactification by mixed volumes with finite segment directions reduces every degenerating realization to a full-dimensional body or a hyperplane body.

1. Introduction

For a convex body $A \subset \mathbb{R}^n$, let $\pi_{ij}A$ be its orthogonal projection onto the coordinate subspace obtained by deleting coordinates i, j, and set

$$p_{ij}(A) = \mathrm{Vol}_{n-2}(\pi_{ij}A), \qquad p_{ii}(A) = 0.$$

We allow nonempty compact convex sets of every affine dimension and write $\mathcal{R}_n$ for the resulting hollow matrices. Alexandrov–Fenchel implies that every matrix in $\mathcal{R}_n$ has at most one positive eigenvalue. In dimension four the converse holds, by the projection-area theorem of Huang, Huh, Michałek, Wang and Wang [2].

The converse in dimension five fails for exact realization. Huh [3] records a counterexample of Zhang and Zhao with unit off-diagonal entries on four labels and zero entries involving the fifth, together with Wang's observation that this matrix nevertheless belongs to $\overline{\mathcal{R}_5}$. The remaining question is whether allowing limits restores the spectral characterization. The obstruction below survives this relaxation and occurs among strictly positive data.

Theorem 1.1. *Let $n \geq 5$. Every symmetric hollow $n \times n$ matrix P of rank three with positive off-diagonal entries lies outside $\overline{\mathcal{R}_n}$. Such a matrix has signature $(1, 2, n-3)$ and is a limit of matrices of signature $(1, n-1)$ outside $\overline{\mathcal{R}_n}$.*

There is therefore a nonempty open set of positive hollow matrices of signature $(1, n-1)$ disjoint from the closure of the projection-volume locus. A particularly simple example in dimension five is

$$P = \begin{pmatrix} 0 & 1 & 3 & 4 & 3 \\ 1 & 0 & 1 & 3 & 4 \\ 3 & 1 & 0 & 1 & 3 \\ 4 & 3 & 1 & 0 & 1 \\ 3 & 4 & 3 & 1 & 0 \end{pmatrix}.$$

Its entries are the squared distances between five consecutive vertices of a regular hexagon of circumradius one. With $\mathbf{1} = (1,1,1,1,1)^T$, $b = (2,1,-1,-2,-1)^T$ and $c = (0,1,1,0,-1)^T$, the factorization

$$P = 2\mathbf{1}\mathbf{1}^T - \tfrac{1}{2} b b^T - \tfrac{3}{2} c c^T$$

shows that P has rank three and signature $(1,2,2)$.

Rank three supplies more than a convenient example. Every matrix in Theorem 1.1 can be written

$$p_{ij} = a_i a_j \big(1 - \cos(\theta_i - \theta_j)\big), \qquad a_i > 0,$$

with distinct points $e^{i\theta_i}$ on a circle. Nonnegative vectors in its column space are therefore the values of affine functions nonnegative on a cyclic polygon. The extreme rays of this cone correspond to the polygon’s edges. If a realizing body has interior, equality in Minkowski’s quadratic inequality converts this polygonal structure into incompatible relations between mixed area measures and the segment directions.

A convergent sequence of projection-volume arrays need not have a bounded sequence of realizing bodies. An affine normalization avoids that difficulty by transferring the degeneration from the body to a finite collection of directions. Let $S_v = [-v/2, v/2]$. We prove the compactification

$$\overline{\mathcal{R}_n} = \big\{ \big(n(n-1) V_n(K[n-2], S_{U_i}, S_{U_j}) \big)_{i,j} \mid K \subset \mathbb{R}^n,\ U_i \in \mathbb{R}^n \big\},$$

and show that K can always be chosen of dimension at least $n-1$. The full-dimensional case is ruled out by the polygonal obstruction. In the remaining hyperplane case, the mixed volumes obey weighted triangle inequalities. At rank three, these inequalities would give five points on a circle with no obtuse triangle, which is impossible.

The compactification works for every coordinate projection dimension. If k coordinates are deleted, a limiting representation requires a body of dimension at least $n - k + 1$. This bound explains why complementary projections admit the particularly sharp division into a full-dimensional case and a hyperplane case.

For projections retaining two coordinates, the same rank-three matrices lead to a different geometric obstruction. The corresponding compatibility conditions and metric realizations are developed in [1].

2. **Rank three and a cyclic polygon**

Lemma 2.1. *Let P be a positive hollow symmetric matrix of rank three. Then it has signature $(1, 2, n-3)$ and admits the representation*

$$p_{ij} = a_i a_j\big(1 - \cos(\theta_i - \theta_j)\big),$$

where $a_i > 0$ and the points $e^{i\theta_i}$ are distinct.

Proof. Every three-label principal determinant is $2p_{ij}p_{ik}p_{jk} > 0$, so that principal matrix has one positive and two negative eigenvalues. Interlacing and rank three give the asserted signature. Write $P = aa^T - bb^T - cc^T$. The zero diagonal gives $a_i^2 = b_i^2 + c_i^2$, while positive off-diagonal entries force all a_i to have the same nonzero sign. Taking them positive and writing $(b_i, c_i) = a_i(\cos\theta_i, \sin\theta_i)$ gives the representation. Positivity excludes coincident circle points. □

Fix an $(n-3)\times n$ matrix W whose rows form a basis of $\ker P$. Then $\ker W = \operatorname{im} P$, and Lemma 2.1 identifies the cone

$$\mathcal{C} = \{y \in \mathbb{R}^n_{\geq 0} \mid Wy = 0\} \tag{1}$$

with affine functions nonnegative on the cyclic polygon, after scaling coordinate i by a_i. Order its vertices cyclically. The cone is three-dimensional, and its extreme rays r^e are indexed by the polygon edges $e = \{i, i+1\}$. Each r^e vanishes exactly at the two endpoints of e and is positive at all other coordinates.

Indeed, an extreme ray has at least two independent active coordinate constraints. The corresponding affine function vanishes on a line through two polygon vertices and is nonnegative at all the others, so that line supports an edge. No third circle point lies on the line. Conversely, the supporting line of each edge gives an extreme ray. This description will convert a relation between vector measures into one measure for each polygon edge.

A different use of circle geometry excludes the weighted metrics that arise from bodies in a hyperplane.

Lemma 2.2. *A positive hollow symmetric matrix of rank three satisfying the triangle inequalities is the squared-distance matrix of at most four points on a circle. If there are four points, they form a rectangle.*

Proof. We first use an elementary property of four-point metrics. If $H = (h_{ij})$ is such a metric, then

$$z^T H z \le 0 \qquad \text{whenever } \sum_i z_i = 0.$$

To see this, relabel so that $h_{12} + h_{34}$ is the largest of the three sums on opposite pairs. For a subset S, let δ_S be the matrix equal to one across the cut $S \mid S^c$ and zero elsewhere. Then

$$H = \sum_{i=1}^{4} x_i \delta_{\{i\}} + y_{13}\delta_{\{1,3\}} + y_{14}\delta_{\{1,4\}},$$

where

$$\begin{aligned} 2x_1 &= h_{13} + h_{14} - h_{34}, & 2x_2 &= h_{23} + h_{24} - h_{34}, \\ 2x_3 &= h_{13} + h_{23} - h_{12}, & 2x_4 &= h_{14} + h_{24} - h_{12}, \\ 2y_{13} &= h_{12} + h_{34} - h_{13} - h_{24}, & 2y_{14} &= h_{12} + h_{34} - h_{14} - h_{23}. \end{aligned}$$

The coefficients are nonnegative, and $z^T \delta_S z = -2(\sum_{i\in S} z_i)^2$ when $\sum_i z_i = 0$.

Now let D have rank three and satisfy the hypotheses. Its principal matrix A on the first three labels is invertible, since $\det A = 2D_{12}D_{13}D_{23} > 0$. Set $\alpha_k = A^{-1}(D_{1k}, D_{2k}, D_{3k})^T$ for $k \ge 4$. The rank condition gives

$$D = M^T A M, \qquad M = (e_1, e_2, e_3, \alpha_4, \dots, \alpha_n).$$

For each $k \ge 4$, the principal matrix on $1, 2, 3, k$ has kernel vector $(-\alpha_k, 1)$. Its coordinate sum must be zero. Otherwise the all-ones vector could be shifted by a multiple of this kernel vector to have coordinate sum zero without changing its strictly positive quadratic value, contrary to the four-point inequality. Thus every column of M has coordinate sum one.

Choose a nondegenerate planar triangle with squared side lengths A_{ij}, and place its circumcenter at the origin. Such a triangle exists because the triangle inequalities for the positive numbers A_{ij} imply strict triangle inequalities for their square roots. Write its vertices as p_1, p_2, p_3, with $|p_i| = r$, and put $p_k = \sum_{i=1}^{3} \alpha_{k,i} p_i$. The sum-one condition gives

$$D_{k\ell} = 2r^2 - 2\langle p_k, p_\ell \rangle.$$

The zero diagonal therefore puts every p_k on the same circle, and the displayed identity becomes $D_{k\ell} = |p_k - p_\ell|^2$. Positive off-diagonal entries make the points distinct.

The metric inequalities now say that every triangle of these points is nonobtuse. For five points in cyclic order, some two consecutive arc gaps sum to less than π, so

their three endpoints form an obtuse triangle. Hence there are at most four points. If there are four, the opposite angles of their inscribed quadrilateral sum to π and are all at most $\pi/2$, so the quadrilateral is a rectangle. □

Corollary 2.3. *A positive hollow symmetric matrix of rank three on five labels admits no vector* $x \in \mathbb{R}^5_{\geq 0} \setminus \{0\}$ *satisfying*

$$p_{ij}x_k \leq p_{ik}x_j + p_{jk}x_i \qquad (i, j, k \text{ distinct}). \tag{2}$$

Proof. Two weights cannot vanish, since an inequality with those weights on the right and a positive weight on the left would fail. If exactly one weight vanishes, replace it by a sufficiently small positive number. Inequalities with that weight on the right are only relaxed, while those with it on the left still hold because their right sides are strictly positive. Thus any nonzero nonnegative solution gives a positive one.

For positive weights, the matrix $D_{ij} = p_{ij}/(x_i x_j)$ has rank three and satisfies the ordinary triangle inequalities. Lemma 2.2 would therefore give five points on a circle with no obtuse triangle, a contradiction. □

3. Compactifying coordinate projection volumes

The affine shape of a convex body can be normalized by a maximal-volume inscribed simplex. Under this normalization, projection volumes become mixed volumes with marked segments, whose directions may diverge. Bounded products of singular values allow the divergence to be absorbed into a contraction of the body without losing too much dimension.

For $1 \leq k < n$, let $\mathcal{R}_{n,k}$ be the range of the tuples

$$p_I(A) = \mathrm{Vol}_{n-k}(\pi_I A), \qquad I \subset \{1, \ldots, n\}, \quad |I| = k,$$

where π_I deletes the coordinates in I. In particular $\mathcal{R}_n = \mathcal{R}_{n,2}$.

Theorem 3.1. *For* $1 \leq k < n$*, the closure of* $\mathcal{R}_{n,k}$ *consists exactly of the tuples*

$$\left(\frac{n!}{(n-k)!} V_n(K[n-k], (S_{U_i})_{i \in I}) \right)_{|I|=k}, \tag{3}$$

where $K \subset \mathbb{R}^n$ *is a convex body and* $U_1, \ldots, U_n \in \mathbb{R}^n$*. Every such tuple has a representation with* $\dim K \geq n - k + 1$*.*

Proof. Let $p(A_m)$ converge. Adding sufficiently small Euclidean balls allows us to assume each A_m has interior without changing the limit. Choose a maximal-volume inscribed simplex and write

$$A_m = x_m + T_m B_m, \qquad \Delta_n \subset B_m \subset [-1,1]^n,$$

where $\Delta_n = \operatorname{conv}(0, e_1, \dots, e_n)$. The upper inclusion follows because replacing any simplex vertex by a point of B_m cannot increase the simplex volume; every barycentric coordinate has absolute value at most one. Pass to a subsequence with $B_m \to B$, where B has interior.

Let $M_m = |\det T_m|^{1/k} T_m^{-1}$ and denote its columns by $v_{m,i}$. Covariance of mixed volume gives

$$p_I(A_m) = \frac{n!}{(n-k)!} V_n(B_m[n-k], (S_{v_{m,i}})_{i\in I}).$$

For arbitrary independent directions, extrusion along k segments gives

$$\frac{n!}{(n-k)!} V_n(B[n-k], S_{v_1}, \dots, S_{v_k}) = |v_1 \wedge \cdots \wedge v_k| \operatorname{Vol}_{n-k}\big(B \mid \operatorname{span}(v_1, \dots, v_k)^\perp\big).$$

The fixed inball in B_m therefore bounds every k-column wedge of M_m. These wedges are the columns of $\bigwedge^k M_m$, so the product of the largest k singular values of M_m is uniformly bounded.

We now suppress the index m and write the singular values as $\sigma_1 \geq \cdots \geq \sigma_n > 0$. Choose a fixed C_0 such that $\sigma_1 \cdots \sigma_k < C_0^k$ for every m. For $g \geq 1$, set

$$f_i(g) = \min\left(1, \frac{C_0}{g\sigma_i}\right), \qquad H(g) = g^k \prod_{i=1}^n f_i(g).$$

At $g = 1$ one has $H(g) \leq 1$. At $g_* = C_0/\sigma_k > 1$ one has $H(g_*) = C_0^k/(\sigma_1 \cdots \sigma_k) > 1$. By continuity, choose $1 \leq g < g_*$ with $H(g) = 1$. In a left singular basis of M, let F have eigenvalues $f_i(g)$. Then

$$0 < F \leq I, \qquad g^k \det F = 1, \qquad \|gFM\| \leq C_0,$$

and at most $k-1$ eigenvalues of F are smaller than one.

Return to the sequence and replace B_m by $F_m B_m$ and its marked matrix by $g_m F_m M_m$. Equation (3) is unchanged, by covariance and $g_m^k \det F_m = 1$. The new bodies and directions are bounded. Take limits $F_m \to F$ and $g_m F_m M_m \to M$. Since at least $n-k+1$ eigenvalues of each F_m are one, $\operatorname{rank} F \geq n-k+1$. Thus $K = FB$ has dimension at least $n-k+1$, and continuity of mixed volumes proves the required representation.

Conversely, start with any K and marked matrix $M = (U_1, \dots, U_n)$ in (3). Choose invertible $M_m \to M$, put $K_m = K + m^{-1}\mathbb{B}$, and define

$$T_m = |\det M_m|^{1/(n-k)} M_m^{-1}.$$

Then $|\det T_m|^{1/k} T_m^{-1} = M_m$. The projection tuples of $T_m K_m$ are therefore the mixed-volume tuples of K_m, M_m, which converge to the prescribed tuple. This proves the converse and the closure formula. □

For complementary projections, $k = 2$, Theorem 3.1 leaves only two possible dimensions for the repeated body, namely n and $n-1$. We treat them in turn.

4. **The obstruction for a body with interior**

Fix a full-dimensional body $K \subset \mathbb{R}^n$ and set

$$q(A, C) = V_n(K[n-2], A, C), \qquad S_K(A) = S^{(n)}_{K[n-2],A}, \qquad \Sigma = \operatorname{supp} S_K(\mathbb{B}).$$

The normalization is $q(A, C) = n^{-1} \int h_C \, dS_K(A)$. Mixed area measures are positive and centered, and are continuous under Hausdorff convergence [5, Section 4.1].

Lemma 4.1. *For every convex body A, one has* $\operatorname{supp} S_K(A) \subset \Sigma$. *If*

$$q(A, A) = q(A, C) = q(C, C) > 0,$$

then $S_K(A) = S_K(C)$ and $h_A - h_C$ is the restriction of a linear function to Σ.

Proof. Approximate h_A uniformly by smooth support functions h_{A_m}. For a sufficiently large constant c_m, the function $c_m - h_{A_m}$ is a support function, since its spherical Hessian plus its value times the identity is positive semidefinite. Thus $c_m\mathbb{B} = A_m + D_m$ for a convex body D_m, and positivity gives $S_K(A_m) \le c_m S_K(\mathbb{B})$. Weak continuity proves the support inclusion. The support-function criterion used here is recalled in [5, Lemma 4.5].

To identify the measures, write the common mixed volume as $a > 0$. Alexandrov–Fenchel applied to $A + sT, C$ gives

$$(a + sq(C, T))^2 \ge a\{a + 2sq(A, T) + s^2 q(T, T)\}.$$

Letting $s \downarrow 0$ and interchanging A, C yields $q(A, T) = q(C, T)$ for every T. Differences of support functions contain all C^2 functions on the sphere, so the measures agree.

The equality theorem of Shenfeld and van Handel [5, Theorems 2.2 and 4.4] gives $h_A - \lambda h_C = \langle t, \cdot \rangle$ on Σ, for some $\lambda \ge 0$ and $t \in \mathbb{R}^n$. Its hypotheses require interior

only for the repeated body K; the comparison bodies may be lower dimensional. Integrate against $S_K(C)$, using the support inclusion and its zero first moment. Since $q(A, C) = q(C, C) > 0$, the resulting identity forces $\lambda = 1$. □

Proposition 4.2. *Let $n \geq 5$ and let P be a positive hollow matrix of rank three. There are no full-dimensional $K \subset \mathbb{R}^n$ and vectors $U_1, \dots, U_n \in \mathbb{R}^n$ such that*

$$p_{ij} = n(n-1)V_n(K[n-2], S_{U_i}, S_{U_j}). \tag{4}$$

Proof. Assume such a representation. Use the cyclic ordering from Lemma 2.1, and let W and r^e describe the polygonal cone (1). Define

$$z_i(u) = \langle U_i, u \rangle, \qquad \mu_i = \frac{n-1}{2} S_K(S_{U_i}).$$

Then

$$p_{ij} = \int |z_j| \, d\mu_i. \tag{5}$$

The μ_i are centered positive measures supported on Σ. Since $p_{ii} = 0$, each μ_i is concentrated on $z_i = 0$.

For every nonzero $w \in \ker P$, its positive and negative parts both contain at least two nonzero coordinates. Otherwise the equation $Pw = 0$ at the sole positive or negative coordinate would contradict positivity of the off-diagonal entries. Since $Pw = 0$, the two zonotopes below have equal, strictly positive q-self-volumes and mixed volume,

$$A_w = \sum_{w_i > 0} w_i S_{U_i}, \qquad C_w = \sum_{w_i < 0} (-w_i) S_{U_i}.$$

Lemma 4.1 applied to each row of W gives a linear map $L \colon \mathbb{R}^n \to \mathbb{R}^{n-3}$ such that

$$W\mu = 0, \qquad W|z(u)| = Lu \quad (u \in \Sigma). \tag{6}$$

The first identity means that the vector of measure densities lies in the polygonal cone. More explicitly, take $\mu = \sum_i \mu_i$ and write $a_i = d\mu_i/d\mu$. Then $a(u) \geq 0$ and $Wa(u) = 0$ almost everywhere. Triangulating a compact section of (1) gives nonnegative measurable coefficients of its extreme rays. Integrating those coefficients against μ produces finite positive measures ν_e such that

$$\mu_i = \sum_e r_i^e \nu_e. \tag{7}$$

For $i \notin e$, the coefficient r_i^e is positive, so domination by μ_i confines ν_e to $z_i = 0$. Thus only the two endpoint functions may be nonzero on the measure assigned to an edge.

Fix $e = \{a, b\}$ and let c be the other cycle neighbor of a. In (7) for μ_c, only the two edges incident to a can contribute to an integral involving z_a. The edge $\{a, c\}$ has coefficient $r_c^{\{a,c\}} = 0$, leaving only e. Centeredness and (5) give

$$\int z_a \, d\nu_e = 0, \qquad \gamma_{ea} = \int |z_a| \, d\nu_e = \frac{p_{ca}}{r_c^e} > 0.$$

The same holds for b.

Let $H = \operatorname{span}(U_1, \ldots, U_n)$ and $m_e = \int u \, d\nu_e$. All signed endpoint moments vanish, and the other coordinate functions vanish on ν_e, so $m_e \in H^\perp$. Moreover, any relation $\sum_i \lambda_i U_i = 0$ with one zero coefficient has all coefficients zero. Indeed, on ν_e the relation reduces to $\lambda_a z_a + \lambda_b z_b = 0$; if one endpoint coefficient is zero, the positive absolute moment at the other endpoint forces its coefficient to vanish too. Propagating around the polygon proves the claim. A vector space of relations with this property has dimension at most one. There are n marked vectors in $\mathbb{R}^n$, hence

$$\dim H^\perp \leq 1.$$

Integrating the second identity in (6) against ν_e gives

$$q_e = L m_e = \gamma_{ea} W_a + \gamma_{eb} W_b, \tag{8}$$

where W_i denotes column i of W. The right-hand sides span a space of dimension at least two. To see this, let $F = \operatorname{span}(q_e)$ and take a row functional η annihilating F. The row $w = \eta W$ satisfies

$$\gamma_{ea} w_a + \gamma_{eb} w_b = 0 \qquad \text{on every edge } e = \{a, b\}.$$

Positive coefficients determine every coordinate from a single one, so the solution space has dimension at most one; for odd n it is zero, since a nonzero solution would alternate signs around an odd cycle. The map $\eta \mapsto \eta W$ is injective. Therefore $\dim F \geq n - 4$, and $\dim F = n - 3$ for odd n. Either bound gives $\dim F \geq 2$ when $n \geq 5$, whereas (8) confines F to $L(H^\perp)$, of dimension at most one. This contradiction proves the proposition. □

5. **The hyperplane limit**

When the repeated body lies in a hyperplane, the marked mixed volumes reduce to a norm in that hyperplane. The reduction explains why the weighted triangle inequalities replace the rigidity argument.

Lemma 5.1. *Let $K \subset H = u^\perp \subset \mathbb{R}^n$, where $|u| = 1$, and write $U_i = \alpha_i u + b_i$ with $b_i \in H$. Define the seminorm*

$$N(v) = (n-1)V_{n-1}(K[n-2], S_v), \qquad v \in H.$$

Then

$$n(n-1)V_n(K[n-2], S_{U_i}, S_{U_j}) = N(\alpha_i b_j - \alpha_j b_i).$$

If this array is nonzero, it satisfies (2) *with nonzero nonnegative weights $x_i = |\alpha_i|$.*

Proof. If $\alpha_i \neq 0$, apply the determinant-one shear fixing H and sending u to $u - b_i/\alpha_i$. It sends U_i to $\alpha_i u$ and the component of U_j in H to $b_j - \alpha_j b_i/\alpha_i$. Extrusion along the normal segment gives

$$nV_n(K[n-2], S_{U_i}, S_{U_j}) = |\alpha_i| V_{n-1}(K[n-2], S_{b_j - \alpha_j b_i/\alpha_i}),$$

which is the claimed identity. Interchange i, j if necessary; if both normal components vanish, all the bodies lie in H and both sides are zero.

The functional N is a seminorm by the Cauchy projection formula [4]. Its triangle inequality, applied to

$$\alpha_k(\alpha_i b_j - \alpha_j b_i) = \alpha_j(\alpha_i b_k - \alpha_k b_i) - \alpha_i(\alpha_j b_k - \alpha_k b_j),$$

gives the weighted inequalities. If all α_i vanished, every entry would vanish, so the weights are nonzero for a nonzero array. □

Proof of Theorem 1.1. Suppose $P \in \overline{\mathcal{R}_n}$. Theorem 3.1 with $k = 2$ gives a representation (4) with $\dim K \geq n - 1$. Proposition 4.2 excludes dimension n. In dimension $n - 1$, translate K into a hyperplane and apply Lemma 5.1. Choose five labels including one with a positive weight. Their principal matrix has rank exactly three, because every three-label determinant is positive, and its inherited weights contradict Corollary 2.3. This proves exclusion from the closure.

For strict perturbations, write $P = D(\mathbf{1}\mathbf{1}^T - cc^T - ss^T)D$ as in Lemma 2.1, where $D = \operatorname{diag}(a_i) > 0$. Set

$$P_\varepsilon = P + \varepsilon D(\mathbf{1}\mathbf{1}^T - I)D.$$

The inner quadratic form is strictly negative on $\mathbf{1}^\perp$ for every $\varepsilon > 0$, since there it equals $-(c^T x)^2 - (s^T x)^2 - \varepsilon|x|^2$. It has a positive value on $\mathbf{1}$, so P_ε has signature $(1, n-1)$. Its entries remain positive, and $P_\varepsilon \to P$ as $\varepsilon \downarrow 0$; because P lies outside the closed realization locus, so do all sufficiently small perturbations. The positive strict-signature locus and the complement of $\overline{\mathcal{R}_n}$ are open, giving the asserted open sets of counterexamples. □

References

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