Small nonrational four-polytopes

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Abstract. We construct a four-polytope with seventeen vertices whose combinatorial type is realizable over a subfield $K \subseteq \mathbb{R}$ exactly when $\sqrt{5} \in K$. We also prove that every four-polytope with at most ten vertices has a rational realization. Thus the smallest number of vertices of a nonrational four-polytope lies between eleven and seventeen. The construction makes a golden-ratio incidence configuration recoverable from faces, while the rationality proof separates projective incidences from a linear system of vertex heights. A general deletion theorem shows that a simple vertex carries no obstruction to the realization field; in particular, every eleven-vertex four-polytope with a simple vertex is rational. The lower bound uses exact finite coverage after the geometric reductions.

1. Introduction

A face lattice can force irrational coordinates in every realization. Perles' original example showed this in dimension eight, and the universality theorem of Richter-Gebert brought arbitrary realization-space phenomena into dimension four [6, 5]. The size of a polytope needed to carry even a quadratic obstruction is a more specific question. The classical four-dimensional construction has thirty-three vertices, whereas Firsching proved that every four-polytope with at most nine vertices is rational [2].

The following results improve both sides of this range.

Theorem 1. *There is a seventeen-vertex four-polytope whose combinatorial type is realizable over a subfield $K \subseteq \mathbb{R}$ if and only if $\sqrt{5} \in K$.*

Theorem 2. *Every four-polytope with at most ten vertices has a rational realization.*

If N denotes the minimum number of vertices of a nonrational four-polytope, these theorems give

$$11 \leq N \leq 17.$$

The two bounds turn on which incidences a small face lattice can enforce. In the construction, seven vertices and one recoverable intersection point carry nine coplanarity relations, two of which force the golden-ratio equation. Ten auxiliary

vertices make the required planes recoverable from faces. Perturbations at the ends of a split vertex expose those faces through a comparison of cap minima.

For the lower bound, we use the freedom that remains when such incidences cannot be forced. Projection from a vertex separates the problem into a three-dimensional incidence configuration and a linear system for reciprocal heights. Most types admit successive joins and intersections followed by a triangular linear solve; the exceptional types reduce to a few explicit projective constructions. Counting relations and nonedges bounds the finite lists to which these arguments must be applied. The accompanying supplement contains the exact lists and their reconstruction data.

A useful principle applies in every dimension and requires no finite input.

Theorem 3. *Let P be a d-polytope with more than $d+1$ vertices, and let v be simple. Deleting v gives a combinatorially determined d-polytope $P \setminus v$, and*

$$\mathrm{Fields}(P) = \mathrm{Fields}(P \setminus v),$$

where Fields *denotes the set of subfields of $\mathbb{R}$ over which the type is realizable. Rational realizations are dense for one type if and only if they are dense for the other.*

Corollary 4. *Every eleven-vertex four-polytope with a simple vertex has a rational realization.*

We prove Theorem 1 first, then develop the rationalization tools and the lower-bound argument. The exceptional linear constructions and the finite coverage are collected in the appendices. We use homogeneous coordinates in projective arguments; juxtaposition of labels denotes their projective span or their vertex set, as indicated by context.

2. The incidence obstruction

The arithmetic is already present in an eight-point configuration. Its role in the construction is clearer if we first recover the equation from its incidences.

Lemma 5. *Suppose seven distinct points $1, 2, 3, 4, 6, 7, 8$ in projective three-space have no three collinear, the first four are projectively independent, and a point $5 \neq 2$ satisfies the coplanarities*

$$1238,\ 1245,\ 1267,\ 1346,\ 1578,\ 2347,\ 2356,\ 3458,\ 4678. \tag{1}$$

Then their field of definition contains $\sqrt{5}$.

Proof. Normalize $1, 2, 3, 4$ to the coordinate points. The incidences involving 6 and 7, followed by diagonal rescaling, give

$$6 = (1, 0, 1, 1), \qquad 7 = (0, 1, t, t).$$

All displayed nonzero entries are forced by the absence of collinear triples. The planes $1245, 2356$ put 5 in the form $(a, b, 0, a)$; here $a \neq 0$ because $5 \neq 2$. The incidence 3458, together with 1238, allows the remaining diagonal rescaling to give

$$5 = (1, 1, 0, 1), \qquad 8 = (1, 1, s, 0), \qquad s, t \neq 0.$$

Use the affine chart $x_0 - x_1 - x_2 + x_3 = 1$, with coordinates $(x_3, x_0, x_3 - x_2)$. The six points $1, \ldots, 6$ become

$$(0, 1, 0),\ (0, 0, 0),\ (0, 0, 1),\ (1, 0, 1),\ (1, 1, 1),\ (1, 1, 0),$$

while $7 = (u, 0, 0)$ and $8 = (0, v, 1)$, where $u = -t$ and $v = -1/s$. The two remaining planes $1578, 4678$ say

$$u(v - 1) = 1, \qquad v(u - 1) = 1.$$

Subtracting gives $u = v$, and hence $u^2 - u - 1 = 0$. Thus $(2u - 1)^2 = 5$. □

This is the golden rank-four configuration; related matroid descriptions appear in [1, Appendix A.3.5]. To turn it into a small polytope, it is not enough to place the eight points in space; several of the planes in (1) are not supporting planes. The construction below makes them recoverable as intersections of spans of actual faces.

3. **A seventeen-vertex realization**

Put $p = (1 + \sqrt{5})/2$. Start with the prism over the kite

$$(0, 0), \quad (p, 0), \quad (1, 1), \quad (0, p)$$

and cut off $(p, 0, 1)$ and $(0, p, 0)$ by

$$x + (p - 1)z \leq p, \qquad y - (p - 1)z \leq 1.$$

The cuts do not interact; violating both gives $x + y > p + 1 > 2$, whereas the kite lies in $x + y \leq 2$. The resulting three-polytope has vertices

$$1 = (0, 1, 0), \qquad 2 = (0, 0, 0), \qquad 3 = (0, 0, 1), \qquad 4 = (1, 0, 1),$$
$$5 = (1, 1, 1), \qquad 6 = (1, 1, 0), \qquad 7 = (p, 0, 0), \qquad 8 = (0, p, 1).$$

Its facets through 5 are $3458, 457, 567, 568$; the other facets are $1238, 2347, 1267, 168$. These follow directly from the two cuts.

Write $v_i = (x_i, y_i, z_i, 1)$ and $q = v_5$. The facets not through q have nonnegative defining forms

$$X = x, \qquad Y = y, \qquad Z = z, \qquad J = h + (p-1)z - y$$

in homogeneous coordinates (x, y, z, h).

3.1. **Splitting a vertex and retaining its incidences**

Replace q by $U = (q, 1)$ and $V = (q, -1)$, in one additional homogeneous coordinate w, and add points close to these two ends. Two elementary observations determine the required faces.

First, if L defines an old facet not through q and m_+ is its minimum on the projected upper cap, then

$$L(x) - m_+ w \geq 0$$

exposes that old facet together with the upper minimizers. It is nonnegative on the upper cap and old vertices, and on the lower cap it tends to $2L(q) > 0$. The lower-cap version uses $L(x) + m_- w$. Facets through q remain supporting whenever all cap projections are positive combinations of old vertices.

Second, suppose a three-face is the convex hull of a polygon R and a segment E outside its plane. If E is coplanar with an edge e of R, then $\mathrm{conv}(e, E)$ is a two-face; its plane meets the polygon plane in the supporting line of e. This observation will turn five direction identities into five pairs of bridge faces.

The directions needed are

$$\begin{aligned} \alpha &= q - v_4 = v_1 - v_2, \\ \beta &= v_4 - v_3 = v_6 - v_1 = (v_7 - v_2)/p, \\ \gamma &= (p-1)v_8 - v_6 = (p-1)v_4 - v_7. \end{aligned}$$

For a sufficiently small positive rational η, retain the seven points $(v_i, 0)$, $i \neq 5$, and add

$$\begin{aligned} U &= (q, 1), & V &= (q, -1), \\ A_0 &= (q + \eta^2(v_4 - 2q), 1), & A_1 &= A_0 + (\eta^3\alpha, 0), \\ B_0 &= (q + \eta(2v_7 - p^2 q), 1), & B_1 &= B_0 - (\eta^2\beta, 0), \\ C &= (q + \eta(v_7 - pq), 1), & D &= (q + \eta^2(v_6 - q), -1), \\ E_0 &= (q + \eta(v_6 + v_7 - 2q), -1), & E_1 &= E_0 + (\eta^2\gamma, 0). \end{aligned}$$

Their projections are positive combinations of old vertices. For example, B_1 replaces a small positive amount of v_7 in B_0 by v_2, while E_1 adds a positive amount of v_8 and decreases the coefficient of v_6. The four old facets through q therefore remain supporting. Consecutive intersections give the planar faces

$$4A_0A_1UV, \qquad 7B_0CUV, \qquad 6DUV,$$

and the old facet 3458 gives the facet $348A_0A_1UV$.

3.2. **The cap minima**

The different orders of η separate the supporting faces. On the upper cap, X is minimized at A_0, A_1, both Y and Z at B_0, B_1, and J at U, C. On the lower cap, X is minimized at V, D, and both Y and J at E_0, E_1.

For an explicit check, the first-order displacements of (X, Y, Z, pJ) at B_0, C, E_0 are respectively

$$(p-1, -p^2, -p^2, p-1), \quad (0, -p, -p, 0), \quad (p-1, -1, -2, p-2).$$

The points A_0, A_1, D move only at second order; moreover $X(A_i - q) = -\eta^2$, $J(A_i - q) > 0$, and $X(D - q) = 0$. The splitting directions preserve each stated minimum. Since $1 < p < 2$ and $p^2 = p + 1$, all remaining comparisons are strict for sufficiently small η.

The first convexity observation now supplies the facets

$$1238A_0A_1, \quad 2347B_0B_1, \quad 1267B_0B_1, \quad 168UC,$$
$$1238VD, \quad 2347E_0E_1, \quad 168E_0E_1.$$

The old faces $1238, 2347, 1267, 168$ survive. Applying the second observation to the direction identities above, together with

$$v_7 - pq = v_1 - pv_8, \qquad v_6 - q = v_2 - v_3,$$

gives the following bridge pairs

First two-face	Second two-face	Recovered plane
$12A_0A_1$	$4A_0A_1UV$	1245
$16B_0B_1$	$34B_0B_1$	1346
$18CU$	$7B_0CUV$	1578
$23VD$	$6DUV$	2356
$47E_0E_1$	$68E_0E_1$	4678

Each pair meets in its auxiliary edge $A_0A_1, B_0B_1, CU, VD, E_0E_1$. Thus all ten auxiliary points are vertices, and the old points are vertices of surviving faces. Their convex hull has exactly seventeen vertices and dimension four. The chart $h > 0$ is valid because all projected points are positive combinations of old vertices.

3.3. **Recovering the obstruction in every realization**

Consider any realization of this face lattice over K. The points $1, 2, 3, 4$ span a projective three-space H; the first three span the face plane 1238, which does not contain 4. The old quadrilaterals put $8, 7, 6$ in H.

An auxiliary vertex in one of the displayed facets over an old two-face cannot lie in H. It lies outside the plane of that two-face, so otherwise it would span H with the face and put every old vertex in the facet. Applying this to $168UC$, $1238A_0A_1$, $1238VD$, $2347B_0B_1$, and $168E_0E_1$ places all ten auxiliary vertices outside H. Define the virtual point $5 = UV \cap H$, which is defined over K and is distinct from every old vertex, since actual polytope vertices cannot be collinear.

The two planes in each bridge pair share an edge and span a projective three-space different from H. Intersecting it with H gives the plane in the last column of the table. The facet $348A_0A_1UV$ gives 3458, and the three surviving quadrilaterals provide the other coplanarities in (1). Lemma 5 now implies $\sqrt{5} \in K$.

Conversely, the construction uses only $p \in \mathbb{Q}(\sqrt{5})$ and a rational η, so is realizable over every field containing $\sqrt{5}$. This proves Theorem 1.

4. **Simple vertices and realization fields**

The arithmetic obstruction just constructed is carried by a nonsimple configuration. Theorem 3 explains why a simple vertex cannot create a new obstruction.

Proof of Theorem 3. Put $v = 0$ and its neighbors at $e_1, \dots, e_d$, so the tangent cone is the nonnegative orthant. Every other vertex satisfies $\sum x_i > 1$; otherwise it belongs

to $\operatorname{conv}(0, e_1, \dots, e_d)$ and cannot be extreme. Deletion therefore removes precisely this corner and creates the simplex facet $T = \operatorname{conv}(e_1, \dots, e_d)$. The remaining vertices span dimension d, since otherwise P would be a pyramid with apex v adjacent to more than d vertices. Applying the same argument inside every face shows that the deletion face lattice is intrinsic.

In restoring v, only T is visible. Any facet of $P \setminus v$ whose supporting hyperplane contains v extends to a facet of P through v. Since v is simple, this hyperplane contains exactly $d-1$ of its neighbors, so its intersection with T is a ridge. The extension is therefore specified by T and the subset S of its d neighboring facets that are coplanar with the new vertex.

Take any realization of $P \setminus v$ over K, polarize, and put the simple vertex dual to T at zero, with neighbors e_i. Cut by

$$\sum_{i \in S} x_i + 2 \sum_{i \notin S} x_i \geq 1.$$

This removes the origin, passes through exactly the neighbors indexed by S, and retains all nonneighbors strictly because $\sum x_i > 1$. Its signs on the old vertices agree with those of the cut dual to the prescribed extension. These signs determine the face lattice of the cut, whose new vertices are the intersections with crossed edges. Polarizing back gives P over K; deletion proves the converse inclusion of fields.

For density, replace the coefficients 2 by freely chosen $a_i > 1$ and approximate the actual cut coefficients together with the core realization, keeping coefficients on S equal to one. All free parameters range over an open orthant. Thus the same construction preserves rational approximation in both directions. □

5. **Rationalizing incidences and heights**

A labeled type is *rationally dense* if every realization is a limit of rational realizations of that type. The distinction from rational existence matters when part of a polytope has already been fixed; an arbitrary rational realization of that part need not admit the required attachment.

5.1. **Projection from a vertex**

Choose a vertex a of a four-polytope. After a real affine normalization, write the remaining vertices as

$$p_i = (q_i, 1)/h_i, \qquad h_i > 0.$$

The convex hull of the q_i is the three-dimensional vertex figure Q. Its vertices are the projected neighbors of a, and every other q_i is a marked point in the relative interior of a prescribed face G_i. This face is the projection of the smallest face of P containing a and i.

The vertex figure and its marks, by themselves, have dense rational realizations. Indeed, the vertex–facet incidence graph of a three-polytope is planar and bipartite, so every subgraph has a node of degree at most three. Delete these nodes and reinsert them by rational linear algebra; three vertices of a polygon are independent, as are three supporting planes at a vertex. The prescribed marks can then be chosen in the relative interiors of their face spans. The difficulty lies in polygon planes not already determined by the vertex figure.

For each facet F through a, make an *ordinary* plane node incident with $F \setminus \{a\}$. For each polygonal two-face R avoiding a, make an *extra* plane node incident with R, unless these incidences already belong to an ordinary plane. Include one point node for each label other than a. We will use deletion orders in this bipartite incidence graph for which reverse insertion always solves independent linear equations.

The following rules give such orders. Incidence throughout means incidence with the nodes still retained.

(i) A plane may be deleted with at most two points. It may be deleted with three points if it is extra, or if those three points together with a consist of three vertices of a two-face and one vertex outside that face.

(ii) A point may be deleted with at most one plane, or with two planes of which at most one is extra.

(iii) A point corresponding to a neighbor of a may be deleted with three ordinary planes. More generally, two or three planes may be used if they are distinct supporting planes at that point of the projection of a single facet avoiding a.

(iv) A point may be deleted with three planes if two are distinct and contain another labeled point w, while the third is an ordinary plane excluding w.

The last rule applies because the first two planes meet in a line through the point being deleted and w, and the third plane does not contain that line. Distinctness in this rule is certified by one ordinary plane and one extra plane, or by distinct supporting planes of a common projected facet. These are the only cases needed below.

Lemma 6. *If these rules delete the incidence graph, its projected configurations have dense rational realizations preserving all required plane incidences.*

Proof. Projection is a projective isomorphism on any facet avoiding a, so any three vertices of a projected polygon are independent. The four-point facial witness in rule (i) likewise makes the three projected points independent. Ordinary planes are distinct supporting planes of Q, and an extra plane differs from every ordinary plane; equality would put its original polygon in a facet through a, in which case the incidence would already have been ordinary.

At a vertex of a three-polytope, any three distinct incident facet normals are independent. Their rays are extreme rays of a pointed three-dimensional cone, whereas a pointed cone in a two-dimensional subspace has only two extreme rays. This proves the independence required in rule (iii), including its application within a projected facet. Rule (iv) was explained above.

Reverse the deletion order, choosing rational homogeneous points and planes near their originals. At each stage, the independent equations define a rational kernel of constant dimension, whose rational points are dense. All strict ordinary supporting inequalities persist. Incidences between points deleted earlier and planes deleted later are imposed when the former are reinserted, so the entire prescribed configuration is recovered. □

Once these incidences are fixed, the heights are governed by affine relations. Suppose first that every facet with more than five vertices contains a, and every polygon avoiding a is a quadrilateral. Take the unique projected relation of each triangular-bipyramid facet avoiding a, together with the relation of each quadrilateral avoiding a, and extend each row by zero to the other labels. If $C(q)$ is the resulting matrix, the required heights satisfy exactly

$$C(q)h = 0. \tag{2}$$

Indeed, these equations say that the reciprocal height is affine on each projected facet; a bipyramid contributes one equation, a square pyramid contributes its base equation, and a tetrahedron contributes none. Every coefficient of a bipyramid or quadrilateral row is nonzero.

A larger facet F avoiding a is handled in the same way. Choose four vertices $T \subset F$ whose independence is certified by three vertices of a two-face and a fourth outside it. For each $v \in F \setminus T$, use the fundamental affine relation on $T \cup \{v\}$. Its coefficient at v is nonzero; a coefficient at $w \in T$ is also nonzero whenever $(T \setminus \{w\}) \cup \{v\}$ has a facial independence witness. These rows impose affine height on all of F, so separate quadrilateral rows contained in F are unnecessary.

Proposition 7 (Projection and height reconstruction)**.** *Suppose the projected incidence graph admits the preceding deletion rules and every polygon avoiding a has at*

most four vertices. Form the height rows as above, allowing larger facets with facially independent four-point frames. If the rows can be deleted successively by choosing in each row a certified nonzero coefficient whose label is absent from every other remaining row, then the type is rationally dense.

Proof. The height rows have full row rank at every realization; a label occurring in only one remaining row forces that row's coefficient to vanish in any dependence, and induction finishes the argument. After applying Lemma 6, choose the free heights rationally near their originals and solve for the remaining heights through one fixed nonsingular minor. The solutions stay positive. The vertex figure preserves facets through a, while (2) preserves every other facet hyperplane. Their ranks and all strict supporting inequalities persist, giving the original face lattice. Rational affine transformations approaching the inverse normalization return realizations near the original coordinates. □

The distinction between a possible row support and a certified nonzero coefficient matters for the fundamental relations; coefficients other than the pivot may vanish. All the data used below record both. Thus the criterion requires no genericity assumption about an unspecified realization.

5.2. **Counting the relations**

A second reduction measures how much arithmetic a face lattice can impose before choosing a projection center. Let $n = f_0(P)$, $m = f_3(P)$, and define

$$E = \sum_{F} (|F| - 4), \qquad D = \binom{n}{2} - f_1(P), \qquad g = \sum_{R} (|R| - 3),$$

where F runs through facets and R through two-faces. Euler's formula, applied also to vertex figures, gives

$$E = \frac{n(n-3)}{2} - m - D + g, \qquad 2g \le D. \tag{3}$$

The second inequality counts diagonals of polygonal two-faces; each contributes at least twice its excess, and a nonedge cannot be a diagonal in two different two-faces.

Choose each polygon's affine relation basis once and supplement it within each facet to the full relation space. If h_F is the rank of the polygon relations inside F, write

$$g_F = \sum_{R \subset F} (|R| - 3), \qquad \epsilon_F = g_F - h_F.$$

The number c of selected rows is

$$c = E - g + \sum_F \epsilon_F. \tag{4}$$

A nonedge belongs to at most two facets. Give it weight one in a unique containing facet, or weight one-half in each of two facets. If d_F is the number of polygon diagonals in an r-vertex facet, its allocated weight satisfies

$$w_F - (r-4) - \epsilon_F = \frac{(r-4)(r-5)}{2} + h_F - \frac{d_F}{2} =: \eta_F \geq 0.$$

For $h_F = 0$ there are no polygon diagonals; for $h_F = 1$ there is one quadrilateral. Otherwise

$$d_F \leq \lfloor r(r-4)/2 \rfloor \leq (r-4)(r-5) + 4 \leq (r-4)(r-5) + 2h_F.$$

Summing the weights, then using (3)–(4), yields

$$E \leq D, \qquad c \leq \frac{n(n-3)}{2} - m - E - \sum_F \eta_F. \tag{5}$$

The following small-core results are proved in Appendix A. Their proofs identify the only eight-element incidence obstruction, rather than assuming that a smooth realization space has rational points.

Proposition 8. (i) *Every four-polytope with at most nine vertices is rationally dense.*

(ii) *A four-polytope with at most nine nonsimplicial facets or at most nine nonsimple vertices is rationally dense.*

(iii) *A four-polytope admitting at most eight selected facet-relation rows is rationally dense.*

6. **Rationality through ten vertices**

For $n = 10$, the master count becomes

$$E = 35 - m - D + g, \qquad E \leq D, \qquad c \leq 35 - m - E - \sum_F \eta_F. \tag{6}$$

Theorem 3 removes simple vertices, and Proposition 8 removes $c \leq 8$ and types with at most nine nonsimplicial facets. The remaining types have $E \geq 10$. If $m \geq 17$, then $c \leq 8$, so only $m \leq 16$ remains.

6.1. **The high-facet boundaries**

When $m = 16$, the only possible excess is $E = 10$, and every nonsimplicial facet has five vertices. If q is the number of quadrilaterals, then

$$D = 9 + q \geq 10, \qquad c = 10 - q \geq 9,$$

so $q = 1$. Thus there are eight triangular bipyramids, two square pyramids with a common base, and six tetrahedra.

When $m = 15$, only $E = 10, 11$ remain. If all nonsimplicial facets have five vertices, the inequalities $c = E - q \geq 9$ and $D = 20 - E + q \geq E$ leave

$$(E, q) = (10, 0), (10, 1), (11, 2).$$

Otherwise there is one six-vertex facet and nine five-vertex facets. No pentagon is possible. The six-facet has allocated weight three, forcing $q \geq 3$, while $c = 11-q+\epsilon_F \geq 9$ and $\epsilon_F \leq 1$ force $q = 3$, $\epsilon_F = 1$. A nonpentagonal six-vertex three-polytope with three quadrilaterals has two triangles and a cubic planar graph, hence is a triangular prism. These four facet-shape families exhaust $m = 15$.

For $10 \leq m \leq 14$, the remaining budget is

$$\#\{F : |F| > 4\} \geq 10, \qquad E \leq 26 - m.$$

The local surplus η_F makes large facets particularly restrictive. Appendix C reduces those cases to small explicit boundary lists. All other lists are obtained by direct facet growth or by coarsening complete simplicial boundaries.

Proposition 9 (Finite reduction)**.** *Every realizable ten-vertex four-polytope not removed by Theorem 3 and Proposition 8 satisfies the projection criterion of Section 5, or is one of the explicit configurations in Appendix B.*

The proof and the complete partition of the finite lists are given in Appendix D; the accompanying exact data supply the incidence insertion orders and height pivots. The computations establish coverage and the required nonzero witnesses. The positive constructions themselves use rational linear algebra, apart from one small Cayley polytope for which rational existence is given directly.

Proof of Theorem 2 and Corollary 4. Proposition 8 settles at most nine vertices. For ten vertices, delete a simple vertex if present, using Theorem 3. The relation count and finite reduction leave either the projection criterion, which gives rational density, or an explicit configuration from Appendix B, each of which has a rational realization. This proves Theorem 2. Deleting a simple vertex from an eleven-vertex type and applying Theorem 3 gives the corollary. □

Appendix A. Small incidence cores

A.1. Plane configurations on eight points

Lemma 10. *Every configuration of at most eight points in the real projective plane has dense rational realizations of its oriented matroid.*

Proof. Call a line rich if it contains at least three points. A noncollinear configuration on $n \le 8$ points has a point on at most two rich lines. Otherwise pair counting and Melchior's inequality give

$$\binom{n}{2} - 3n \ge 3 + \sum_{k\ge 4} \frac{(k-3)(k+2)}{2} t_k,$$

where t_k counts k-point lines. Melchior's inequality itself follows from Euler's formula for the dual arrangement in $\mathbb{RP}^2$, with $3F \le 2E$.

Only $n = 8$ remains, and then there are eight rich triples, three through each point, and four ordinary pairs. The two triples omitting an ordinary pair must be disjoint; otherwise their common point could lie on no third rich line. Avoiding repeated pairs forces, up to relabeling,

$$135, 246, 147, 238, 168, 257, 367, 458.$$

Normalize the first four points to a frame. The first six incidences give

$$5 = (u, 0, 1), \quad 6 = (1, v, 1), \quad 7 = (u, 1, 1), \quad 8 = (0, v, 1),$$

and the last two give $uv = 1 = u + v$, impossible over $\mathbb{R}$.

Delete a point on at most two rich lines and proceed inductively. Reinsert it freely, on one rational line, or at the intersection of two rational lines. Rational choices sufficiently close to the original retain the signs of all nonzero minors. Collinear configurations, repeated projective points, and zero vectors are immediate. □

Gale duality gives rational density for four-polytopes with at most eight vertices, since their homogeneous kernels have rank at most three. Rational kernel bases and the inverse transformation vary continuously in a fixed nonzero-minor chart.

A.2. Relation spaces and the golden exception

Let the rows of C span all facet relation spaces, with c rows, and choose L with $\ker L = \operatorname{col} C$. Preserve the rank of L and its ranks on the support of every column of C. The corresponding kernels then vary continuously. Choose nearby rational

columns in these kernels to form C'. Its rank is bounded above by $\dim\ker L$ and below by a fixed nonzero minor, hence is unchanged. Five nearby rational homogeneous coordinate rows in $\ker C'$ preserve all facet equations, ranks, and strict supports.

By Lemma 10 and duality, this works whenever $c \le 7$. For $c = 8$, only rank four remains. The exact eight-element incidence input is the following.

Lemma 11. *Every real rank-four configuration on eight elements is rationally dense, except the matroid represented by*

$$[I_4 \mid D(t)], \qquad D(t) = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & t & t+1 \\ 1 & 1 & t & 0 \end{pmatrix}, \qquad t^2 + t - 1 = 0.$$

Its dependent four-sets are those in (1)*; every triple is independent and every five-set spans.*

The finite proof uses the 731 connected rank-four eight-element matroids in the small-matroid catalog [8]. Normalize a basis to I_4 and a spanning tree of nonzero matrix entries to one; every realization admits this normalization. Eliminate a variable only through an equation linear in it whose coefficient is a product of entries or minors prescribed nonzero. These substitutions hold throughout the allowed chart. The exact data reduce the list to 636 charts with no equations, 61 covered by the following lemma, two rational curves, 31 systems with no real points, and the displayed exception. Disconnected cases reduce to configurations on at most seven elements.

Lemma 12 (Multiaffine density)**.** *Let $f(x_1, \dots, x_s, z)$ have rational coefficients and degree at most one in each x_i. If $f(\,\cdot\,, z^0)$ is not identically zero and $f(x^0, z^0) = 0$, then (x^0, z^0) is a limit of rational zeros of f.*

Proof. Expand in the increments $x_i - x_i^0$, holding $z = z^0$. Choose a nonzero monomial of smallest degree, with variable set S, and set all increments outside S to zero. The restriction is exactly

$$a_S \prod_{i \in S} (x_i - x_i^0).$$

Make all but one of these increments small and nonzero, leaving the last zero. This gives a nearby zero at which some x_j-derivative is nonzero. Writing $f = ax_j + b$, rationally approximate all other coordinates and set $x_j = -b/a$. The resulting rational zeros approach the nearby zero, and hence the original point. □

For an irreducible residual polynomial with at most one unrestricted variable z, the coefficients as a polynomial in the x_i have greatest common divisor one in $\mathbb{Q}[z]$. They cannot all vanish at any z^0. A coefficient certified as a nonzero minor factor gives the same conclusion. Thus the degree and factor checks in the data establish density even at singular real points, where a particular attempted denominator might vanish.

The two remaining curves have the parametrizations

$$x^2+xy+y^2-x-y=0,\quad x=\frac{1+u}{1+u+u^2},\quad y=\frac{u(1+u)}{1+u+u^2},$$
$$x^2y^2-2xy^2+3xy-x+y^2-2y+1=0,\quad x=u^2-u+1,\quad y=-\frac{u}{(u-1)^2}.$$

Their inverse parameters $u=y/x$ and $u=xy/(y-1)$ are defined at every allowed point. The remaining nonsplit quadratic has discriminant $-3y^2(y-1)^2$ and no allowed real point. This completes the mathematical justification of the chart checks used in Lemma 11; the exact reductions are supplied in the electronic supplement [3].

If one of the exceptional four-flats is not required, rational density is restored. Replace t by rational b approaching it and make the following entry changes in the full matrix

Omitted flat	Changed entries
1238	$(4,8)=b+1-1/b$
1245	$(3,5)=b+1-1/b$
1267	$(4,7)=1/(b+1)$
1578	none
2356	$(4,6)=1/[b(b+1)]$, $(4,7)=1/(b+1)$

The involution (1 4)(2 3)(5 6)(7 8) gives the other four cases. Thus the relation-space argument can fail with eight rows only when the column supports require all nine exceptional four-flats. Every nonzero column then has at least four entries, and every nonzero row-space vector of the exceptional L has support at least four.

A.3. **Polarity and nine vertices**

Keep in P^* only the vertices corresponding to nonsimplicial facets of P, forming its *polar core*. Rational density of this labeled core implies density of P. Indeed, at an original vertex the selected normals expose a face of the core. Preserving its dimension preserves the rank and consistency of the equations $a_F\cdot x=1$, so nearby

rational solutions exist. The remaining tetrahedral supports persist by openness. By polarity, the same argument applies to the core of nonsimple vertices.

For $n = 9$, the quantity $n(n-3)/2$ is 27. The polar-core argument allows us to assume nine nonsimple vertices and at least nine nonsimplicial facets, so $m, E \geq 9$. The row count leaves only $E = 9$ or $(m, E) = (9, 10)$.

If $E = 9$, all nine nonsimplicial facets have five vertices. With q quadrilaterals, $c = 9 - q$. For $q \geq 2$, the relation-space argument applies. For $q = 1$, the seven bipyramid rows and one quadrilateral row have 39 nonzero entries, whereas nine distinct exceptional four-supports on nine columns would require exactly 36. The golden exception is therefore impossible. If $q = 0$ and $m = 10$, then $D = 8 < E$; thus only nine bipyramid facets remain.

Their 45 incidences force every vertex to belong to five facets. Each vertex figure is a prism or square pyramid, hence has at most six vertices; since $f_1 = 27$, all are prisms. The nonedge graph is therefore 2-regular. Each nonedge is the apex pair of exactly one bipyramid. At a vertex, the two such facets meet only there and together use all nine labels, as their triangular faces in the prism figure are disjoint. Membership of a fixed label alternates around each nonedge cycle except at that label. Its own cycle must be odd and every other cycle even; varying the label forces a single nine-cycle. Labeling it cyclically gives

$$F_{i,i+1} = \{i, i+1, i+3, i+5, i+7\}.$$

But $F_{0,1} \cap F_{2,3}$ has four vertices, whereas any proper intersection of these bipyramid facets is triangular. This is impossible.

For $(m, E) = (9, 10)$, there is one six-vertex facet and eight five-vertex facets. No pentagon can occur. The six-facet has relation excess $\epsilon \leq 1$, so

$$c = 10 - q + \epsilon, \qquad D = 8 + q.$$

Its allocated weight is three; the other facets contribute eight, giving $D \geq 11$, hence $q \geq 3$ and $c \leq 8$. Equality requires three dependent quadrilateral rows in the six-facet. They give a nonzero vector of $\ker C^T = \operatorname{row} L$ supported on three positions, excluding the golden exception. This proves rational density through nine vertices, and the polar-core argument proves Proposition 8(ii).

A.4. **Eight relation rows in any four-polytope**

Retain a vertex set I and suppose every omitted vertex is an apex in each incident facet. Every facet is then the join of its retained base with the simplex on its omitted vertices. Rational density of $\operatorname{conv} I$ preserves the base dimensions; restore the omitted

vertices freely and rationally nearby. All facet dimensions and strict supporting inequalities persist.

Choose the polygon relations as four-point circuits and supplement them within each facet by circuits on at most five points. Then

$$\operatorname{nnz}(C) \le 4g + 5(c - g) = 5c - g.$$

For $c \le 8$, only the golden rank-four case remains. If $g > 0$, there are at most 39 nonzero entries, and every nonzero column has at least four, so at most nine columns are nonzero. Retain those vertices. A zero column in every facet relation means exactly that its vertex is an apex in that facet. The retained core is rationally dense by the result just proved, and the preceding restoration argument applies. If $g = 0$, then $c = E \le 8$ and the polar core suffices. This proves Proposition 8(iii).

Appendix B. The exceptional rational constructions

All constructions in this appendix are made near a specified realization. A fixed nonzero minor keeps each join, intersection, and linear solve of the stated dimension. The strict supporting inequalities are then open, so the face lattice is retained. Rationally approximating the inverse normalization returns rational points near the original realization. This also explains why no new facet appears; its limiting vertex set would lie in an old supporting face, whose perturbed points already lie in a retained facet hyperplane.

B.1. A symmetric Cayley polytope

The one case for which only rational existence is needed is the fourteen-facet type $A73$. In $\mathbb{R}^3$, take the triangular bipyramids

A	x	y	z	B	x	y	z
0	1	0	1	1	0	1	-1
9	-1	0	1	2	0	-1	-1
8	0	0	0	7	0	0	0
5	21/25	18/25	3/5	3	7/10	$-9/10$	$-1/2$
6	$-21/25$	$-18/25$	3/5	4	$-7/10$	9/10	$-1/2$

Their apex pairs are $5, 6$ and $3, 4$, since each midpoint lies inside its base triangle. Set

$$P = \operatorname{conv}\big((A, 1) \cup (B, -1)\big).$$

Reflection $\sigma(x,y,z)=(-x,-y,z)$ exchanges $0,9$; $1,2$; $3,4$; and $5,6$. For $n=(x,y,z)$, the support functions are

$$h_A(n)=\max\bigl(0,\,z+|x|,\,\tfrac{3}{5}z+|\tfrac{21}{25}x+\tfrac{18}{25}y|\bigr),$$
$$h_B(n)=\max\bigl(0,\,-z+|y|,\,-\tfrac{1}{2}z+|\tfrac{7}{10}x-\tfrac{9}{10}y|\bigr).$$

The following directions, with their reflections, give the lateral facets

Direction	A-face	B-face	Facet
$a_1=(3,-1,-3)$	058	3	0358
$a_2=(1,-2,-1)$	068	23	02368
$a_3=(0,5,9)$	059	47	04579
$b_1=(2,1,1)$	05	137	01357
$b_2=(4,7,7)$	5	147	1457
$b_3=(5,0,-7)$	58	123	12358

A triangle–segment row gives a bipyramid; the segment is parallel to the triangle plane but to none of its edges, so no four of the five vertices are coplanar. A triangle–point row gives a tetrahedron.

To see completeness, the a-rays and their reflections form the normal fan of A, and the b-rays form that of B. On the nine edges of the first fan, only $b_1,\bar b_1,b_3,\bar b_3$ occur, on the edges dual to $05,96,58,68$. Indeed,

$$b_1=(2a_1+a_3)/3,\qquad b_3=2a_1+\bar a_2,$$

whereas $b_2=(8a_1+4\bar a_2+7a_3)/5$ lies inside a vertex cone. Subdivide these four edges. On $05,58,68,96$, the successive arcs expose the common B-vertices $3,7$; $3,1$; $2,4$; and $7,4$, respectively. The unsplit edges $08,06,09,59,89$ expose $3,3,7,4,4$. Convexity of the vertex cones excludes further crossings. Thus the common refinement has exactly the twelve listed rays.

With the two base facets, the bipyramids are

$$01357,02368,03679,04579,05689,12347,12358,12468,14589,24679,$$

and the tetrahedra are $0358,1457,2367,4689$. This is the required rational realization of $A73$.

B.2. **Independent line intersections**

For type $A40$, relabel

$$(A_0,A_1,A_2,A_3,A_4)=(0,1,3,2,9),\qquad (B_0,B_1,B_2,B_3,B_4)=(4,6,7,5,8).$$

Indices are modulo five. Its ten bipyramid facets are

$$F_i = \{A_i, A_{i+1}, A_{i+3}, B_i, B_{i-1}\}, \qquad G_i = \{A_i, A_{i+3}, B_{i+2}, B_{i+3}, B_{i+4}\},$$

and its tetrahedra are $T_i = \{A_i, A_{i+3}, B_i, B_{i+1}\}$.

The five A_i are affinely independent in every realization. Applying a nonnegative defining form of F_i to a homogeneous dependence shows that its coefficients at A_{i+2}, A_{i+4} have opposite signs or both vanish. These pairs form an odd cycle, forcing every coefficient to vanish.

Normalize the A_i to the coordinate points of $\mathbb{P}^4$. If H_i is the hyperplane of F_i and $L_j = A_{j+1} \vee B_j$, then $L_j \subset H_j \cap H_{j+1}$ and

$$B_j = L_j \cap (L_{j-1} \vee L_{j+1}). \tag{7}$$

The join on the right is the hyperplane of G_{j-3}; its four defining vertices are independent, and it excludes A_{j+1}, so the intersection is unique. Choose the H_i rationally near their originals while retaining their three coordinate points, then choose the L_j independently in the rational planes $H_j \cap H_{j+1}$ through A_{j+1}. Formula (7) gives all B_j rationally. Every F_i incidence holds by containment of the lines, and every G_i incidence follows from the same intersection formula. This proves rational density.

For type $A14$, normalize the affine frame $1, 2, 3, 4, 9$; four of these points lie in the bipyramid 12369 and 4 lies outside. Choose rational lines near $09, 36, 48$, subject to

$$9 \in L_0, \qquad 3 \in L_6 \subset 1239, \qquad 4 \in L_8.$$

Construct successively

$$\begin{aligned}
H &= 24 \vee L_0, & 8 &= L_8 \cap (13 \vee L_0),\\
M_7 &= H \cap (L_0 \vee L_6) \cap (L_6 \vee L_8), & 7 &\in M_7,\\
L_7 &= 27, & M_5 &= 1249 \cap (L_0 \vee L_8) \cap (L_7 \vee L_8),\\
5 &= M_5 \cap (1 \vee L_6 \vee 8), & 6 &= L_6 \cap (15 \vee L_7),
\end{aligned}$$

then choose $0 \in L_0$. The choices of 7 and 0 are free and rational near their originals.

Every displayed join is a hyperplane and M_7, M_5 are lines. The needed independence comes from four vertices of a bipyramid, or from three distinct supporting planes at a vertex. The lines defining 8,5,6 contain respectively the fixed points 4,4,3, which are excluded from the target facets $01389, 13568, 12567$, so the intersections are transverse. The ten bipyramid hyperplanes are

Facet	Hyperplane	Facet	Hyperplane
01389	$13 \vee L_0$	02479	H
03679	$L_0 \vee L_6$	04589	$L_0 \vee L_8$
12369	1239	12459	1249
12567	$15 \vee L_7$	13568	$1 \vee L_6 \vee 8$
24578	$L_7 \vee L_8$	34678	$L_6 \vee L_8$

The tetrahedra $0347, 0348, 1589, 2679, 5678$ impose only open conditions. Thus this construction proves rational density without a cyclic compatibility equation.

B.3. **A prism frame and its axis**

Every triangular prism is projectively equivalent to

$$A_i = e_i, \qquad B_i = e_i + x \qquad (i = 1, 2, 3),$$

with x, e_1, e_2, e_3 independent in its hyperplane H. Its three longitudinal lines are pairwise coplanar, hence concurrent; they cannot all lie in one plane. Rescaling the point pairs with respect to their common point gives the displayed form.

The two end triangles span $U = \operatorname{span}(e_1, e_2, e_3)$ and $U' = \operatorname{span}(e_1+x, e_2+x, e_3+x)$. Their axis is

$$L = \mathbb{P}(U \cap U') = \mathbb{P}\operatorname{span}(y_{12}, y_{23}), \qquad y_{ij} = e_i - e_j, \quad y_{13} = y_{12} + y_{23}.$$

The opposite end edges meet at y_{ij}. Put a common outside vertex at $O = e_4$ and write other outside vertices as $e_4 + u, e_4 + v, e_4 + w$, with $u, v, w \in H$.

If two bipyramids consist of the same three outside vertices and the opposite edges A_iA_j, B_iB_j, the trace in H of the outside triangle plane passes through y_{ij}. Indeed, a line meeting both end edges either passes through their intersection or lies in their quadrilateral plane. The latter alternative would put the other end edge into the first bipyramid hyperplane, contradicting its exact vertex set.

For the fifteen-facet type $D4$, use

$$(A_1, A_2, A_3) = (0, 3, 4), \quad (B_1, B_2, B_3) = (9, 2, 5), \quad O = 8,$$

and $1 = e_4 + u$, $6 = e_4 + v$, $7 = e_4 + w$. The bipyramid pairs

$$(01478, 15789), \quad (03678, 26789), \quad (12568, 13468)$$

become

$$y_{13} \in \operatorname{span}(u, w), \quad y_{12} \in \operatorname{span}(v, w), \quad y_{23} \in \operatorname{span}(u, v).$$

None of u, v, w lies on L; otherwise two of these lines coincide with L or force two distinct y_{ij} to coincide, and a bipyramid acquires an extra end vertex. No two vectors are proportional, since three actual vertices cannot be collinear. Consequently every realization has the form

$$v = au + by_{23}, \qquad w = k(au + by_{13}), \qquad abk \neq 0.$$

Conversely, these formulas imply all incidences because $w - kv = kby_{12}$. Rational approximation of u, a, b, k preserves the type.

For type $D0$, use

$$(A_1, A_2, A_3) = (0, 3, 5), \quad (B_1, B_2, B_3) = (9, 2, 6), \quad O = 8,$$

and $1 = e_4+u$, $4 = e_4+v$, $7 = e_4+w$. The pairs $(01578, 16789)$ and $(12468, 13458)$ give $y_{13} \in \operatorname{span}(u, w)$ and $y_{23} \in \operatorname{span}(u, v)$; the bipyramids $03457, 24679$ give $v-w \in U \cap U'$. As $u \notin L$, every realization is locally parametrized by

$$w = au + by_{13}, \qquad v = au + cy_{23}.$$

The difference $v - w$ lies on the axis identically. Rational approximation proves density, while the fixed prism preserves all square-pyramid bases.

B.4. **A second large facet**

Type 85 has nonsimplicial facets

$$012345, \quad 134567, \quad 01367, \quad 45678$$

and bipyramids

$$24589, 35789, 06789, 14689, 01269, 02379.$$

Normalize its prism 134567 by

$$(A_1, A_2, A_3) = (1, 4, 6) = (e_1, e_2, e_3),$$
$$(B_1, B_2, B_3) = (3, 5, 7) = (e_1 + x, e_2 + x, e_3 + x),$$

and put $9 = e_4$, $8 = e_4 + w$, $0 = e_4 + u$, $2 = e_4 + v$. The bipyramids $14689, 35789$ put

$$w = ae_1 + be_2 + ce_3 \in U \cap U', \qquad a + b + c = 0.$$

All three coefficients are nonzero, since four vertices of a bipyramid are independent. The next incidences give

$$u \in \Pi_3 = \operatorname{span}(w, x, e_3), \quad v \in \Pi_2 = \operatorname{span}(w, x, e_2), \quad y_{13} \in \operatorname{span}(u, v).$$

The six-vertex facet 012345, containing the quadrilateral 1345, requires $u - v \in \operatorname{span}(x, e_1, e_2)$, so $u_3 = v_3$.

Choose w rationally on the axis, then u rationally in Π_3, and define

$$[v] = \mathbb{P}\operatorname{span}(u, y_{13}) \cap \mathbb{P}\Pi_2.$$

Scale v to make $v_3 = u_3$. The point y_{13} is outside Π_2 because $b \neq 0$; if $u \in \Pi_2$, the final scaling would make 0 and 2 coincide. Finally $v_3 \neq 0$, since otherwise $v \in \operatorname{span}(x, e_2)$ and four vertices $2, 4, 5, 9$ of 24589 would be coplanar. All choices are therefore rational and transverse near every realization, proving density.

B.5. **Two crossed prisms**

Suppose two prism facets share the quadrilateral $q_1q_2q_3q_4$, their outside pairs are a, c and b, d, and their other quadrilaterals are

$$acq_1q_2, \quad acq_3q_4, \quad bdq_2q_3, \quad bdq_4q_1.$$

The eight points admit the following normalization in an auxiliary affine chart

q_1	q_2	q_3	q_4	a	c	b	d
0	e_1	$e_1 + e_2$	e_2	e_3	$e_3 + e_1$	e_4	$e_4 + e_2$.

To prove it, choose representatives $q, q+u, q+u+v, q+v$ for the common quadrilateral. The line ac meets its plane at u, so its pair can be scaled to $A, A + u$; similarly b, d become $B, B + v$. The five vectors q, u, v, A, B are independent because the prism hyperplanes are distinct. The chart with value one on q, A, B and zero on u, v gives the normalization.

The missing grid point

$$z = e_3 + e_4 = a + b - q_1 = b + c - q_2 = c + d - q_3 = d + a - q_4$$

lies in all four planes

$$\Pi_1 = abq_1, \quad \Pi_2 = bcq_2, \quad \Pi_3 = cdq_3, \quad \Pi_4 = daq_4.$$

The points z, a, b, c, d are projectively independent. Projection from z turns these planes into the four cyclic edges of a tetrahedron, and the only lines meeting all four cyclic edges are its two remaining edges. Hence a line xy meeting all four Π_i satisfies

$$z \in xy, \qquad xy \subset zac, \qquad \text{or} \qquad xy \subset zbd. \tag{8}$$

Conversely each condition gives the required intersections.

Suppose the remaining equations require x, y, Π_i to span a hyperplane. These say exactly that xy meets every Π_i. We may therefore approximate xy among lines through the rational point z, or approximate x, y within one of the rational planes in (8). All facet ranks and strict inequalities persist. The three relevant two-prism types have labels

Type	x, y	a, b, c, d	q_1, q_2, q_3, q_4
0	3,7	4,5,8,6	0,2,1,9
1	7,8	4,1,5,6	2,3,9,0
2	6,7	0,1,8,9	2,4,5,3

Their four bipyramids are $xyabq_1, xybcq_2, xycdq_3, xydaq_4$; the square-pyramid bases are grid quadrilaterals and impose no additional equations.

The same grid handles the two types with three or four prism facets. For the four-prism type

$$135789, \quad 012389, \quad 012459, \quad 023478,$$

normalize the first two with $(a, b, c, d) = (5, 2, 7, 0)$ and $(q_1, q_2, q_3, q_4) = (1, 8, 3, 9)$. The quadrilaterals $1245, 2478, 0347, 0459$ put 4 in all four grid triangle planes, whose intersection is z. Vertex 6 is only a square-pyramid apex or tetrahedron vertex, so is free.

For the three-prism type $014569, 012379, 023569$, use $(a, b, c, d) = (1, 2, 4, 3)$ and $(q_1, q_2, q_3, q_4) = (0, 5, 6, 9)$. The quadrilaterals $0127, 1379$ put 7 on za. It is not z, since four vertices of 24578 would then be coplanar. Its two bipyramid hyperplanes are $zabc, zacd$, whose intersection forces $8 \in zac$. Choose 7 on za and 8 in zac rationally near their originals; every required incidence follows.

B.6. **Two compatible prisms**

If two prism facets share two longitudinal edges, their concurrency points agree. Their eight vertices lie in four independent pairs on concurrent lines and normalize to

$$[e_i], \quad [e_i + e_0], \qquad i = 1, 2, 3, 4.$$

Thus this core is rational and projectively unique.

For type 52, the prism pairs are $(16, 27, 48)$ and $(16, 39, 48)$. Normalize their common core and put $H = 1234$, $B = 6789$. Choose 0 in H and 5 in $H \cap B$ rationally nearby. This preserves 012345 and 56789; the bases $1267, 1369, 2478, 3489, 2379$ are fixed by the concurrent-line configuration.

For type 53, the prism pairs are $(01, 67, 89)$ and $(45, 67, 89)$. Set $\Pi = 0145$, $L = 68$, $B = 1479$. The plane Π is disjoint from L, and B excludes the concurrence point.

Choose a rational hyperplane $H \supset \Pi$, then $3 \in H \cap B$, then $2 \in H \cap (3 \vee L)$. These intersections are a plane and a line because $H \neq B$, $3 \notin L$, and $L \not\subset H$. They preserve 012345, 13479, and 2368; all other quadrilateral incidences belong to the fixed prisms.

B.7. **Concurrent planes and reciprocal heights**

The remaining type with one six-vertex facet admits a linear construction after passing to Gale coordinates. The useful observation is quite general. Write a four-dimensional vector space as

$$H = \Pi \oplus \mathbb{R}g_0, \qquad \dim \Pi = 3,$$

and let K be the graph of a linear form ℓ on Π. Suppose three distinct planes $M_i = \ker n_i$ in Π meet in a line $\mathbb{R}k$, with $\ell(k) \neq 0$. For $g_i = q_i + h_i g_0 \in K$, where $h_i \neq 0$, a point $g_* = q_* + g_0 \in K$ belongs to every $M_i + \mathbb{R}g_i$ precisely when

$$n_i(q_*) = \frac{n_i(q_i)}{h_i}.$$

The unique relation among the three normals therefore gives one linear equation in the reciprocal heights h_i^{-1}. Once this holds, two of the displayed equations and $\ell(q_*) = 1$ determine q_* uniquely. Concurrence is what makes the remaining compatibility linear.

For type 679, the facets imposing equations are the six-vertex facet 012345, whose quadrilateral is 1345, and the eight bipyramids

$$01367,\ 01268,\ 12469,\ 06789,\ 02378,\ 45679,\ 24589,\ 35789.$$

The other facets are the square pyramid 13456 and tetrahedra $2689, 3567, 2358$. Let $g_i \in \mathbb{R}^5$ be homogeneous Gale vectors. If P is the homogeneous vertex matrix and G is a matrix whose rows form a basis of its kernel, then

$$\operatorname{rank} G_I = |I| - 5 + \operatorname{rank} P_{I^c}. \tag{9}$$

Thus $g_0, g_2, g_6, g_7, g_8, g_9$ span a four-space H, while g_6, g_7, g_8, g_9 span a three-space $K \subset H$. Normalize the other four vectors as

$$g_1 = e_5, \qquad g_3 = e_5 + u, \qquad g_4 = e_5 + v, \qquad g_5 = e_5 + w.$$

The vectors u, v, w are independent; put $\Pi = \langle u, v, w \rangle$. The bipyramid 06789 gives $g_2 = q_2 \in \Pi$, and $g_0 \notin \Pi \cup K$. Projection along g_0 identifies K with Π, so

$$g_i = q_i + \ell(q_i) g_0 \quad (i = 6, 7, 8, 9).$$

We use u, v, w, g_0, g_1 as a fixed rational coordinate basis.

Four bipyramid equations become the following construction in $\mathbb{P}(\Pi)$

$$q_6 \in q_2 \vee v, \qquad q_8 \in q_2 \vee u, \qquad q_7 = (q_6 \vee u) \cap (q_8 \vee (w-u)). \tag{10}$$

Choose q_2, then q_6, q_8 on the indicated lines, and construct q_7. For the last three equations put

$$M_6 = \langle v, w\rangle, \quad M_7 = \langle v-u, w-u\rangle, \quad M_8 = \langle q_2, w-v\rangle.$$

These planes are distinct and share the line $\mathbb{R}(w-v)$. Choose their normals with relation $c_6n_6 + c_7n_7 + c_8n_8 = 0$. Each coefficient is nonzero. With $h_i = \ell(q_i)$, the preceding observation gives the sole compatibility condition

$$\frac{c_6n_6(q_6)}{h_6} + \frac{c_7n_7(q_7)}{h_7} + \frac{c_8n_8(q_8)}{h_8} = 0. \tag{11}$$

After rational choices in (10), choose h_6^{-1}, h_7^{-1} rationally nearby and solve (11) for h_8^{-1}. The independent vectors q_6, q_7, q_8 then determine ℓ. Finally normalize $\ell(q_9) = 1$ and solve

$$n_6(q_9) = n_6(q_6)/h_6, \quad n_7(q_9) = n_7(q_7)/h_7, \quad \ell(q_9) = 1.$$

The coefficient matrix is nonsingular because $\ell(w-v) \neq 0$. The six-vertex and square-pyramid equations are already imposed by K and H, so all facet equations are preserved.

For this construction to prove density, the denominators and intersections must be nondegenerate at every realization. They follow from facial ranks, rather than a generic choice. Every four vertices of a triangular bipyramid facet are independent, and adjoining a vertex outside that facet gives rank five. Applying (9) gives the following checks. Here a primal set containing a whole facet and an outside vertex also has rank five.

Condition	Primal rank witness
u, v, w independent	The complement of 1345 contains 06789 and 2.
$g_0 \notin \Pi$	6789 is a four-subset of 06789.
$g_0 \notin K$	12345 spans 012345; 2 is outside the plane 1345.
Triples in g_6, g_7, g_8, g_9 independent	Their complements contain 012345 and an outside vertex.
$h_6, h_7, h_8, h_9 \neq 0$	The complementary sets $0789, 0689, 0679, 0678$ are independent in 06789.
$M_8 \neq M_6, M_7$	The complements of $1245, 2345$ contain 06789 and respectively 3, 1.
$n_i(q_i) \neq 0$	The complements of $01456, 03457, 02458$ are $23789, 12689, 13679$; each contains four vertices of $02378, 01268, 01367$ and the outside vertex 9.
$\ell(w - v) \neq 0$	0123 is independent, since 0 has neighbors $1, 2, 3$ in 012345.

The joins $q_6 \vee u$ and $q_8 \vee (w - u)$ are genuine lines; otherwise 0136 or 0358 would have Gale rank at most three, whereas its complement contains the bipyramid 24589 or 12469 and the outside vertex 7. They cannot coincide, since q_6, q_7, q_8 are independent. The initial joins are defined for the same reason, using the complements of 123 and 124. These witnesses also show that all choices above vary continuously in a neighborhood of each realization. Rational approximation in the fixed coordinates, followed by a rational kernel basis and a nearby rational affine chart, proves rational density of the primal type.

B.8. **The sixteen-facet residue**

The sixteen-facet list leaves 74 systems with eight bipyramids, two square pyramids on a common quadrilateral, and six tetrahedra. Ordinary projection covers 69 of them. Two more, labeled 444 and 446, use the three-plane insertion from Section 5; project from 8, delete the two ordinary triangular plane nodes, then point 1 and its three now triangular plane nodes. The remaining incidence system peels. The only extra independence checks are

Type	Two planes sharing a line	Third plane
444	ordinary 1369, extra 1235, sharing 13	0179, excluding 3
446	ordinary 0149, extra 1346, sharing 14	1579, excluding 4

Both height systems have a leaf order at center 8. It remains to exclude types 31 and 32 and rationalize type 452.

All three arguments take place in Gale space. The quadrilateral circuit puts its six complementary Gale points in a four-dimensional vector space H, while the other four points lie outside H. Each bipyramid gives a hyperplane incidence on its five complementary labels. Formula (9) translates all required independence statements back into facial ranks.

A common obstruction for types 31 and 32. Let $A, B, C, D, E, F \in \mathbb{P}(H)$ and let $O, X, Y, Z \notin \mathbb{P}(H)$. Put $\{R, S\} = \{C, E\}$, and suppose the four-sets

$$ABDR, \quad BDRS, \quad ABDF, \quad ACEF$$

and the triples OXB, YZD, XZS are independent. We show that the seven hyperplane incidences

$$BDROX,\ ABFOX,\ BDRYZ,\ ADFYZ,\ BSOXZ,\ DSXYZ,\ ACEXZ \tag{12}$$

are incompatible.

Write $X = O + x$, $Y = O + y$, $Z = O + z$ with $x, y, z \in H$, and set

$$M = \langle B, D, R\rangle, \qquad T = \langle A, F\rangle \cap M.$$

Here T is a vector line. The independent four-sets give

$$H = M \oplus \langle S\rangle, \qquad T \not\subset \langle B, D\rangle, \qquad T \not\subset \langle A, C, E\rangle.$$

Indeed, $A \notin M$; the independence of A, B, D, F separates T from BD; and $AF \cap ACE = A$. The first four incidences in (12) now imply

$$x \in \langle B, T\rangle, \qquad r := z - y \in \langle D, T\rangle.$$

Neither vector is proportional to the indicated point B or D, by the independent triples. Project H onto M along S, writing a bar. The next two incidences give

$$\bar{z} \in \langle B, x\rangle = \langle B, T\rangle, \qquad \bar{z} - x \in \langle D, r\rangle = \langle D, T\rangle.$$

Their intersection puts $\bar{z} - x$ in T, and hence $z - x \in ST$. The last incidence also puts $z - x$ in ACE. Since $S \in ACE$ and $T \not\subset ACE$, it follows that $z - x \in S$, contradicting the independence of X, Z, S.

For the two types the substitutions are

Type	(A, B, C, D, E, F)	(O, X, Y, Z)	(R, S)
31	$(0, 1, 3, 5, 8, 9)$	$(7, 4, 6, 2)$	(E, C)
32	$(0, 2, 4, 6, 8, 9)$	$(7, 3, 5, 1)$	(C, E)

Each incidence is the complement of a bipyramid. To record the rank checks compactly, write $B + v$ (F) when B is an independent four-subset of the bipyramid F and $v \notin F$. The four-set witnesses are

Gale set	Type 31	Type 32
$ABDR$	$2369 + 4$ (02369)	$1589 + 3$ (01589)
$BDRS$	$0269 + 4$ (02369)	$0159 + 3$ (01589)
$ABDF$	$2368 + 4$ (23568)	$1458 + 3$ (14568)
$ACEF$	$1245 + 6$ (12458)	$1236 + 5$ (12368)

For the triples OXB, YZD, XZS, the complementary witnesses in type 31 are $02369 + 5$, $13478 + 0$, and $01789 + 5$; in type 32 they are $04569 + 1$, $02479 + 3$, and $02479 + 5$. Each consists of a whole facet and an outside vertex. Thus the obstruction applies to every putative realization.

Three affine lines for type 452. Relabel the Gale points by

$$(A, B, C, D, E, F) = (0, 4, 5, 9, 8, 7), \quad (O, X, Y, Z) = (6, 3, 2, 1).$$

The first six lie in H, complementary to the quadrilateral 1236. The points A, B, C, D are independent. Otherwise the primal six-set 123678 would lie in a three-flat, and its convex hull would be a six-vertex three-polytope with the quadrilateral face 1236 and the eight distinct triangular facets

$$127,\ 278,\ 367,\ 678,\ 138,\ 178,\ 268,\ 378$$

cut out by the bipyramids. Nine facets exceed the bound $2f_0 - 4 = 8$.

Normalize A, B, C, D to the coordinate points, and write

$$E = (1, 1, s, t), \qquad F = (p, q, 1, 1).$$

The coefficients used to normalize are nonzero. In order, the Gale sets 4589, 0589, 0479, and 0457 have the primal rank witnesses

$$0127+3\ (01247),\quad 1247+3\ (01247),\quad 1358+2\ (13589),\quad 1389+2\ (13589).$$

Put $U=CDE$, $V=ABF$, with equations $x_0=x_1$, $x_2=x_3$, and write $X=O+x$, $Y=O+y$, $Z=O+z$. The eight bipyramid equations become

$$\begin{aligned} 01247,\ 03467: &\quad x, z-y\in U,\\ 13589,\ 25689: &\quad y, z-x\in V,\\ 14578,\ 02578,\ 34789,\ 06789: &\quad ADOXY,\ BDOXZ,\ ACOYZ,\ BCXYZ. \end{aligned}$$

The incidence $ADOXY$ makes the lines $Dx\subset U$ and $Ay\subset V$ meet at a point $T\in U\cap V$. They are genuine lines; a collapse would make OXD or OYA collinear, but the complementary primal sets contain respectively $01247+5$ and $13589+4$. They are distinct because $A\notin U$.

Both coordinate pairs of T are nonzero. If $T\in CD$, then $x\in CD$, contradicting the independence of Gale $OXCD=3569$, whose complement contains $01247+8$. The analogous exclusion of $T\in AB$ uses $OYAB=0246$ and $13589+7$. We may therefore write

$$T=A+B+\lambda(C+D),\qquad \lambda\neq 0.$$

The next two incidences put z on $BDT\cap ACT$, the line through T and $A+\lambda C$. The conditions $z-y\in U$ and $z-x\in V$ then give

$$x=aT+\lambda kD,\qquad y=bT-kA,\qquad z=cT-k(A+\lambda C).$$

Here $k\neq 0$, since otherwise O,X,Y would be collinear, although their primal complement contains $14578+0$. The last incidence is automatic

$$y-x=(b-a-k)T+kB+\lambda kC,\qquad z-x=(c-a-k)T+kB.$$

Diagonal rescaling sets $\lambda=1$, and a homology fixing H pointwise and fixing O sets $k=1$. Thus

$$x=(a,a,a,a+1),\quad y=(b-1,b,b,b),\quad z=(c-1,c,c-1,c).$$

All required incidences hold identically in the seven free parameters s,t,p,q,a,b,c. Rational approximation preserves the original ranks and strict supporting inequalities, proving rational density.

Appendix C. The contribution of large facets

The surplus η_F in (5) measures the relations saved by the polygonal faces of a facet. We first estimate it locally, then use the estimates to delimit the large-facet lists.

C.1. Local surplus

For a three-polytope F on r vertices, let h_F be the rank of its polygon relation spaces and d_F the number of polygon diagonals. Then

$$\eta_F = \frac{(r-4)(r-5)}{2} + h_F - \frac{d_F}{2}, \qquad \eta_F \geq \begin{cases} 3/2, & r = 7, \\ 3, & r = 8. \end{cases} \tag{13}$$

If every face has at most four vertices, the lower bounds improve to two and four, respectively. The verification also identifies the equality configurations needed later. Euler's formula and minimum degree give polygon excess at most four for $r = 7$, and at most six for $r = 8$. Two quadrilateral relation rows are independent. Four have rank at least three; if they had rank two on N used columns, the zero sets of the four rows would be disjoint, so $4(N-4) \leq N$. This gives $N \leq 5$, although two quadrilateral faces already use at least six vertices.

The following table gives the possibilities containing a larger polygon, with q denoting the number of quadrilateral faces.

r	Larger faces	$h_F,\ d_F$	Bound for η_F
7	hexagon	$3,\ 9$	$3/2$
7	one pentagon, $q \leq 2$	2 if $q = 0$, else 3; $5 + 2q$	$3/2$
8	heptagon	$4,\ 14$	3
8	hexagon, $q \leq 2$	3 if $q = 0$, else 4; $9 + 2q$	$7/2$
8	two pentagons, $q \leq 2$	$4,\ 10 + 2q$	3
8	one pentagon, $q \leq 4$	≥ 3 if $q > 0$, 4 if $q = 4$	$7/2$

A polygon on $r-1$ vertices makes a pyramid. A hexagon in an eight-vertex polytope can have at most two quadrilateral companions, since each uses the two outside vertices as an edge. Two pentagons have disjoint relation spaces, and three pentagons would require at least nine vertices. For the last row, if $q = 4$ and $h_F \leq 3$, the four quadrilateral rows would be proportional on the three coordinates outside the pentagon. Their outside supports would coincide; a common triple violates proper face intersections, while a common pair puts four faces on one edge. Thus $h_F = 4$.

If all nonsimplicial faces are quadrilaterals, then $\eta_F = 3 + h_F - q$ for $r = 7$ and $\eta_F = 6 + h_F - q$ for $r = 8$. The preceding rank bound settles $q \leq 5$. The remaining case is $r = 8, q = 6$, where the graph is cubic and bipartite, hence $K_{4,4}$ minus a

matching, the cube. Starting with a corner and its three neighbors, four face relations successively introduce the other four vertices, so $h_F = 4$. This proves (13).

C.2. **Global consequences**

Assume that at least ten facets are nonsimplicial; all other types were removed by the polar-core argument. A nine-vertex facet makes the four-polytope a pyramid over a three-polytope, which is rationally dense. For an eight-vertex facet, $E \geq 13$. At equality every other nonsimplicial facet has five vertices, so the large facet has no pentagon and contributes $\eta_F \geq 4$; otherwise $E \geq 14$ and it contributes at least three. In either case (6) gives $c \leq 8$.

Two seven-vertex facets give $E \geq 14$ and total surplus at least three. One seven-vertex facet and at least three six-vertex facets give $E \geq 15$ and surplus at least $3/2$. Again $c \leq 8$. With one seven-vertex facet F and exactly two six-vertex facets, the only remaining budget is

$$m = 10, \qquad E = 14.$$

Every facet is nonsimplicial, and we may assume that no vertex is simple.

Let a, b, c be the three vertices outside F. A nonsimplicial facet not adjacent to F contains all three. At most two facets contain three fixed noncollinear vertices, so F has at least seven two-faces. It therefore has no hexagon; a pentagon permits at most one quadrilateral, and without a pentagon there are at most three quadrilaterals.

If F has a pentagon, it contributes at least $5/2$, and the adjacent pentagonal pyramid contributes another $1/2$. A pentagon outside F makes both six-vertex facets pentagonal pyramids; together with $\eta_F \geq 2$ this again gives total surplus at least three. Without pentagons, each nonprismatic six-vertex facet contributes one, so the same conclusion holds unless both six-vertex facets are prisms. All these cases have $c \leq 8$.

Suppose both are prisms. Their allocated nonedge weights, together with F and seven five-vertex facets, sum to 19. If q is the global quadrilateral count, then

$$D = 11 + q \geq 19, \qquad 2q \leq 3 + 6 + 7 = 16.$$

Hence $q = 8$, the facet F has three quadrilaterals, and the other seven facets are square pyramids. To have $c > 8$, the three quadrilateral rows in F must be dependent. The rank-two zero-set argument now gives exactly six used vertices, each in two quadrilaterals. These faces form a prism belt, and the seventh vertex subdivides one triangular cap. Thus F is a triangular prism with a tetrahedron stacked on one cap; it has twelve edges, three vertices of degree four, and four of degree three.

Exactly two facets are not adjacent to F. They are square pyramids containing a, b, c, and their common face is abc or $abcw$. The whole graph has $45 - 19 = 26$ edges. Since every vertex of F has total degree at least five, at least

$$3 \cdot 1 + 4 \cdot 2 = 11$$

edges join F to a, b, c. If the common face is $abcw$, the outside graph is a two-edge path, leaving twelve such edges. Both square-pyramid apices lie in F and are adjacent to all three outside vertices, exceeding the minimum demand by at least one each; this would require thirteen cross edges.

If the common face is abc, exactly eleven cross edges remain, so every vertex of F has total degree five. A prism cannot meet F in a triangle, since it would then be a third facet containing a, b, c. Each of the three cap triangles oxy, oyz, ozx therefore has a square-pyramid neighbor. Its apex cannot be x, y, z, which already have degree four in F and would acquire two outside neighbors. Its apex is consequently o. All three square pyramids use the same two outside neighbors of o, giving three facets through three fixed vertices, again impossible.

We have proved that any remaining large-facet type has exactly one seven-vertex facet and at most one six-vertex facet.

C.3. **The remaining seven-vertex facet budgets**

Let F be the unique seven-vertex facet, let r count its quadrilaterals, and let s count nonsimplicial facets. The adjacency argument above gives $f_2(F) \geq s-3$. The relevant quantities are

Other large facet	s	$f_2(F)$	Allocated weight	g	c
None	$E-2$	$10-r$	$E+3$	q	$E-q+\epsilon_7$
Six vertices, no pentagon	$E-3$	$10-r$	$E+4$	q	$E-q+\epsilon_7+\epsilon_6$
Six vertices, a pentagon	$E-3$	$8-r$	$E+3$	$q+2$	$E-q-2$

Here ϵ_j is the polygon-relation excess of the j-vertex facet. The allocated weight is at most $D = 35 - m - E + g$, and $m \geq s$. In the first row, $E \geq 13$ forces $r \leq 2$, hence $\epsilon_7 = 0$, and

$$c \leq 32 - m - E \leq 8.$$

In the second row, $E \geq 14$ gives $r \leq 2$ and $\eta_F = 3$, whence $c \leq 35 - 2E \leq 7$. The third row gives the same conclusion for $E \geq 14$. The complete boundary budgets are therefore the following.

(i) With no six-vertex facet, $E = 12$,

$$10 \le m \le 12, \quad \max(0, 11-m) \le r \le 3, \quad \max(r, m-8) \le q \le 3+\epsilon_7.$$

The numbers of square pyramids, bipyramids, and tetrahedra are $2q-r$, $9-2q+r$, and $m-10$. There are 14 budgets over 27 facet templates.

(ii) With one six-vertex facet and no pentagon, $E = 13$. Either $(m, q) = (10, 5)$ with $\epsilon_7 + \epsilon_6 \ge 1$, or $(m, q) = (10, 6), (11, 6)$ with both excesses equal to one. If r_6 is the number of quadrilaterals in the six-vertex facet, the small-facet counts are

$$2q - r - r_6, \qquad 8 - 2q + r + r_6, \qquad m - 10,$$

with $r \ge 11 - m$. There are 34 budgets.

(iii) With a pentagon, $E = 13$, $m = 10$, $q = 2$, and $r \le 1$. The six-vertex facet is a pentagonal pyramid, and there are four possible seven-vertex facet templates.

All facet counts must be nonnegative. The condition $\epsilon_7 = 1$ describes the capped prism found above, and $\epsilon_6 = 1$ describes a triangular prism. Thus these inequalities specify the entire remaining enumeration range.

Appendix D. Finite coverage

We finish the proof of Proposition 9. The geometric reductions leave finitely many boundary complexes, with facet sizes and counts fixed by Section 6 and Appendix C. The exact data [3] contain their complete enumerations, sign exclusions, and incidence reconstruction orders. The following table gives the resulting partition. Except in its first two rows, $10 \le m \le 14$.

Facet family	Finite reduction	Remaining argument
Sixteen facets	801 coarsenings; 727 sign exclusions; 74 survivors	69 ordinary projection orders; two shared-line insertions; 31 and 32 excluded, and 452 rationalized, in Appendix B.8.
Fifteen facets	Four shape families with $9, 257, 1, 4$ survivors	267 projection orders; the two line constructions $A14, A40$ and prism constructions $D0, D4$ in Appendix B.

Facet family	Finite reduction	Remaining argument
All facets have at most five vertices, $E = 10$	339 surviving systems	338 projection orders; the rational Cayley realization $A73$.
Other small-facet residue	Five survivors for $m = 14$, four for $m = 13$	Projection and height reconstruction in every case.
One nonprismatic six-vertex facet	1846 systems; 432 simple-vertex cases, 104 boundary failures, 1087 sign exclusions, 223 survivors	217 projection orders with large facet through the center, five with the large facet away from the center, and type 679 from Appendix B.7.
One prism facet	Eight survivors	Projection and height reconstruction.
At least two prisms, no other facet larger than five	27 low-facet count families, and three two-prism types for $m = 14$	Two simple-vertex cases; all others use the grid, common-concurrence, or three- and four-prism constructions.
Several six-vertex facets with a pentagon	22 count families, one residual survivor	Projection and height reconstruction.
Other several-six-vertex-facet types	228 systems; 53 simple-vertex cases, two boundary failures, 157 sign exclusions, sixteen survivors	Eleven projection orders with all large facets through the center, two with one large facet away, and the constructions for 52, 53, 85.
A facet with at least seven vertices	The surplus reductions leave five pure-seven candidates; four have a simple vertex and one is excluded by signs. No mixed seven/six candidate survives.	Appendix C and simple-vertex deletion.

For clarity, the four fifteen-facet families have respectively ten bipyramids; eight bipyramids and two square pyramids; seven bipyramids and four square pyramids; or one prism, three square pyramids, and six bipyramids. The first, second, and fourth families have five tetrahedra, and the third has four. The remaining several-six-vertex-facet branch comprises 31 count families and 270 facet-type budgets; the

all-prism and pentagonal branches are listed separately.

There are two exhaustive enumeration methods. In direct growth, start with a facet of an allowed type and attach every locally admissible facet across each open ridge, retaining the prescribed vertex and facet budgets. Reject only a violated necessary incidence or manifold condition. Every polytopal boundary is reached because its facet adjacency graph is connected; canonical relabeling removes duplicates without removing an isomorphism class. The coarsening branches instead start from every compatible pulling triangulation in Lutz's complete ten-vertex simplicial three-sphere catalog [4, 7]. The input contains 247882 three-spheres, with the type and index conversion recorded explicitly. A pulling triangulation uses the original vertices, so every required polytopal boundary occurs among these coarsenings.

For each candidate, orient its triangulated boundary. Facet incidences prescribe zero determinants and the signs of determinants obtained by adjoining an outside vertex. The three-term Grassmann–Plücker identities then propagate the allowed signs. A contradiction is an exact obstruction to realization; candidates with a consistent sign system are retained. The boundary failures in the table are failures of the necessary coherent orientation of the proposed triangulated boundary.

A positive certificate records a projection center, an incidence deletion order, and a height-row deletion order. A two-face independence witness justifies each three-point plane insertion, while every height pivot is a nonzero polygon or bipyramid circuit coefficient, or a fundamental-relation coefficient witnessed by a four-point facial frame. Lemma 6 and Proposition 7 turn these into rational constructions near every realization. Appendix B.8 displays the two shared-line witnesses, and Appendix B treats the other residual types.

The supplement contains the complete catalog inputs and enumeration programs, together with independent checkers for the reconstruction orders and determinant-sign exclusions. These reconstruct the 338 projection certificates in the first two table rows, cover the low-facet branches, and verify the eight-element incidence input of Lemma 11. Every candidate is therefore either excluded by an exact necessary condition or rationalized by the constructions above, proving Proposition 9.

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