Normal–projective equivalence and the exact cube bound

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Abstract. We characterize normal–projective equivalence of simple polytopes by the connected components of their labelled realization spaces modulo affine transformations. An explicit local construction realizes every sufficiently small change of facet normals in at most $2d$ steps, implying that every simple three-polytope is normal–projectively unique. For combinatorial d-cubes, the exact worst-case number of transformations to the standard cube is $2d-3$ for every $d \geq 2$. The upper bound comes from making opposite diagonals concurrent in the polar, while a rank obstruction forces the same numbers of projective and normal moves.

1. Introduction

Normal and projective transformations move a polytope while preserving its face lattice. Adiprasito, Kalmanovich, and Nevo [1] showed that these operations connect all realizations of a product of simplices, using at most $8d$ operations for a d-cube, and asked for the optimal count and for uniqueness of the dodecahedron. We answer both questions by relating the operations to the topology of realization spaces; for a simple polytope, they connect precisely the connected components modulo affine maps. Every simple three-polytope is therefore unique under these operations, and for cubes the exact count is $2d-3$.

A *normal transformation* changes the facet support numbers while retaining the labelled face lattice and the corresponding outward unit facet normals. A *projective transformation* is an invertible map

$$x \longmapsto \frac{Lx+b}{c^T x+\alpha}$$

whose denominator is nonzero on the entire polytope. All polytopes here are bounded, convex, and full dimensional. A sequence of these operations is a *PN-sequence*; each operation, including an affine map, counts as one step. A combinatorial type is *PN-unique* if all its realizations are connected by such sequences.

The local assertion behind the component theorem has a uniform length.

Theorem 1. *Two realizations of a simple d-polytope are PN-equivalent if and only if their classes lie in the same connected component of its labelled realization space modulo affine transformations. After putting the origin in the interior and normalizing every facet support number to one, every sufficiently small change of the facet-normal matrix can be achieved in at most $2d$ steps.*

A local PN-class is therefore open, and a finite cover turns a path of realizations into a finite sequence of operations. Conversely, normal moves interpolate support numbers with a common normal fan, and projective moves admit paths modulo affine transformations. Thus the component description is exact in both directions.

Corollary 2. *Every simple convex three-polytope is PN-unique.*

Steinitz's isotopy theorem gives the required connectedness in dimension three. To make the passage from a path to a finite PN-sequence explicit, we also construct paths by interpolating positive stresses on convex caps; the resulting vertex coordinates are rational functions of the path parameter.

For cubes, the quantitative question is the least uniform number of operations. Let $C_d = [-1,1]^d$, and let k_d be the largest PN-distance from a combinatorial d-cube to C_d.

Theorem 3. *For every $d \ge 2$, one has $k_d = 2d - 3$. Every combinatorial d-cube can be taken to C_d by $d-1$ projective and $d-2$ normal transformations. For $0 < \varepsilon < 1$, the cube*

$$P_{d,\varepsilon} = \{x \in \mathbb{R}^d : x_i \ge 0, \quad x_i + \varepsilon x_{i+1} \le 1 \ (1 \le i \le d)\}, \qquad x_{d+1} = x_1,$$

requires at least $d-1$ projective and at least $d-2$ normal transformations in any such sequence.

The upper and lower bounds describe the two operations in complementary coordinates. In the polar, a projective move changes the center of polarity and a normal move dilates vertices along their rays; successive moves make the opposite diagonals meet. In a rooted facet matrix, normal moves are diagonal row scalings and projective moves are rank-one additions. The cyclic family in the theorem then forces exactly the two counts used by the construction.

2. **Local reachability**

For an $m \times d$ matrix A and $h \in \mathbb{R}^m$, write

$$P(A,h) = \{x \in \mathbb{R}^d : Ax \le h\}.$$

The rows of A are raw outward normals. Keeping these rows fixed while changing h is a normal transformation whenever the face lattice is preserved.

The elementary projectivity

$$T_u(x) = \frac{x}{1 + u^T x}$$

has inverse T_{-u}. Substitution gives

$$T_u(P(A,h)) = P(A + hu^T, h), \tag{1}$$

provided that the forward and inverse denominators are positive on the two displayed polytopes. The support vector thus controls a rank-one change of the normal matrix. Also $T_u T_v = T_{u+v}$, so projectivities whose parameters sum to zero form a closed loop. We will change the supports between its steps to turn this loop into a prescribed small change of the normal matrix.

Lemma 4. *Suppose $P(A,\mathbf{1})$ is simple and its inequalities are precisely its facets. For every B sufficiently close to A, there is a PN-sequence of at most $2d$ steps from $P(A,\mathbf{1})$ to $P(B,\mathbf{1})$.*

Proof. Simplicity gives an open neighborhood $\mathcal{U}$ of $(A,\mathbf{1})$ in which $P(C,h)$ is bounded, contains the origin, and has the same labelled face lattice; after shrinking $\mathcal{U}$, all these polytopes lie in one ball. Indeed, the old vertex matrices remain invertible and all nonincident facet slacks remain positive, while boundedness persists because the facet normals positively span $\mathbb{R}^d$. Any additional vertices along a convergent sequence of perturbations would have a limit in the original polytope; their active facets would then be the d facets of an old simple vertex, giving exactly its already counted continuation.

Choose the last coordinate direction outside the kernels of the finitely many facet normals, so that $A_{id} \neq 0$ for every i. This is a simultaneous change of coordinates at both endpoints; conjugation by a fixed linear map preserves both types of move and adds no steps. For B close to A, define

$$h_{d,i} = \frac{A_{id}}{B_{id}} > 0, \qquad D = \operatorname{diag}(h_d).$$

Positive row scaling does not change a system of inequalities when the supports are scaled with the rows. We may therefore aim for $P(DB, h_d) = P(B, \mathbf{1})$; its last normal column already agrees with that of A. The remaining $d-1$ columns can be corrected separately. Set

$$h_j = h_d + \eta^{-1}(DB_{\bullet j} - A_{\bullet j}) \quad (1 \le j < d),$$

where $\eta > 0$ is small and fixed. For each $j < d$, set the support vector to h_j and apply $T_{\eta e_j}$. Finally set the support vector to h_d and apply $T_{-\eta(e_1+\cdots+e_{d-1})}$. By (1), the resulting normal matrix is

$$A + \eta \sum_{j<d} (h_j - h_d) e_j^T = DB,$$

where the last column agrees because $A_{\bullet d} = DB_{\bullet d}$. The final polytope is $P(DB, h_d) = P(B, \mathbf{1})$. There are d normal and d projective moves.

It remains to choose the parameters so that every intermediate polytope is admissible. First set $B = A$; all support vectors are $\mathbf{1}$ and the projectivities form a closed loop. Choose η small enough that this loop stays in $\mathcal{U}$ and its denominators are positive on the common bounding ball. Now let B vary near A. The intermediate normal matrices are within $O(\eta) + O(\|B - A\|)$ of A, and the supports are within $O(\|B - A\|/\eta) + O(\|B - A\|)$ of $\mathbf{1}$. With η fixed and B sufficiently close, they remain in $\mathcal{U}$ and both denominators in (1) remain positive. Every step is consequently admissible. □

Proof of Theorem 1. In normalized support coordinates, each PN-equivalence class is open by Lemma 4. Its complement is a union of the other open classes. A connected component therefore meets only one class. Translations into the interior and affine frame changes are projective steps, so the conclusion is independent of the normalization.

Conversely, every allowed move stays in one component modulo affine maps. A normal move preserves the normal fan; Minkowski interpolation between its endpoints preserves that fan and interpolates their support numbers. For a projective move, put the origin in the interior and normalize the denominator to $1 + c^T x > 0$. Subtracting the image of the origin factors the map as an invertible linear map after T_c. The path T_{tc}, $0 \le t \le 1$, is admissible because

$$1 + tc^T x = (1 - t) + t(1 + c^T x) > 0$$

throughout the original polytope. Thus every PN-class is path connected in the affine quotient, proving the converse. □

The proof also gives finite words along a prescribed path. If $A(t)$, $0 \le t \le 1$, is a continuous path of normalized simple realizations, cover it by finitely many neighborhoods on each of which one coordinate frame satisfies the nonvanishing condition above. Compactness and uniform continuity give a finite mesh subordinate to this cover, with consecutive matrices close enough for the local construction. A mesh of N intervals uses at most $2dN$ steps, in addition to endpoint normalizations. For a rational path without poles, quantitative margins follow by bounding the vertex matrices and minimizing the positive nonincident vertex slacks and absolute last normal coordinates on each closed chart interval.

3. **The three-dimensional path**

Steinitz's isotopy theorem says that the realization space of a three-polytope, with an affine frame fixed, is connected [4, Theorem 4.11]. Together with Theorem 1, this proves Corollary 2. For simple three-polytopes, a constructive path can be obtained as follows. It also makes the finite-word procedure explicit.

Lemma 5. *Two strictly convex polyhedral caps over the same triangle, with the same labelled triangulation, can be joined through such caps by a path whose vertex coordinates are rational functions of the path parameter without poles on* $[0,1]$.

Proof. We use positive equilibrium weights as coordinates for the cap. Their positivity is preserved under interpolation; the horizontal positions are then recovered from a linear Dirichlet problem, and the heights from the jumps between adjacent roof planes.

Fix the base triangle and write each roof plane as $z = a_f \cdot p + b_f$. For an interior edge $i \to j$, with triangles f, g on its left and right, strict concavity gives a positive symmetric weight ω_{ij} such that

$$(a_f, b_f) - (a_g, b_g) = -\omega_{ij}\big(J(p_j - p_i), \det(p_i, p_j)\big), \qquad J(x,y) = (-y, x).$$

Summing around an interior vertex gives

$$\sum_{j \sim i} \omega_{ij}(p_i - p_j) = 0.$$

Interpolate the two sets of positive weights linearly. For each parameter, solve these equilibrium equations with the boundary triangle fixed. The interior Dirichlet Laplacian is positive definite, so the solution is unique and rational in the weights. The positive-weight form of Tutte's embedding theorem [3], proved for triangulations

in [2, Theorem 4.1], gives a nondegenerate straight-line triangulation of the base for every parameter.

Use the displayed plane jumps to reconstruct the roof. They integrate around every dual cycle; it suffices to check cycles around interior vertices, where the two gradient coordinates sum to zero by equilibrium and the constant coordinate sums to

$$-\det\Big(p_i, \sum_{j\sim i} \omega_{ij}(p_j - p_i)\Big) = 0.$$

The roof is therefore well defined up to an affine function. Subtract the affine function matching its three boundary values. Every interior edge has a strict concave jump, so the result is concave on the whole triangle, vanishes on its boundary, and is positive in its interior. The jumps also prevent neighboring roof facets from merging. Thus the caps retain the labelled triangulation. Integration along a fixed dual spanning tree shows rational dependence, and uniqueness of the jumps and boundary values recovers the two endpoint caps. □

Here is an explicit reduction to these caps. Root a simple three-polytope at a vertex and choose affine coordinates in which its incident facets are $x_i \geq 0$ and its remaining facets are $a_j \cdot x \leq 1$. For sufficiently small $\varepsilon > 0$, $c = \varepsilon(1,1,1)$ is interior, and the vertices of $\varepsilon(P-c)^\circ$ are

$$-e_1, -e_2, -e_3, \qquad \frac{\varepsilon a_j}{1 - \varepsilon \sum_i a_{ji}}.$$

The affine map

$$U(y) = \left(-y_2 + \frac{1 + \sum_i y_i}{3},\ -y_3 + \frac{1 + \sum_i y_i}{3},\ 1 + \sum_i y_i \right)$$

takes the first three points to the standard base triangle at height zero. The others have height $(1 - \varepsilon \sum_i a_{ji})^{-1}$ and projected barycentric coordinates

$$\frac{1/3 - \varepsilon a_{ji}}{1 - \varepsilon \sum_k a_{jk}} \quad (i = 1, 2, 3),$$

which are positive when ε is small. The resulting simplicial polytope is consequently a cap with a fixed triangular base. Apply this construction to both endpoints and use Lemma 5. Subtracting the vertex average of each cap and taking polars gives a rational path of normalized simple realizations.

The endpoint changes on the primal are projective. Indeed, for any polytope K containing zero and any $O \in \operatorname{int} K$,

$$H_O(x) = \frac{x}{1 - O \cdot x}, \qquad H_O(K^\circ) = (K - O)^\circ. \tag{2}$$

The denominators are positive because O is interior to K and $-O$ is interior to $K-O$. An affine change $K \mapsto GK + b$, followed by centering at the image of an interior point O, therefore induces $G^{-T}H_O$ on the polar. This accounts for the normalization above. Lemma 4 replaces the resulting path by finite six-step words.

4. **Making the opposite diagonals concurrent**

We now prove the upper bound in Theorem 3. The polar of a combinatorial cube is a crosspolytope, with d labelled opposite pairs of vertices. A realization need not be centrally symmetric.

Lemma 6. *Suppose a bounded full-dimensional d-polytope is described by d pairs of inequalities, and no feasible point makes both members of any pair tight. Then all the inequalities are facets, and their labelled face lattice is that of a cube.*

Proof. At each vertex exactly one inequality from each pair is tight, and their normals are independent. Retain any $d-1$ of these equalities and follow the resulting edge to its other endpoint. The newly tight inequality must be the partner of the released one; it cannot be the partner of a retained inequality, nor can the released inequality become tight again on this line. Thus every single-coordinate flip of an occurring binary choice also occurs. Starting at any vertex gives all 2^d choices, each at a unique vertex. The vertex–facet incidences are exactly those of the cube. □

Corollary 7. *If the full-dimensional convex hull of d labelled pairs (p_i, q_i) has an interior point in each open segment (p_i, q_i), it is a crosspolytope with precisely these vertices and opposite pairs.*

Proof. Translate an interior point to zero and polarize. Two inequalities from one pair cannot both be tight; their exposed face would contain the entire paired segment and hence an interior point. Apply Lemma 6. □

Lemma 8. *For a crosspolytope $K = \operatorname{conv}\{p_i, q_i : 1 \le i \le d\}$, there are sites $s_i \in (p_i, q_i)$ such that s_1 is a strictly positive convex combination of $s_2, \dots, s_d$.*

Proof. The open segment (p_1, q_1) lies in $\operatorname{int} K$, since no proper face contains opposite vertices. Write its midpoint as a convex combination of all vertices with positive coefficients

$$\frac{p_1 + q_1}{2} = \sum_v a_v v, \qquad a_v > 0.$$

Both a_{p_1} and a_{q_1} are less than $1/2$. Otherwise rearranging this identity would express the other endpoint as a convex combination of the remaining vertices. Hence

$$(1/2 - a_{p_1})p_1 + (1/2 - a_{q_1})q_1 = \sum_{i=2}^{d} (a_{p_i} p_i + a_{q_i} q_i)$$

is a relation with positive coefficients on both sides and equal total weight. Normalize each pair and then the total weight. □

The sites provide an invariant for the construction; the processed diagonals will meet at a positive average of the sites on the remaining diagonals. This relation determines a center from which radial dilations merge the next diagonal, and when only one remains, it says that the last diagonal already passes through the common point.

Lemma 9. *Suppose a group G of opposite diagonals passes through a common point s in their open segments, and s is a positive convex combination of sites on every remaining diagonal. If at least two diagonals remain, one projective move followed by one normal move on the primal cube enlarges G by one and preserves this property.*

Proof. Choose one remaining site s_j and write

$$s = (1 - \lambda)z + \lambda s_j, \qquad 0 < \lambda < 1,$$

where z is the positive average of the other remaining sites. Put $u = s_j - z$ and choose $t > 1$ close enough to one that $O = z + tu \in \operatorname{int} K$. Dilate both endpoints of each pair about O by the factor

$$\gamma_i = \begin{cases} t/(t - \lambda), & i \in G, \\ t/(t - 1), & i = j, \\ 1, & i \notin G \cup \{j\}. \end{cases}$$

The old common site $z + \lambda u$ and the selected site $z + u$ both move to z. The other sites are unchanged, so the new group $G \cup \{j\}$ has the required positive-average relation.

Let K' be the hull of the new endpoints. Since every factor is at least one and $O \in K$, one has $K \subseteq K'$. To see this without assuming the inclusion, write $O = \sum_v a_v v$; weights proportional to a_v/γ_v express O as a convex combination of the new vertices. Each old vertex then lies on its new radial segment. The processed pairs contain $z \in \operatorname{int} K$, and the unchanged pairs retain their interior sites. Corollary 7 shows that K' is still a crosspolytope with the same labels.

On the primal, first apply (2), obtaining inequalities $(v - O) \cdot x \leq 1$. Then change their supports to $1/\gamma_v$, keeping the normals fixed. The result is $(K' - O)^\circ$. These are the asserted projective and normal moves; both are admissible because $O \in \operatorname{int} K \subseteq \operatorname{int} K'$. □

Upper bound in Theorem 3. Translate the input cube to contain zero and take its polar. By Lemma 8, start with $G = \{1\}$ and a positive-average relation among the sites. Apply Lemma 9 $d - 2$ times. Only one diagonal remains, and the common site of the group equals its site. Thus all d diagonals meet at an interior point z.

One final projectivity recenters the polar at z. Its pairs now have the form $\alpha_i u_i, -\beta_i u_i$, where $\alpha_i, \beta_i > 0$ and the u_i form a basis. The primal is the parallelotope

$$-\beta_i^{-1} \leq u_i \cdot x \leq \alpha_i^{-1} \qquad (1 \leq i \leq d),$$

which an affine map takes to C_d. Absorb this affine map into the last projectivity and the initial translation into the first. The counted word is

$$(\mathsf{PN})^{d-2}\mathsf{P},$$

with $d-1$ projective and $d-2$ normal steps. For $d = 2$, both endpoint normalizations are included in the sole projectivity. □

5. **A rank obstruction**

The upper bound uses one projective move each time a new diagonal is brought into the common configuration. To prove optimality, we need a quantity that can change by at most one under a projective move and is unaffected by any number of normal moves. The appropriate quantity is the minimum rank of a rooted facet matrix after subtracting a diagonal matrix.

Track a labelled vertex of a cube through a PN-sequence. At each stage choose affine coordinates in which this vertex is zero, its incident facets are $x_i \geq 0$, and the opposite facets have support one. The resulting *rooted matrix* A is defined by

$$x_i \geq 0, \qquad Ax \leq \mathbf{1}_d.$$

These are coordinate choices for the argument, not extra operations.

Lemma 10. *After r projective steps and any number of normal steps, the rooted matrices can be chosen so that*

$$A_{\text{final}} = DA_{\text{initial}} + R, \qquad D > 0 \text{ diagonal}, \qquad \operatorname{rank} R \leq r.$$

Consequently, if the final cube is a parallelotope, there is a diagonal matrix E with $\operatorname{rank}(A_{\text{initial}} - E) \le r$.

Proof. For a normal move, retain the linear frame and translate the tracked vertex to zero. The opposite raw normals are unchanged and their supports are positive. Rescaling these supports to one gives $A \mapsto DA$ with D positive diagonal.

For a projective move, subtract the image of the root and normalize the denominator there. The map has the form

$$x \longmapsto \frac{Lx}{1 + c^T x}, \qquad L \text{ invertible.}$$

In target coordinates obtained by applying L^{-1}, substitution gives $A \mapsto A + \mathbf{1}_d c^T$. Thus a normal move preserves the rank of the accumulated error and a projective move increases it by at most one. This proves the formula by induction. A parallelotope has a diagonal rooted matrix, so multiplication by D^{-1} gives the last assertion. □

Lower bound in Theorem 3. The region $P_{d,\varepsilon}$ lies in $[0,1]^d$ and is full dimensional. No feasible point has both $x_i = 0$ and $x_i + \varepsilon x_{i+1} = 1$, since that would give $x_{i+1} = 1/\varepsilon > 1$. Lemma 6 therefore makes it a labelled cube. Its rooted matrix at zero is $A = I + \varepsilon S$, where S is the cyclic shift matrix. For every diagonal E, the submatrix of $A - E$ on rows $1, \dots, d-1$ and columns $2, \dots, d$ is lower triangular with diagonal ε. Its determinant is ε^{d-1}, so

$$\operatorname{rank}(A - E) \ge d - 1.$$

Lemma 10 forces at least $d - 1$ projective steps.

Since projectivities compose admissibly on the polytope and normal equivalence is transitive, we may compress each consecutive block of one type into a single move. The resulting word alternates and still has at least $d - 1$ projective steps, so it has at least $d - 2$ normal steps as well. These lower bounds also hold for the original word, and the construction above attains both, proving $k_d = 2d - 3$. □

References

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