Universal cold spots of E_8 and related lattices

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Abstract. We prove that the deep holes of E_6, E_8, E_7^*, the sixteen-dimensional Barnes–Wall lattice, and D_n^+ in every even dimension $n \geq 8$ are precisely the minimizers of every Gaussian lattice sum. We determine the exact minima and equality points, which also gives the minimizers for every convergent positive mixture of Gaussian potentials. The analytic core is a sharp hyperbolic inequality for geometric means; reduction to two-point measures proves the E_8 bound, and a power inequality extends it to the entire D_n^+ family. Projection and diagonal gluing transfer sharp bounds to E_7^* and Barnes–Wall, while a positive cubic theta identity gives the E_6 theorem. A uniform rounding argument identifies the geometric equality sets of E_8 and D_n^+.

1. Introduction

For a Euclidean lattice $L \subset \mathbb{R}^n$ and $\alpha > 0$, consider the periodic Gaussian sum

$$\Theta_L(\alpha, x) = \sum_{v \in L} e^{-\alpha \|v-x\|^2}.$$

Its minimizers are called *cold spots*. When α is large, the nearest lattice points dominate the sum, so the geometry of the deepest holes governs the limiting problem. A cold spot that remains a minimizer for every $\alpha > 0$ must also control the contributions of all the more distant shells. This is the universal cold-spot problem.

Bachoc, Moustrou, Vallentin, and Zimmermann [1] proved that the deep holes of several highly symmetric lattices are stable cold spots; they minimize the Gaussian sum once α is sufficiently large. They asked whether these minima persist for every positive parameter. We answer that question for four exceptional lattices and obtain an infinite family as well.

Theorem 1.1. *For each of the lattices*

$$E_6, \qquad E_8, \qquad E_7^*, \qquad \mathrm{BW}_{16}, \qquad D_n^+ \quad (n \geq 8 \text{ even}),$$

the minimizers of $\Theta_L(\alpha, \cdot)$, for every $\alpha > 0$, are exactly the deep holes of L.

Here a *deep hole* is a point at maximal distance from the lattice. The root lattices are normalized to have minimal squared norm two, E_7^* is the dual of that normalization of E_7, and BW_{16} has minimal squared norm four. We use

$$D_n = \{m \in \mathbb{Z}^n : \textstyle\sum_i m_i \equiv 0 \pmod 2\}, \qquad D_n^+ = D_n \cup (D_n + \tfrac{1}{2}\mathbf{1}).$$

This union is a lattice precisely when n is even, and $D_8^+ = E_8$. It is integral when $4 \mid n$ and even when $8 \mid n$; no integrality assumption is needed for the theorem.

The proofs give exact polarization values and explicit equality loci. For reference, the squared covering radii and the numbers of deep-hole classes modulo lattice translations are

L	squared radius	number of classes
E_6	$4/3$	2
E_8	1	135
E_7^*	$7/8$	36
BW_{16}	3	120
D_n^+ ($n > 8$ even)	$n/8$	$\binom{n}{n/2}$

For $n > 8$, the deep holes of D_n^+ have a particularly simple description; precisely half of their coordinates are integers and half are half-integers. The additional equality families at $n = 8$ will explain both the exceptional count 135 and the transfer to the other lattices.

Corollary 1.2. *Let L be one of the lattices in Theorem 1.1, and let*

$$f(r) = \int_{(0,\infty)} e^{-\alpha r}\, d\nu(\alpha)$$

for a nonzero positive measure ν, with the lattice sums convergent. Then the minimizers of $\sum_{v\in L} f(\|v - x\|^2)$ are exactly the deep holes of L.

Proof. Tonelli's theorem integrates the pointwise Gaussian inequality. It is strict off the common minimum set for every positive parameter, so integration introduces no additional equality points. □

The main analytic step is a geometric-mean inequality on a hyperbolic arc. Classical quadratic theta identities put the four one-dimensional factors of the E_8 sum on this arc, and the inequality compares an arbitrary collection of coordinates with its diagonal values. Once this eight-dimensional inequality is available for arbitrary probability measures, a power inequality extends it to every even dimension above

eight. We give this reduction and its equality analysis in Sections 2 and 3, before proving the analytic inequality in Sections 4 and 5.

The remaining lattices are connected by two operations that preserve sharp Gaussian bounds. On a suitable affine slice of E_8, the two fiber theta functions agree; projection therefore gives the complete minimum set of E_7^*. For Barnes–Wall, a diagonal gluing of two copies of E_8 turns the theta sum into an inner product of coset profiles, and equality is a finite character calculation. The E_6 argument uses a different identity, arising from the three-point Fourier transform over the Eisenstein integers. Its positive terms give a short minimum iteration and determine equality without a separate theorem on triangular cold spots. These constructions occupy Sections 6–8.

2. **Quadratic theta identities**

Fix $\alpha > 0$ and define

$$A_\alpha(t) = \sum_{m\in\mathbb{Z}} e^{-\alpha(m-t)^2}, \qquad B_\alpha(t) = \sum_{m\in\mathbb{Z}} (-1)^m e^{-\alpha(m-t)^2}.$$

Write $a = A_\alpha(0)$, $b = B_\alpha(0)$, and $c = A_\alpha(1/2)$. The function A_α is even and one-periodic; B_α is even and satisfies $B_\alpha(t+1) = -B_\alpha(t)$. Poisson summation gives

$$A_\alpha(t) = \sqrt{\pi/\alpha}\,\theta_3(\pi t, q), \qquad B_\alpha(t) = \sqrt{\pi/\alpha}\,\theta_2(\pi t, q), \quad q = e^{-\pi^2/\alpha}.$$

By Jacobi's product formula [4, Section 20.5],

$$A_\alpha(t) = \sqrt{\pi/\alpha}\prod_{j\ge 1}(1-q^{2j})\big(1+2q^{2j-1}\cos(2\pi t)+q^{4j-2}\big).$$

Each variable factor is strictly increasing in $\cos(2\pi t)$ and positive. Thus the minima of A_α occur exactly at $\mathbb{Z}+1/2$, and its maxima exactly at $\mathbb{Z}$. The analogous product for θ_2 shows $B_\alpha(t) > 0$ for $0 \le t < 1/2$, with $B_\alpha(1/2) = 0$; in particular $b > 0$.

2.1. **Duplication and the hyperbolic arc**

For a fixed t, abbreviate

$$A = A_\alpha(t), \quad B = B_\alpha(t), \quad C = A_\alpha(t-1/2), \quad D = B_\alpha(t-1/2).$$

Set $P = A_{2\alpha}(0)$, $Q = A_{2\alpha}(1/2)$, $U = A_{2\alpha}(t)$, and $W = A_{2\alpha}(t-1/2)$. Splitting the double sums for $A^2 \pm B^2$ by parity of the sum of their integer indices, and using half-sum and half-difference coordinates, gives

$$\begin{aligned} A^2 &= PU + QW, \quad B^2 = PU - QW, \\ C^2 &= PW + QU, \quad D^2 = PW - QU. \end{aligned} \tag{1}$$

At $t = 0$ these identities read

$$a^2 = P^2 + Q^2, \qquad b^2 = P^2 - Q^2, \qquad c^2 = 2PQ,$$

so $a^4 = b^4 + c^4$. Substitution in (1) also gives

$$a^2A^2 = b^2B^2 + c^2C^2, \qquad a^2C^2 = c^2A^2 + b^2D^2. \tag{2}$$

These relations will reduce the four positive theta factors to a single parameter.

2.2. **Product and quartic identities**

Changing the two summation indices to their sum and difference gives, for every $\beta > 0$,

$$A_\beta(t)A_\beta(t-1/2) = A_{2\beta}(1/4)A_{\beta/2}(2t-1/2). \tag{3}$$

For either parity of the sum, the remaining difference sum is $A_{2\beta}(1/4)$; this accounts for the constant. Similarly, combining the integer and half-integer diagonal cosets gives

$$A_\beta(t)^2 + A_\beta(t-1/2)^2 = A_{\beta/2}(0)A_{\beta/2}(2t).$$

Separation into even and odd indices gives $A_\alpha(1/2) = 2A_{4\alpha}(1/4)$. Together with duplication, these formulas imply

$$\begin{aligned} s(t) := A^4 - B^4 = C^4 - D^4 = 4PQUW = c^3A_\alpha(2t-1/2) > 0, \\ A^4 + D^4 = (P^2+Q^2)(U^2+W^2) = a^3A_\alpha(2t). \end{aligned} \tag{4}$$

The strict minimum of A and (3) further give

$$A_\alpha(v)A_\alpha(v-1/2) \geq ac, \qquad \text{with equality exactly when } v \in \tfrac{1}{2}\mathbb{Z}. \tag{5}$$

All rearrangements are justified by absolute, locally uniform convergence of the Gaussian series.

3. The D_n^+ family and the exceptional dimension eight

The four theta factors in Section 2 satisfy two quadratic relations. Their common parametrization leads to the following inequality, whose proof occupies Sections 4 and 5. We state it for probability measures because this makes the passage from dimension eight to higher dimensions immediate.

For a nonnegative function f on a probability space, write $Gf = \exp \int \log f \, d\mu$, with $\exp(-\infty) = 0$.

Theorem 3.1 (Hyperbolic geometric means)**.** *For every probability measure μ on $[0, R]$, where $R > 0$,*

$$(G\cosh p)^4 + (G\cosh(R-p))^4 - (G\sinh p)^4 - (G\sinh(R-p))^4 \\ \geq 2\cosh R \, G\cosh(2p-R).$$

Equality holds exactly when μ is a point mass or a balanced reflected pair $\frac{1}{2}(\delta_p + \delta_{R-p})$.

Proposition 3.2. *For every probability measure μ on $[0, 1/2]$, with the notation of Section 2,*

$$(GA)^8 + (GC)^8 - (GB)^8 - (GD)^8 \geq 2(ac)^4. \tag{6}$$

Its equality measures are precisely

$$\delta_0, \qquad \delta_{1/4}, \qquad \delta_{1/2}, \qquad \tfrac{1}{2}(\delta_0 + \delta_{1/2}). \tag{7}$$

Proof. Define $R > 0$ by $\cosh R = a^2/c^2$ and $\sinh R = b^2/c^2$, which are consistent because $a^4 - b^4 = c^4$. Since $A^4 - B^4 = s(t) > 0$, define $p \geq 0$ by

$$A^2/\sqrt{s} = \cosh p, \qquad B^2/\sqrt{s} = \sinh p.$$

The quadratic identities (2) give

$$C^2/\sqrt{s} = \cosh(R-p), \qquad D^2/\sqrt{s} = \sinh(R-p), \qquad 0 \leq p \leq R.$$

For the range assertion, the first identity initially gives the hyperbolic cosine; the second would give a negative value for $D^2/\sqrt{s}$ if $p > R$. In particular,

$$t = 0,\ 1/4,\ 1/2 \quad \longmapsto \quad p = R,\ R/2,\ 0.$$

Applying Theorem 3.1 to the pushforward measure gives

$$\tfrac{1}{2}\big((GA)^8 + (GC)^8 - (GB)^8 - (GD)^8\big) \geq G\big[s^2 \cosh R \cosh(2p-R)\big].$$

The expression inside the geometric mean is the diagonal theta value

$$\begin{aligned} F_{\text{diag}}(t) &= \tfrac{1}{2}(A^8 + C^8 - B^8 - D^8) \\ &= s(A^4 + D^4) = (ac)^3 A_\alpha(2t) A_\alpha(2t - 1/2) \geq (ac)^4, \end{aligned}$$

by (4) and (5). Equality in this last step restricts t to $0, 1/4, 1/2$; their distinct hyperbolic parameters and the equality statement of Theorem 3.1 then give exactly (7). □

3.1. **The Gaussian minimum in every even dimension**

For $L_n = D_n^+$, parity decomposition gives

$$\Theta_{L_n}(\alpha, x) = \tfrac{1}{2}\left(\prod_i A_\alpha(x_i) + \prod_i B_\alpha(x_i) + \prod_i A_\alpha(x_i - 1/2) + \prod_i B_\alpha(x_i - 1/2)\right). \tag{8}$$

Fold each coordinate as $x_i = m_i + \varepsilon_i t_i$, where $m_i \in \mathbb{Z}$, $\varepsilon_i \in \{\pm 1\}$, and $0 \leq t_i \leq 1/2$. Then $A_i = A_\alpha(t_i)$ and $C_i = A_\alpha(t_i - 1/2)$ are the original nonalternating factors, while B_i, D_i are the absolute values of their alternating counterparts. Therefore

$$\Theta_{L_n}(\alpha, x) \geq \tfrac{1}{2}\left(\prod_i A_i + \prod_i C_i - \prod_i B_i - \prod_i D_i\right). \tag{9}$$

Equality requires each nonzero signed alternating product to have negative sign.

Theorem 3.3. *For every even $n \geq 8$ and every $\alpha > 0$,*

$$\min_x \Theta_{D_n^+}(\alpha, x) = (ac)^{n/2}.$$

When $n > 8$, the complete minimum set is

$$\left\{x \in \tfrac{1}{2}\mathbb{Z}^n : \#\{i : x_i \notin \mathbb{Z}\} = n/2\right\}. \tag{10}$$

When $n = 8$, the minimum set consists of the following three families

(i) $e_1 + E_8$*;*

(ii) *points with four integer and four half-integer coordinates;*

(iii) $m + \varepsilon/4$*, where $m \in \mathbb{Z}^8$ has odd coordinate sum and $\varepsilon \in \{\pm 1\}^8$ has an even number of negative signs.*

All these minimum sets are independent of α.

Proof. Set $k = n/8 \geq 1$ and

$$X = (\prod_i A_i)^{8/n}, \quad Y = (\prod_i C_i)^{8/n}, \quad U = (\prod_i B_i)^{8/n}, \quad V = (\prod_i D_i)^{8/n}.$$

Proposition 3.2, applied to $n^{-1}\sum_i \delta_{t_i}$, gives $X + Y - U - V \geq 2(ac)^4$. Moreover $X > U$ and $Y > V$. For $x \geq u \geq 0$ and $k \geq 1$,

$$x^k - u^k \geq (x-u)^k,$$

with equality for $k > 1$, $x > u$, exactly when $u = 0$. Thus

$$\begin{aligned} X^k + Y^k - U^k - V^k &\geq (X-U)^k + (Y-V)^k \\ &\geq 2^{1-k}(X+Y-U-V)^k \geq 2(ac)^{n/2}. \end{aligned}$$

Combining this with (9) proves the bound.

For $n > 8$, equality forces $U = V = 0$, $X = Y$, and equality in (6). Of its four equality measures, only the balanced endpoint measure has both alternating geometric means zero. Hence exactly half the folded coordinates are zero and half are $1/2$. Conversely, every such point has both alternating products zero and both nonalternating products $(ac)^{n/2}$, so it attains equality regardless of integer-part parity.

For $n = 8$, all four measures in (7) are possible. The all-zero and all-half folded tuples require odd integer-part sum, and together form $e_1 + E_8$. For four zero and four half coordinates, both alternating products vanish, so no sign restriction remains. For the all-quarter tuple, the first alternating product has sign $(-1)^{\sum m_i}$ and the second has sign $(-1)^{\sum m_i + \#\{i:\varepsilon_i=-1\}}$. Both are negative exactly under the conditions in (iii). These families attain the bound, proving completeness. □

3.2. **A uniform covering argument**

The equality loci above can be identified geometrically by a single rounding estimate. In dimension eight, both endpoints of the estimate are sharp, accounting for the extra families of deep holes.

Proposition 3.4. *For every even $n \geq 8$, the squared covering radius of D_n^+ is $n/8$. Its deep holes are exactly the points in Theorem 3.3.*

Proof. Let $r_i \in [0, 1/2]$ be the distance of x_i from the nearest integer, and set

$$m = \min_i r_i, \quad M = \max_i r_i, \quad \delta = M - m, \quad s = m + M, \quad T = \sum_i r_i.$$

Rounding to the integer lattice costs $\sum_i r_i^2$. If the rounded point has odd coordinate sum, changing the choice in a coordinate with maximal r_i repairs parity at additional cost $1 - 2M$. The analogous repair for the half-integer coset costs at most $2m$. Hence

$$\operatorname{dist}(x, D_n)^2 \leq \sum_i r_i^2 + 1 - 2M, \qquad \operatorname{dist}(x, D_n + \tfrac{1}{2}\mathbf{1})^2 \leq \sum_i (1/2 - r_i)^2 + 2m.$$

The secant inequality $r_i^2 \le (m+M)r_i - mM$ now gives

$$\operatorname{dist}(x, D_n^+)^2 \le \min\{sT - nmM + 1 - 2M,\ (s-1)T + n/4 - nmM + 2m\}. \quad (11)$$

The first affine function is nondecreasing in T and the second nonincreasing, so their minimum is at most their common value at $T_* = n/4 - 1 + 2s$. That value is

$$\frac{n}{16} + \frac{1}{2} - \delta + \left(2 - \frac{n}{4}\right)(s - 1/2)^2 + \frac{n}{4}\delta^2. \quad (12)$$

The coefficient of $(s-1/2)^2$ is nonpositive, and the remaining expression is convex in $\delta \in [0, 1/2]$. Its endpoint maximum is $\max\{n/16 + 1/2, n/8\} = n/8$.

If $n > 8$, equality forces $\delta = 1/2$, hence $m = 0$, $M = 1/2$, and $s = 1/2$. Both affine slopes in (11) are then nonzero, so equality requires $T = T_* = n/4$. Equality in the secant estimate requires every r_i to be 0 or $1/2$, giving exactly $n/2$ of each. The parity repairs cost nothing at tied rounding coordinates, so every point of (10) has squared distance $n/8$.

For $n = 8$, expression (12) becomes $1 - \delta + 2\delta^2$, whose maximum one occurs precisely at $\delta = 0$ or $\delta = 1/2$. The second case gives four zero and four half folded coordinates as before. In the first, all $r_i = t$. For $0 < t < 1/2$, both affine slopes are nonzero and equality requires $8t = 1 + 4t$, so $t = 1/4$. For this quarter-grid case, both nearest-coset distances equal one exactly when both parity repairs are required; writing $x = m + \varepsilon/4$, this means $\sum m_i$ odd and an even number of negative signs in ε.

The endpoints $t = 0, 1/2$ must be treated separately, because one affine slope vanishes. An integer point has distance one from E_8 exactly when its coordinate sum is odd; the all-half case has the same condition on its integer part. These are precisely $e_1 + E_8$. Thus every equality family agrees with the analytic classification, completing the proof. □

3.3. **Translation classes and symmetry**

For $n > 8$, each half-coordinate subset of size $n/2$ has two integer-parity classes modulo D_n. Translation by $\frac{1}{2}\mathbf{1}$ complements the subset and pairs these classes, giving $\binom{n}{n/2}$ classes modulo D_n^+. Coordinate permutations, even sign changes, and translations act transitively; an odd integer-part parity can be reversed by changing the signs of one half-integer and one integer coordinate.

For E_8, family (i) gives one class. In family (ii), the $\binom{8}{4}$ choices of half-coordinate support each have two integer parities, and translation by $\frac{1}{2}\mathbf{1}$ identifies complementary supports, leaving 70 classes. Family (iii) has 128 allowed sign vectors, paired with

their negatives by the same half-translation, leaving 64 classes. Thus the total is $1 + 70 + 64 = 135$.

These classes form one affine Weyl orbit. Reflection in the root $\frac{1}{2}\mathbf{1}$ sends e_1 to $(3/4, -1/4, \dots, -1/4)$. Reflection of this point in a half root with first sign positive, three further positive signs, and four negative signs gives a vector with four half coordinates and four zeros. Permutations, even sign changes, and lattice translations then give all the equality families described above. This proves Theorem 1.1 for E_8 and D_n^+, subject only to the hyperbolic inequality proved next.

4. **Localization and an aligned two-point inequality**

Theorem 3.1 is a minimization statement over probability measures. Its first variation has a restricted shape; after division by a positive factor, the second derivative is strictly convex. This forces any minimizing measure to have at most two support points. The resulting two-point inequality has two regimes, one controlled by divided differences and the other by variation of the interval length.

4.1. **Reduction to two atoms**

For a probability measure μ on $[0, R]$, put

$$A = \exp\left(4\int \log\cosh p\, d\mu\right), \qquad B = \exp\left(4\int \log\sinh p\, d\mu\right),$$
$$C = \exp\left(4\int \log\cosh(R-p)\, d\mu\right), \quad D = \exp\left(4\int \log\sinh(R-p)\, d\mu\right),$$

where $\exp(-\infty) = 0$, and define

$$F = A + C - B - D, \qquad \mathcal{J}(\mu) = \log F - \log(2\cosh R) - \int \log\cosh(2p - R)\, d\mu.$$

Thus the desired inequality is $\mathcal{J} \geq 0$.

Lemma 4.1. *Every global minimizer of $\mathcal{J}$ is supported on at most two points.*

Proof. The probability measures on $[0, R]$ form a weakly compact space. The quantities A, C and the last integral are continuous. The function $\log\sinh p$, extended by $-\infty$ at zero, is a decreasing limit of continuous truncations, so its integral, and hence B, is upper semicontinuous; the same holds for D. Thus F is lower semicontinuous. Moreover,

$$B/A \leq \tanh^4 R < 1, \qquad D/C \leq \tanh^4 R < 1, \qquad A, C \geq 1,$$

so $F \geq 2(1 - \tanh^4 R) > 0$. Consequently $\mathcal{J}$ is finite and lower semicontinuous and attains a minimum.

Let μ be any minimizer. On $(0, R)$ define

$$g(p) = \frac{4}{F}\big(A \log\cosh p + C \log\cosh(R-p) \\ - B \log\sinh p - D \log\sinh(R-p)\big) - \log\cosh(2p - R),$$

omitting a term when its coefficient B or D is zero. Variation toward a point mass gives

$$\frac{d}{d\varepsilon}\mathcal{J}((1-\varepsilon)\mu + \varepsilon\delta_p)\Big|_{\varepsilon=0+} = g(p) - \int g\, d\mu \geq 0.$$

This formula remains valid when $B = 0$; its logarithmic integral is then $-\infty$ and remains so along the variation for every $\varepsilon < 1$, leaving the exponential identically zero. The same applies to D.

Extend g by endpoint limits. At zero the limit is $+\infty$ if $B > 0$ and finite otherwise, with the analogous statement at R. The variation inequality extends to these limits, so $g \geq \int g\, d\mu$ everywhere. Since this lower bound is the μ-average, μ is supported on the global minimum set of g.

Write $V = 2p - R$. Differentiation gives

$$\frac{g''(p)}{4\operatorname{sech}^2 V} = \frac{1}{F}\left[A\frac{\cosh^2 V}{\cosh^2 p} + C\frac{\cosh^2 V}{\cosh^2(R-p)} + B\frac{\cosh^2 V}{\sinh^2 p} + D\frac{\cosh^2 V}{\sinh^2(R-p)}\right] - 1. \tag{13}$$

Each displayed quotient is strictly convex. For $f(p) = \cosh^2 V/\cosh^2 p$, writing $T = \tanh V$ and $t = \tanh p$ gives

$$f''/f = 6 + 8T^2 + 6t^2 - 16Tt = 8(T-t)^2 + 6 - 2t^2 > 4.$$

For $f(p) = \cosh^2 V/\sinh^2 p$, with $u = \coth p \geq 1$,

$$f''/f = 6 + 8T^2 + 6u^2 - 16Tu \geq 6(u - 4/3)^2 + 10/3 > 0.$$

Reflection treats the remaining quotients. Since $A, C > 0$, the right side of (13) is strictly convex even when B or D vanishes. Its negative set is an interval, possibly empty or touching an endpoint. Therefore g is strictly convex on the two outer pieces and strictly concave on the intervening piece. Each outer piece contributes at most one global minimum, and the middle contributes none except possibly its endpoints. Isolated zeros cannot produce a flat interval, so there are at most two minima. $\square$

It remains to prove the two-point assertion with its equality cases. For $0 \le p \le q \le R$ and $0 \le w \le 1$, write

$$\begin{aligned} F(R,p,q,w) = {} & \cosh^{4w} p \cosh^{4(1-w)} q + \cosh^{4w}(R-p)\cosh^{4(1-w)}(R-q) \\ & - \sinh^{4w} p \sinh^{4(1-w)} q - \sinh^{4w}(R-p)\sinh^{4(1-w)}(R-q), \end{aligned}$$

and normalize it by

$$J(R,p,q,w) = \frac{F(R,p,q,w)}{2\cosh R\,\cosh(2p-R)^{w}\cosh(2q-R)^{1-w}}. \tag{14}$$

Lemma 4.2. *The ratio J is at least one. Equality holds exactly when $w\delta_p + (1-w)\delta_q$ is a point mass, or when $w = 1/2$ and $p + q = R$.*

The point-mass case follows from $\cosh^4 t - \sinh^4 t = \cosh 2t$. Reflection sends (p,q,w) to $(R-q, R-p, 1-w)$ without changing J, so we may assume $0 < w \le 1/2$ and $p < q$. We first treat $p + q \ge R$; Section 5 treats the complementary range and completes the proof.

4.2. **Two arithmetic deficits**

The numerator of J separates into two terms after the geometric mean in its denominator is replaced by the corresponding arithmetic mean. The weights appear in opposite orders in these terms. We compare their deficits from the arithmetic bound, keeping the distance between the two support points fixed; one deficit decreases under translation, while the sum of the two is positive.

Fix $\delta > 0$ and $0 < r < 1$. For $a \ge 0$ put $b = a + \delta$, and define

$$\begin{aligned} D_+(a) &= \cosh^{2-2r} a \cosh^{2+2r} b - \sinh^{2-2r} a \sinh^{2+2r} b, \\ D_-(a) &= \cosh^{2+2r} a \cosh^{2-2r} b - \sinh^{2+2r} a \sinh^{2-2r} b, \\ E_+(a) &= D_+(a) - \tfrac{1-r}{2}\cosh(2a) - \tfrac{1+r}{2}\cosh(2b), \\ E_-(a) &= D_-(a) - \tfrac{1+r}{2}\cosh(2a) - \tfrac{1-r}{2}\cosh(2b). \end{aligned}$$

The two facts we need are

$$E_+(a) + E_-(a) > 0, \qquad E_-'(a) < 0 \quad (a > 0). \tag{15}$$

The derivative assertion extends to strict decrease on $[0,\infty)$ by continuity.

For the first, let $a > 0$ and set

$$X = \cosh a \cosh b, \quad Y = \sinh a \sinh b, \quad A = \log\frac{\cosh b}{\cosh a}, \quad B = \log\frac{\sinh b}{\sinh a}.$$

Then $0 < A < B$ and

$$X \sinh A = Y \sinh B = \tfrac{1}{2}(\sinh^2 b - \sinh^2 a).$$

Expansion gives

$$E_+(a) + E_-(a) = 4\big[X^2 \sinh^2(rA) - Y^2 \sinh^2(rB)\big] > 0,$$

because $t \mapsto \sinh(rt)/\sinh t$ is strictly decreasing; its logarithmic derivative is $r\coth(rt) - \coth t < 0$, as $t\coth t$ is increasing. At $a = 0$, the term $Y\sinh(rB)$ tends to zero while $X\sinh(rA)$ remains positive, so strictness persists.

For the second fact put $C = \cosh a$, $S = \sinh a$, $D = \cosh b$, $T = \sinh b$, and

$$u = (C/D)^2, \qquad v = (S/T)^2, \qquad 0 < v < u < 1.$$

Define

$$\phi(z) = D^2 u^z - T^2 v^z = \frac{(1-v)u^z - (1-u)v^z}{u - v}.$$

Differentiating the deficit, with $b = a + \delta$, gives

$$\tfrac{1}{2}E'_-(a) = (1+r)CS[\phi(r) - 1] + (1-r)DT[\phi(1+r) - 1].$$

As $CS/(DT) = \sqrt{uv}$, its negativity is equivalent to

$$(1+r)\sqrt{uv}\,[\phi(r) - 1] < (1-r)[1 - \phi(1+r)]. \tag{16}$$

For $f_z(t) = t^z$, let $f_z[v, u, 1]$ denote the second divided difference. With $\Omega = (1-u)(1-v)$,

$$\phi(r) - 1 = -\Omega f_r[v, u, 1], \qquad 1 - \phi(1+r) = \Omega f_{1+r}[v, u, 1].$$

The Hermite–Genocchi formula is

$$f_z[v, u, 1] = z(z-1)\int_\Delta \tau^{z-2}\, d\lambda, \qquad \tau = \lambda_1 v + \lambda_2 u + \lambda_3,$$

where $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$, and $d\lambda = d\lambda_1 d\lambda_2$ has total mass $1/2$. Cancelling positive factors reduces (16) to

$$\int_\Delta (\tau - \sqrt{uv})\tau^{r-2}\, d\lambda > 0. \tag{17}$$

The pushforward density under τ is

$$\rho(\tau) = \begin{cases} (\tau - v)/((u-v)(1-v)), & v \leq \tau \leq u, \\ (1-\tau)/((1-u)(1-v)), & u \leq \tau \leq 1. \end{cases}$$

Write $g = \sqrt{uv}$. First take exponent $r = 0$. The contribution of $[u, 1]$ is positive; on $[v, u]$, pair $\tau \in [g, u]$ with $uv/\tau \in [v, g]$. After omitting the common positive denominator, the paired integrands sum to

$$\frac{(\tau - g)(\tau - v)}{\tau^2} - \sqrt{v/u}\frac{(\tau - g)(u - \tau)}{\tau^2} = (1 + \sqrt{v/u})\frac{(\tau - g)^2}{\tau^2} \geq 0.$$

Thus $\int_\Delta (\tau - g)\tau^{-2}\, d\lambda > 0$. For every $r \geq 0$, subtracting g^r times this integral from (17) leaves the nonnegative integrand $(\tau - g)(\tau^r - g^r)\tau^{-2}$. This proves the second fact in (15).

4.3. **The aligned range**

Suppose $0 < w < 1/2$, $p < q$, and $p + q \geq R$. Use the preceding deficits with

$$r = 1 - 2w, \qquad \delta = q - p, \qquad a = p, \qquad a' = R - q.$$

The hypothesis is $a \geq a'$. The difference between $F(R, p, q, w)$ and

$$2\cosh R\,[w\cosh(2p - R) + (1 - w)\cosh(2q - R)]$$

is $E_+(a) + E_-(a')$, which is positive by (15). Weighted arithmetic–geometric mean therefore gives $J > 1$.

When $w = 1/2$, the arithmetic bound is an identity, and the last arithmetic–geometric mean is an equality precisely when $p + q = R$. Reflection gives the corresponding statement for $w \geq 1/2$ and $p + q \leq R$. It remains to treat $0 < w < 1/2$, $p < q$, and $p + q < R$.

5. **Scale comparison and the two-point minimum**

We complete Lemma 4.2 in the range

$$0 < w < 1/2, \qquad p < q, \qquad p + q < R. \tag{18}$$

Instead of differentiating simultaneously in the two support points, hold their positions relative to the midpoint fixed and vary the interval length. A Cauchy representation makes the sign of this derivative explicit. For half of the weights, the limiting configuration already gives the required lower bound; the other weights reduce to an endpoint configuration, where a stationary minimum has a decreasing inward direction.

5.1. **A Cauchy representation**

Fix $s_1 = p - R/2 < s_2 = q - R/2$ and set

$$a = e^{2s_1}, \quad b = e^{2s_2}, \quad r = 4w, \quad z = e^{-R}, \quad x = z^2,$$

$$c = a^{-r} b^{-(4-r)}, \qquad M = e^{4(ws_1 + (1-w)s_2)}, \qquad H = \cosh(2s_1)^w \cosh(2s_2)^{1-w}.$$

Thus $0 < a < b$, $0 < r < 2$, and the last condition in (18) is $ab < 1$. The open admissible range is $R > 2\max(|s_1|, |s_2|)$, equivalently $0 < z < \min(a, 1/b)$.

Let $g(t) = c(t+a)^r (t+b)^{4-r}$, with its branch analytic off $[-b, -a]$ and positive for $t > -a$. Since the exponents sum to four, it continues to real positive values for $t < -b$ as well. If P_4 is its polynomial part at infinity, the upper boundary value on the cut has imaginary part

$$\operatorname{Im} g(-s + i0) = c \sin(\pi r)(s-a)^r (b-s)^{4-r}.$$

The Cauchy formula applied to $g - P_4 = O(1/t)$ therefore gives

$$g(t) = P_4(t) - \frac{c \sin(\pi r)}{\pi} \int_a^b \frac{\rho(s)}{t+s}\, ds, \qquad \rho(s) = (s-a)^r (b-s)^{4-r}. \tag{19}$$

The endpoint powers are integrable. At integer r, g is a polynomial and the integral coefficient vanishes.

Substituting the exponential formulas for the hyperbolic functions in (14) gives

$$J = \frac{M}{16H} T_z g, \qquad T_z g = \frac{g(z) - g(-z) + z^4[g(1/z) - g(-1/z)]}{z + z^3}. \tag{20}$$

If $P_4(t) = \sum_{j=0}^4 p_j t^j$, then $T_z P_4 = 2(p_1 + p_3)$, independent of R. Thus all variation is carried by the resolvent term. For $t \mapsto 1/(t+s)$ its image under T_z is

$$K_s(x) = -\frac{2}{s^2} + \frac{2(s^2-1)x}{s^2(s^2-x)(1-s^2x)},$$

with derivative

$$K_s'(x) = \frac{2(s^2-1)(1-x^2)}{(s^2-x)^2(1-s^2x)^2}. \tag{21}$$

All denominators are positive in the admissible range. Using $dx/dR = -2x$ in (19)–(20) yields

$$\frac{\partial J}{\partial R} = \frac{Mc \sin(\pi r)}{8\pi H}\, x I(x), \qquad I(x) = \int_a^b \rho(s) K_s'(x)\, ds. \tag{22}$$

5.2. **The sign of the kernel integral**

We claim $I(x) < 0$ when $ab < 1$ and $0 < r < 2$. If $b \le 1$, every term in (21) is negative. Otherwise $a < 1 < b$ and $a < 1/b$. Pair $(1/b, 1)$ with $(1, b)$ by reciprocation. Since

$$K'_{1/s}(x) = -s^6 K'_s(x),$$

the paired integrand for $s \in (1, b)$ is

$$K'_s(x)\,[\rho(s) - s^4\rho(1/s)].$$

Put $U = (s-a)/(1-as)$ and $V = (bs-1)/(b-s)$. Then $V > U > 1$, since

$$V - U = \frac{(1-ab)(s^2-1)}{(b-s)(1-as)} > 0,$$

and

$$\frac{\rho(s)}{s^4\rho(1/s)} = \frac{U^r}{V^{4-r}} < 1.$$

The paired contribution is therefore negative, and the leftover interval $(a, 1/b)$ also contributes negatively. This proves the claim.

Equation (22) now gives a complete monotonicity statement at fixed centered support; J decreases strictly with R for $0 < w < 1/4$, increases strictly for $1/4 < w < 1/2$, and is independent of R for $w = 1/4$.

5.3. **The limit at large scale**

Let V be the random variable taking values $2s_1, 2s_2$ with probabilities $w, 1-w$. Expansion at $z = 0$ in (20) gives

$$\lim_{R\to\infty} J = \frac{\mathbb{E}\cosh(V - 2\mathbb{E}V)}{\exp(\mathbb{E}\log\cosh V)}. \tag{23}$$

Indeed, $g'(0) = 4\mathbb{E}e^{-V}$ and the coefficient of t^3 in P_4 is $4c\mathbb{E}e^{V}$, while $M = e^{2\mathbb{E}V}$ and $c = e^{-4\mathbb{E}V}$.

The function $t \mapsto \cosh\sqrt{t}$ is strictly convex on $[0,\infty)$, and $t \mapsto \log\cosh\sqrt{t}$ is strictly concave there. For the latter, the derivative of $\tanh v/v$ is negative because $\sinh v \cosh v > v$. Since $\mathbb{E}(V - 2\mathbb{E}V)^2 = \mathbb{E}V^2$, Jensen's inequality twice gives

$$\mathbb{E}\cosh(V - 2\mathbb{E}V) \ge \cosh\sqrt{\mathbb{E}V^2} \ge \exp(\mathbb{E}\log\cosh V).$$

For distinct atoms, simultaneous equality requires opposite values with equal probabilities. Hence the limit in (23) is strictly greater than one throughout (18). Monotonicity proves $J > 1$ for $0 < w \le 1/4$.

5.4. **The endpoint configuration**

For $1/4 < w < 1/2$, monotonicity reduces the remaining case to the smallest admissible R. Since $ab < 1$, that endpoint has $p = 0$. Fix the weight and set $t = 2(1 - w) \in (1, 3/2)$. The variables

$$u = \tanh^2 R \in (0, 1), \qquad z = \frac{\tanh(R - q)}{\tanh R} \in (0, 1)$$

parametrize $p = 0$, $0 < q < R$. Put

$$a = 1 - u, \quad b = 1 - uz, \quad A = a^{1-t}b^t, \quad B = uz^t, \quad k = \frac{uz(1 - z)}{b}.$$

Direct substitution gives the boundary ratio

$$J_b(u, z; t) = \frac{1 + A^2 - B^2}{2A(1 - k^2)^{t/2}}. \tag{24}$$

Here $0 < k < 1$. With $c = 1 - 2uz + uz^2$ and $d = 1 - uz^2$, the useful factorizations are

$$1 - k^2 = cd/b^2, \qquad 2A(1 - k^2)^{t/2} = 2a^{1-t}(cd)^{t/2}.$$

For example, $\cosh(R - q) = d^{-1/2}$ and $\cosh(2q - R) = c/(\sqrt{a}\, d)$, which give the formulas immediately.

Lemma 5.1. *A stationary point of J_b in $(0, 1)^2$ has a strictly negative inward derivative in p for the original ratio J, with R, q, w fixed.*

Proof. Write

$$L = 1 + A^2 - B^2, \quad \eta = \frac{k^2}{1 - k^2} > 0, \quad e = 1 - 2z + uz^2,$$

$$X = \frac{2A^2}{L} - 1, \qquad Y = \frac{2B^2}{L}.$$

At a stationary point, logarithmic differentiation of (24) gives

$$0 = X\, d\log A - Y\, d\log B + t\eta\, d\log k.$$

The needed derivatives are

$$\begin{aligned} &\partial_u \log A = (t - 1)/a - tz/b, \quad \partial_u \log B = 1/u, \quad \partial_u \log k = 1/(ub), \\ &\partial_z \log A = -tu/b, \qquad\qquad \partial_z \log B = t/z, \quad \partial_z \log k = e/[z(1 - z)b]. \end{aligned}$$

The z equation gives

$$Xuz + Yb = \frac{\eta e}{1 - z}.$$

Substitution into the u equation yields

$$\frac{Xu(t-1)(1-z)}{a} = \eta\left(\frac{e}{1-z} - t\right).$$

Eliminating X, Y between these equations gives

$$X + \frac{a}{u}Y = -\frac{t\eta a z}{u(t-1)(1-z)} < 0.$$

Now differentiate the original J inward in p. Because $4w > 1$, the derivative of the term containing $\sinh^{4w} p$ vanishes at zero, and the other terms give

$$\partial_p \log J\big|_{p=0} = 2w\sqrt{u}\,\frac{A^2 - 1 + (2/u - 1)B^2}{L} = 2w\sqrt{u}\left(X + \frac{a}{u}Y\right) < 0.$$

□

Lemma 5.2. *For fixed $t \in (1, 3/2)$, if the infimum of J_b is below one, it is attained at an interior point of $(0,1)^2$.*

Proof. All boundary limits except the corner $(u, z) = (1, 1)$ follow directly from (24). As $u \to 0$, the ratio tends uniformly to one. As $z \to 1$ with u bounded away from one, it also tends to one. If $z \to 0$ and $u \to u_0 < 1$, it tends to $(A_0 + A_0^{-1})/2 \geq 1$, where $A_0 = (1 - u_0)^{1-t}$. If $u \to 1$ with z bounded away from one, it tends to infinity.

At the remaining corner write $a = 1 - u$, $h = 1 - z$. Then

$$b = a + uh, \quad c = a + uh^2, \quad d = a + u(2h - h^2),$$
$$A = a^{1-t}(a + uh)^t, \qquad B = u(1 - h)^t.$$

If $h/a \to \lambda < \infty$, expansion gives

$$J_b \longrightarrow \frac{1 + t\lambda}{(1 + 2\lambda)^{t/2}} \geq 1,$$

by concavity of the power $t/2 < 1$.

Suppose $h/a \to \infty$. If h^2/a is bounded along a subsequence, then $c \asymp a$, $d \asymp h$, $b \asymp h$, and $1 - B^2 \asymp h$. Dropping A^2 from the numerator gives

$$J_b \geq C(h/a)^{1-t/2} \longrightarrow \infty.$$

If instead $h^2/a \to \infty$, then $1 - k^2 \sim 2h$ and $1 - B^2 \sim 2th$. Arithmetic–geometric mean in the numerator gives

$$J_b \geq \frac{\sqrt{1 - B^2}}{(1 - k^2)^{t/2}} \sim \sqrt{t}\,(2h)^{(1-t)/2} \longrightarrow \infty.$$

Every sequence at the corner has a subsequence of one of these types. Thus the lower limit on the whole boundary is at least one, which proves the assertion by compactness. □

5.5. **Completion of the hyperbolic inequality**

Fix $1/4 < w < 1/2$ and suppose some configuration has $J < 1$. The aligned range is already excluded. In the unaligned range, decreasing R at fixed centered support reduces J to a no-larger endpoint value. Hence any infimum below one over all configurations of this weight equals the boundary infimum. By Lemma 5.2, it is attained at an interior boundary point, where it is stationary in u, z. Lemma 5.1 then gives an inward perturbation with a strictly smaller value, a contradiction.

This proves $J \geq 1$. In the unaligned range the inequality is strict; for $w \leq 1/4$ this follows from the strictly larger limiting value, and for $w > 1/4$ a finite interior value is strictly larger than its boundary value. A nondegenerate boundary equality in the latter range would itself be a global minimum and contradict Lemma 5.1. Together with the aligned equality analysis and reflection, this proves Lemma 4.2.

Finally, Lemma 4.1 places every minimizing measure for $\mathcal{J}$ on at most two points. The two-point inequality therefore proves Theorem 3.1; conversely, every equality measure is a global minimizer and hence has the two-point equality form. This also covers endpoint atoms and vanishing geometric means.

6. **Projection and the dual lattice E_7^***

An orthogonal projection need not preserve a Gaussian minimum, because different fiber cosets generally contribute different weights. The following criterion identifies exactly the situation needed here; on one affine slice, all fiber weights coincide.

Proposition 6.1 (Projection transfer)**.** *Let L be a full lattice in an orthogonal decomposition $H \oplus W$, both subspaces rational for L. Put $N = L \cap H$, $F = L \cap W$, and $M = \pi_H(L)$, and write the finite glue decomposition as*

$$L = \bigcup_{g \in G} (N + a_g) \oplus (F + b_g), \qquad M = \bigcup_{g \in G} (N + a_g),$$

with distinct projected cosets. Suppose $t_0 \in W$ satisfies

$$\Theta_{F+b_g}(\alpha, t_0) = c(\alpha) \quad \textit{for every } g \in G \textit{ and every } \alpha > 0.$$

If L has a universal minimum set C_L meeting $H + t_0$, then M has universal minimum set

$$C_M = \{x \in H : x + t_0 \in C_L\},$$

and its minimum is $\min \Theta_L(\alpha, \cdot)/c(\alpha)$. *If C_L is the deep-hole set of L, then C_M is the deep-hole set of M.*

Proof. Orthogonal factorization gives the exact identity

$$\Theta_L(\alpha, x + t_0) = c(\alpha)\Theta_M(\alpha, x).$$

The asserted minimum and equality set follow because the slice meets C_L.

Equality of the fiber theta functions for all parameters implies equality of their nearest squared distances, say ρ^2, by taking the leading exponential as $\alpha \to \infty$. Therefore

$$\operatorname{dist}(x + t_0, L)^2 = \operatorname{dist}(x, M)^2 + \rho^2.$$

If the slice contains a deep hole of L, this gives $R_M^2 = R_L^2 - \rho^2$ and identifies all projected deep holes. □

Take $L = E_8 = D_8 \cup (D_8 + \frac{1}{2}\mathbf{1})$, $r = \frac{1}{2}\mathbf{1}$, and $H = r^\perp$. Then $\|r\|^2 = 2$, $N = E_8 \cap H$ is E_7, and $F = \mathbb{Z}r$. Unimodularity gives $M = \pi_H(E_8) = E_7^*$; integrality gives the inclusion in the dual, while

$$\operatorname{covol}(M) = \frac{\operatorname{covol}(E_8)}{\|r\|} = \frac{1}{\sqrt{2}} = \operatorname{covol}(E_7^*).$$

The two fiber cosets are $\mathbb{Z}r$ and $(\mathbb{Z} + 1/2)r$. At $t_0 = r/4$ their Gaussian factors agree by reflection, so

$$\Theta_{E_8}(\alpha, x + r/4) = A_{2\alpha}(1/4)\Theta_{E_7^*}(\alpha, x). \tag{25}$$

The deep hole e_1 lies on this slice, since $\langle e_1, r\rangle = 1/2$. Proposition 6.1 therefore yields

Theorem 6.2. *For every $\alpha > 0$,*

$$\min_x \Theta_{E_7^*}(\alpha, x) = \frac{A_\alpha(0)^4 A_\alpha(1/2)^4}{A_{2\alpha}(1/4)}.$$

The equality points are exactly the deep holes of E_7^*, with squared radius $7/8$.

Proof. The minimum follows from (25). The nearest squared fiber distance is $\|r/4\|^2 = 1/8$, so the radius formula is $1 - 1/8 = 7/8$. □

6.1. **The equality classes**

Let P denote projection onto H, and put $x_0 = e_1 - \frac{1}{8}\mathbf{1}$. Since $P(r) = 0$,

$$M = E_7^* = P(D_8) = \left\{ z - \frac{\sum_i z_i}{8}\mathbf{1} : z \in \mathbb{Z}^8,\ \sum_i z_i \text{ even} \right\}.$$

Modulo M, all the deep holes are

$$x_0 + M, \qquad x_0 + y_S + M, \qquad y_S = \tfrac{1}{2}\mathbf{1}_S - \tfrac{1}{4}\mathbf{1}, \quad |S| = 4, \tag{26}$$

where complementary subsets give the same class.

To see completeness, impose on the E_8 equality families the fiber congruence $\langle c, r\rangle \in 1/2 + \mathbb{Z}$. Integer holes project to $x_0 + M$. A four-half-coordinate hole with odd integer-part sum projects to $x_0 + y_S + M$. A quarter-coordinate hole satisfies the congruence precisely when the number of negative quarter signs is divisible by four; subtracting $r/2$ then turns it into one of the preceding two types without changing its projection.

The classes in (26) are distinct except for complements. Indeed $2y_S = P\mathbf{1}_S \in M$, so $y_{S^c} \equiv y_S$. For $S \neq T, T^c$, the coordinate differences of $y_S - y_T$ include both integers and half-integers, excluding membership even in $P(\mathbb{Z}^8)$; the same observation gives $y_S \notin M$. There are therefore $1 + \binom{8}{4}/2 = 36$ classes.

They form one affine-isometry orbit. The vector $\alpha_S = 2y_S$ is an E_7 root of squared norm two, and $\langle x_0, \alpha_S\rangle = \pm 1/2$. Reflection in α_S sends x_0 to $x_0 \pm y_S$, which gives the required class modulo M.

7. Diagonal gluing and the Barnes–Wall lattice

The second transfer uses an inner product of theta profiles. It applies to a pair of lattices of the same rank, rather than to an orthogonal projection.

Proposition 7.1 (Diagonal gluing)**.** *Let $L \subset M$ have finite index N, and define*

$$\Gamma(L, M) = \bigcup_{g \in M/L} (L + g) \times (L + g).$$

If the Gaussian minima of L, M at a fixed parameter are κ, σ, respectively, then

$$\Theta_{\Gamma(L,M)}(\alpha, (x, y)) \geq 2\kappa\sigma - N\kappa^2.$$

Equality holds exactly when x, y both minimize the M-sum and, in each coset g, at least one of $x - g, y - g$ minimizes the L-sum.

Proof. Put $\theta_g(x) = \Theta_L(\alpha, x - g) \geq \kappa$. Then

$$\Theta_{\Gamma(L,M)}(\alpha, (x, y)) = \sum_g \theta_g(x)\theta_g(y), \qquad \sum_g \theta_g(x) = \Theta_M(\alpha, x).$$

Summing $(\theta_g(x) - \kappa)(\theta_g(y) - \kappa) \geq 0$ gives the bound and its equality conditions. □

If both lattices have universal minima and the equality conditions can be met simultaneously, their diagonal gluing inherits universal minima. For Barnes–Wall, the relevant coset profiles take only two values, and their compatibility becomes a character calculation.

7.1. The lattices and the glue

Let $C_m = \mathrm{RM}(1, m)$ be the binary code of affine functions on $\mathbb{F}_2^m$. Thus C_3 has parameters $[8, 4, 4]$ and C_4 has parameters $[16, 5, 8]$. Set

$$L = E_8, \qquad M = \mathbb{Z}^8 + \tfrac{1}{2}C_3, \qquad \Gamma = D_{16} + \tfrac{1}{2}C_4.$$

Equivalently,

$$\Gamma = \left\{ \tfrac{1}{2}z : z \in \mathbb{Z}^{16},\ z \bmod 2 \in C_4, \sum_i z_i \equiv 0 \pmod 4 \right\}.$$

This is the Barnes–Wall lattice scaled to minimal squared norm two and covolume $1/16$; the usual integral normalization is $\mathrm{BW}_{16} = \sqrt{2}\,\Gamma$ [2, Section 7].

We have $L \subset M$ and $[M : L] = 16$. To identify M geometrically, pair the eight coordinates by their last binary coordinate and apply the orthogonal map T with block

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

on each pair. Then $TL = \sqrt{2}M$. For $x \in D_8$, the integral vector $\sqrt{2}Tx$ has equal parities in each pair, and its four pair parities have even sum; these are the words of $\mathrm{RM}(1, 2)$, hence the words of C_3 independent of the last coordinate. For $x \in D_8 + \frac{1}{2}\mathbf{1}$, each pair has opposite parities and the first-entry parities again have even sum, giving the remaining words. The inverse transformation verifies the converse. Thus M is isometric to $E_8/\sqrt{2}$ and has squared covering radius $1/2$.

Every class of $V = M/L$ has a unique expression

$$g = \tfrac{1}{2}u + \varepsilon e_1 \pmod{L}, \qquad u \in C_3/\langle \mathbf{1} \rangle, \quad \varepsilon \in \mathbb{F}_2. \tag{27}$$

The code C_3 is self-dual; it has dimension four, and intersections of supports of affine functions have even size. This also shows that addition in (27) is binary, so $V \cong \mathbb{F}_2^4$.

The Plotkin description $C_4 = \{(u, u + \eta\mathbf{1}) : u \in C_3, \eta \in \mathbb{F}_2\}$ gives

$$\Gamma = \bigcup_{g \in V} (L + g) \times (L + g).$$

Indeed, the half-coordinate words in the two blocks are independently u or its complement, and their integer-part parities are both ε. This is exactly the even total integer parity required by D_{16}, and proves the converse as well.

7.2. **The sharp bound**

Retain $a = A_\alpha(0)$ and $c = A_\alpha(1/2)$, and write $k = A_\alpha(1/4)$, $\ell = B_\alpha(1/4)$. The E_8 theorem and the similarity above give

$$\theta_g(x) \geq K := (ac)^4, \qquad \sum_g \theta_g(x) \geq S := \big(A_{\alpha/2}(0)A_{\alpha/2}(1/2)\big)^4 = 16k^8.$$

The last equality follows from (3) and $A_\gamma(1/2) = 2A_{4\gamma}(1/4)$. At $t = 1/4$, the quartic identities give $k^8 - \ell^8 = K$, so Proposition 7.1 yields

$$\Theta_\Gamma(\alpha, (x, y)) \geq 2KS - 16K^2 = 16(k^{16} - \ell^{16}). \tag{28}$$

This is attained at $y = \frac{1}{4}\mathbf{1}$, $x = y + e_1$. For the representatives (27), direct substitution in (8) gives

$$\theta_g(y) = k^8 + (-1)^\varepsilon \ell^8, \qquad \theta_g(x) = k^8 - (-1)^\varepsilon \ell^8.$$

There are eight cosets of each parity, so their inner product is exactly the bound.

The equality criterion is already independent of α; both x, y must be deep holes of M, and every quotient coset must make at least one of $x - g, y - g$ an L-deep hole. We now identify this set with all the deep holes of Γ.

7.3. **Eligible holes and fifteen characters**

Call an M-deep hole eligible if one of its translates by M is an L-deep hole. The eligible classes modulo M are exactly

$$r/2 + M, \qquad r \in \mathcal{R}(E_8), \tag{29}$$

where the roots are $\pm e_i \pm e_j$ and the vectors $\frac{1}{2}(\pm 1)^8$ with an even number of negative signs.

Here is a direct verification from Theorem 3.3. Integer and all-half L-holes lie in M, so are not M-deep. An L-hole with four half coordinates has support word u_0 of weight four; it lies in M if $u_0 \in C_3$, and otherwise has squared distance $1/2$ from M. In the latter case its binary distance from C_3 is two; self-duality supplies a codeword meeting its support oddly, and complementation makes the intersection size three. Every quarter-coordinate L-hole has squared distance $1/2$ from M, since its nearest integer point attains the lower bound from the half-integer coordinate grid. These are all the eligible holes. A half root divided by two is a quarter vector, and addition of $e_1 \in M$ gives the required odd parity. A divided integer root has two half coordinates; addition of a half-codeword gives the eligible four-half pattern. The distance-two argument also proves the converse, establishing (29).

For a root r, define

$$\chi_r(g) = (-1)^{2\langle r, g\rangle}, \qquad g \in V.$$

The exponent is integral for $g \in M$ and changes by an even integer under $g \mapsto g + L$, so this is a character of V. These are exactly the fifteen nontrivial characters, and two root halves represent the same class modulo M precisely when their characters agree.

For an integer root $r = \pm e_i \pm e_j$, the character on (27) is $(-1)^{u_i + u_j}$. The seven nonzero linear functionals on $C_3/\langle \mathbf{1}\rangle$ all occur. Equality of these functionals is, by self-duality, equality of the corresponding even binary words modulo C_3, which is equality of the root-half classes. For a half root $r = \varepsilon/2$ with even negative-sign word v,

$$\chi_r(u/2 + \eta e_1) = (-1)^{v\cdot u + \eta}.$$

The even binary code modulo C_3 has dimension three, giving eight further characters, all nontrivial on the e_1 factor. Again their equality is equivalent to equality of the root-half classes. The two lists are disjoint and exhaust the nonzero characters of V.

7.4. **Theta and distance profiles**

For every root r and $g \in V$,

$$\theta_g(r/2) = k^8 + \chi_r(g)\ell^8, \qquad \operatorname{dist}(r/2 - g, L)^2 = \begin{cases} 1/2, & \chi_r(g) = 1, \\ 1, & \chi_r(g) = -1. \end{cases}$$

For a half root, the theta formula is the quarter-coordinate sign calculation. For an integer root, subtraction of a half-codeword gives two, four, or six half coordinates, so the alternating products vanish. Four half coordinates give $K = k^8 - \ell^8$; two or six give

$$\tfrac{1}{2}a^2c^2(a^4 + c^4) = k^8 + \ell^8,$$

using $k^4 = ac(a^2 + c^2)/2$ and $\ell^4 = ac(a^2 - c^2)/2$. The four-half case is exactly $u_i + u_j = 1$. Rounding these same coordinate patterns gives the distance formula. Each value occurs in eight cosets, and translating $r/2$ by $s \in M$ replaces $\chi_r(g)$ by $\chi_r(g - s)$.

7.5. **All Barnes–Wall deep holes**

For arbitrary (x, y), choose a coset nearest to x in M. The two covering radii give

$$\operatorname{dist}((x, y), \Gamma)^2 = \min_g \big(\operatorname{dist}(x - g, L)^2 + \operatorname{dist}(y - g, L)^2\big) \le \tfrac{1}{2} + 1 = \tfrac{3}{2}.$$

The complementary profiles of the point attaining (28) attain this distance as well.

If equality holds, the same argument in each block forces both x, y to be M-deep. Choosing a nearest coset for the other block then shows each is eligible. Their classes are therefore among (29). Distinct characters have intersecting affine near-coset hyperplanes, which would give total squared distance one in some coset. Equal characters give depth $3/2$ exactly when the two near-coset hyperplanes are complementary. Thus all deep holes are

$$\bigcup_{r \in \mathcal{R}(E_8)} \{(r/2 + s, r/2 + t) : s, t \in M,\ \chi_r(t - s) = -1\}.$$

This is also exactly the equality set of Proposition 7.1, since at least one profile in each coset must equal K. Every deep hole, and only a deep hole, is therefore a Gaussian minimizer for every positive parameter.

Modulo Γ, subtract the common diagonal translation (s, s) and reduce $t-s$ modulo L. There are fifteen root-half classes and eight quotient classes with $\chi_r(t-s) = -1$, giving 120 deep-hole classes. For $\mathrm{BW}_{16} = \sqrt{2}\Gamma$, replace α by 2α in (28) and multiply hole coordinates by $\sqrt{2}$; the squared covering radius becomes three. This proves the Barnes–Wall part of Theorem 1.1.

8. **A cubic theta identity and E_6**

The E_6 argument rests on a positive identity for three translates of the triangular theta function. Positivity allows the product to be compared with itself at a dilated argument, and a minimum iteration then gives the sharp bound. The underlying three-point Fourier transform is the Eisenstein counterpart of the half-sum and half-difference transformation used earlier; cubic theta identities of this kind have a classical literature, including [3].

Let $\omega = e^{2\pi i/3}$, $\mathcal{E} = \mathbb{Z}[\omega] \subset \mathbb{C}$, and put

$$\theta = 1 + 2\omega = i\sqrt{3}, \qquad c = (1-\omega)/3.$$

The triangular lattice $\mathcal{E}$ has minimal squared norm one, deep-hole classes $\pm c + \mathcal{E}$, and squared covering radius $1/3$, as its regular hexagonal Voronoi cell shows. Also $\theta c = 1 + \omega \in \mathcal{E}$. Fix $\alpha > 0$ and write

$$F(z) = \sum_{m \in \mathcal{E}} e^{-\alpha |m-z|^2}, \qquad a = F(0), \quad b = F(c) = F(-c) > 0.$$

Proposition 8.1. *For every $z \in \mathbb{C}$,*

$$F(z)F(z+c)F(z-c) = \frac{b}{3}\big\{a[F(\theta z + c) + F(\theta z - c)] + bF(\theta z)\big\}. \tag{30}$$

Consequently,

$$F(z)F(z+c)F(z-c) \geq ab^2,$$

with equality exactly when $z \in \mathcal{E} + \mathbb{Z}c$.

8.1. **The Fourier identity**

For $0 < q < 1$ define

$$f_q(u, v) = \sum_{m,n \in \mathbb{Z}} q^{m^2+mn+n^2} e^{2\pi i(mu+nv)},$$

and set $c_0 = (1/3, -1/3)$, $A = f_q(0)$, $B = f_q(c_0)$, and

$$T = \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}.$$

We first prove

$$f_q(z)f_q(z+c_0)f_q(z-c_0) = \frac{B}{3}\big\{A[f_q(Tz+c_0) + f_q(Tz-c_0)] + Bf_q(Tz)\big\}. \quad (31)$$

Use the frequency lattice $\mathcal{E} = \mathbb{Z}[\zeta]$, where $\zeta = 1+\omega = e^{\pi i/3}$ and $|m+n\zeta|^2 = m^2+mn+n^2$. Reduction modulo θ is

$$\operatorname{res}(m+n\zeta) = m - n \pmod 3.$$

The product on the left of (31) is invariant under $z \mapsto z + c_0$, so only total frequencies $k \in \theta\mathcal{E}$ occur. Fix $k = \theta h$. For $n_1 + n_2 + n_3 = k$, put

$$u = \frac{n_1 + \omega n_2 + \omega^2 n_3}{\theta}, \qquad v = \frac{n_1 + \omega^2 n_2 + \omega n_3}{\theta}.$$

This is a bijection onto $u, v \in \mathcal{E}$ with $u + v \equiv -h \pmod\theta$. Its inverse is

$$\begin{aligned} n_1 &= -\frac{h+u+v}{\theta}, \\ n_2 &= -\frac{h+\omega^2 u+\omega v}{\theta}, \\ n_3 &= -\frac{h+\omega u+\omega^2 v}{\theta}. \end{aligned}$$

The one congruence makes all three expressions integral because $\omega \equiv 1 \pmod\theta$. Orthogonality of the three-point Fourier transform gives

$$\sum_j |n_j|^2 = |k|^2/3 + |u|^2 + |v|^2, \qquad n_2 - n_3 = u - v.$$

Hence the translation phase is $\omega^{\operatorname{res}(u-v)}$.

Define the two residue sums

$$a_0 = \sum_{u\in\theta\mathcal{E}} q^{|u|^2}, \qquad b_0 = \sum_{u\in\theta\mathcal{E}+1} q^{|u|^2} = \sum_{u\in\theta\mathcal{E}-1} q^{|u|^2}.$$

For $h \equiv 0 \pmod\theta$, the three residue pairs $(0,0), (1,-1), (-1,1)$ contribute $a_0^2 - b_0^2$. For either nonzero residue of h, their contribution is $b_0^2 - a_0 b_0$. Since $A = a_0 + 2b_0$ and $B = a_0 - b_0$, grouping the total frequencies gives

$$f_q(z)f_q(z+c_0)f_q(z-c_0) = B\big[Af_{q^3}(3z) - b_0 f_q(Tz)\big].$$

Here multiplication by θ on frequencies has matrix T^{T} and $T^2 = -3I$. The index-three Fourier filter is

$$f_{q^3}(3z) = \tfrac{1}{3}\big[f_q(Tz) + f_q(Tz + c_0) + f_q(Tz - c_0)\big].$$

Substitution and $A - 3b_0 = B$ prove (31).

For a physical point $z = u + v\omega$, Poisson summation reads

$$F(z) = \frac{2\pi}{\sqrt{3}\alpha} f_q(u, v), \qquad q = e^{-4\pi^2/(3\alpha)}.$$

The matrix T is multiplication by $-\theta$ in these coordinates; evenness removes the minus sign. Both sides of (31) have degree three in theta values, giving (30). Absolute convergence justifies all regroupings.

8.2. **Minimum iteration and equality**

Put $P(z) = F(z)F(z + c)F(z - c)$. Arithmetic–geometric mean in (30) gives

$$P(z) \geq (ab^2)^{2/3} P(\theta z)^{1/3}.$$

The positive continuous periodic function P has a positive minimum m. At a minimizing point, $m \geq (ab^2)^{2/3} m^{1/3}$, so $m \geq ab^2$; equality is attained at zero.

If $P(z) = ab^2$, the same chain forces $P(\theta z) = ab^2$ and

$$aF(\theta z + c) = aF(\theta z - c) = bF(\theta z).$$

Multiplication gives $F(\theta z) = a$. Every Fourier coefficient of F is positive, so its maximum a occurs exactly at lattice points. Thus $\theta z \in \mathcal{E}$, equivalently $z \in \mathcal{E} + \mathbb{Z}c$. Conversely, these three classes all attain ab^2. This proves Proposition 8.1 without using a separate minimum theorem for F.

8.3. **Polarization and the deep holes of E_6**

Let

$$L_6 = \mathcal{E}^3 + \mathbb{Z}(c, c, c).$$

Then $\sqrt{2}L_6$ is the standard E_6 root lattice. For completeness, a simple-root basis is

$$-\sqrt{2}(c, c, c), \quad \sqrt{2}(1, 0, 0), \quad -\sqrt{2}(1 + \omega, 0, 0),$$
$$\sqrt{2}(0, 1, 0), \quad -\sqrt{2}(0, 1 + \omega, 0), \quad \sqrt{2}(0, 0, 1).$$

Its Gram graph is the E_6 diagram with arms of lengths two, two, and one. The full arms generate two copies of $\mathcal{E}$, and $3c = 1 - \omega$ supplies the remaining direction in the third copy, so the roots generate $\sqrt{2}L_6$.

The three-coset decomposition and Proposition 8.1 give

$$\Theta_{L_6}(\alpha,(z_1,z_2,z_3)) = \sum_{r=0}^{2}\prod_{i=1}^{3} F(z_i - rc) \geq 3\prod_{i=1}^{3} P(z_i)^{1/3} \geq 3ab^2.$$

A permutation of $(0,c,-c)$ attains equality. Conversely, scalar equality places each coordinate in one of the phases $0,c,-c$ modulo $\mathcal{E}$. If their multiplicities are n_0,n_1,n_2, the three summands are $a^{n_r}b^{3-n_r}$. Since $a>b$, equality in arithmetic–geometric mean requires $n_0=n_1=n_2=1$. Hence the complete minimum set is

$$\{\text{permutations of } (0,c,-c)\} + L_6, \tag{32}$$

and the minimum is $3\Theta_{\mathcal{E}}(\alpha,0)\Theta_{\mathcal{E}}(\alpha,c)^2$. There are two classes modulo L_6, interchanged by negation.

The same positive identity identifies all the deep holes. For a fixed z,

$$\lim_{\alpha\to\infty} -\alpha^{-1}\log\Theta_{\mathcal{E}}(\alpha,z) = \operatorname{dist}(z,\mathcal{E})^2;$$

one isolates the finitely many nearest points and bounds the Gaussian tail. Applying this limit to (30), with $d(z)=\operatorname{dist}(z,\mathcal{E})$, gives

$$\sum_{r=0}^{2} d(z+rc)^2 = \min\{1/3 + d(\theta z + c)^2,\ 1/3 + d(\theta z - c)^2,\\ 2/3 + d(\theta z)^2\} \leq 2/3.$$

Equality requires both $\theta z + c$ and $\theta z - c$ to be triangular deep holes. Their two phase conditions intersect only at $\theta z \in \mathcal{E}$, so equality is exactly $z \in \mathcal{E} + \mathbb{Z}c$.

Therefore

$$\operatorname{dist}((z_1,z_2,z_3),L_6)^2 = \min_r \sum_i d(z_i - rc)^2 \leq \tfrac{1}{3}\sum_i\sum_r d(z_i - rc)^2 \leq 2/3.$$

The points in (32) attain $2/3$. Equality forces each coordinate into the three phase classes and then forces the phase counts to be equal, so these are all deep holes. Scaling by $\sqrt{2}$ gives squared radius $4/3$ for E_6; replace α by 2α in its minimum formula and scale the hole coordinates likewise. This proves the remaining case of Theorem 1.1.

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