Ellipsoids are determined by their intrinsic volumes

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Abstract. In every dimension, an ellipsoid is determined up to congruence by its intrinsic volumes. The proof passes to the elementary symmetric functions of the squared semiaxes and extends the intrinsic-volume map to a full coefficient orthant. A Gram representation for an ordered moment kernel makes every principal Jacobian minor positive, so global univalence gives reconstruction, including at repeated semiaxes. The same coefficient map gives bounds from the measured volumes and the optimal uniform Hölder exponent $1/n$ for recovering semiaxes on bounded nondegenerate families.

1. Introduction

The intrinsic volumes of a convex body $K \subset \mathbb{R}^n$ are the coefficients in Steiner's formula

$$\mathrm{vol}_n(K + tB_n) = \sum_{k=0}^{n} \kappa_{n-k} V_k(K) t^{n-k}, \qquad \kappa_j = \mathrm{vol}_j(B_j), \quad \kappa_0 = 1.$$

They include volume, half the surface area, and a multiple of mean width. Gusakova and Zaporozhets conjectured that, within the class of ellipsoids, these quantities determine the body up to a rigid motion [4]. Petrov and Tarasov proved the conjecture in dimension three; Myroshnychenko, Tatarko, and Yaskin subsequently established the corresponding reconstruction theorem for dual volumes in every dimension [3]. We prove the intrinsic-volume conjecture in all dimensions.

Theorem 1.1. *Let $E, F \subset \mathbb{R}^n$ be full-dimensional ellipsoids. If*

$$V_k(E) = V_k(F) \qquad (1 \le k \le n),$$

then E and F are congruent.

The proof uses the one-dimensional integral formula for ellipsoidal intrinsic volumes obtained by Gusakova, Spodarev, and Zaporozhets [2]. Its useful variables are the elementary symmetric functions of the squared semiaxes. In these variables the

formula extends beyond ellipsoids to every polynomial with positive coefficients and fixed constant and leading terms. This extension removes the real-rootedness constraint, leaving a rectangular domain on which a global univalence theorem applies.

The essential point is positivity of the coefficient Jacobian. After a positive diagonal rescaling, its symmetric part splits into moment matrices indexed by the coefficients of the polynomial. Each such matrix has a Gram representation, built from the stochastic ordering of power-tilted measures. The constant coefficient alone supplies strict positivity in every direction. Thus the argument remains nonsingular when semiaxes coincide, although recovering the individual semiaxes from the coefficients necessarily loses regularity.

That distinction leads to a quantitative form of reconstruction. Write

$$d_{\mathrm{ax}}(E,F) = \min_{\pi\in S_n} \max_{1\le i\le n} |a_i - b_{\pi(i)}|$$

for the distance between the unordered semiaxis lists.

Theorem 1.2. *For $0 < a \le b < \infty$ there is a constant $C = C(n,a,b)$ such that ellipsoids whose semiaxes lie in $[a,b]$ satisfy*

$$d_{\mathrm{ax}}(E,F) \le C \left(\max_{1\le k\le n} |V_k(E) - V_k(F)| \right)^{1/n}.$$

If $a < b$, no larger exponent gives a uniform two-point estimate in any neighborhood of a sphere of radius in (a,b). For $n \ge 2$, this remains true when E and F are required to have equal volume.

We first express the intrinsic volumes in coefficient coordinates, then prove the moment-kernel positivity and apply it to the Jacobian. The final section extracts coefficient bounds and the stability theorem from the same map.

2. The coefficient transform

Let $a_1,\dots,a_n > 0$ be the semiaxes of an ellipsoid, put $x_i = a_i^2$, and denote the elementary symmetric functions of $x_1,\dots,x_n$ by $e_0,\dots,e_n$, with $e_0 = 1$. Set

$$Q(t) = \prod_{i=1}^{n}(1 + x_i t^2) = \sum_{j=0}^{n} e_j t^{2j}, \qquad Q_k(t) = \sum_{j<k} e_j t^{2j}.$$

The formula of [2, Theorem 1] reads

$$\frac{V_k(E)}{\kappa_k} = \int_0^\infty \frac{t^{k-1}}{\sqrt{Q(t)}} \sum_{i=1}^{n} \frac{x_i e_{k-1}(x_1,\dots,\widehat{x_i},\dots,x_n)}{1 + x_i t^2}\, dt. \tag{1}$$

The hat denotes an omitted variable. The sum in this integrand can be written using only the head Q_k and the full polynomial

$$\sum_{i=1}^{n} \frac{x_i e_{k-1}(x_1, \ldots, \widehat{x_i}, \ldots, x_n)}{1 + x_i t^2} = \frac{Q_k Q' - Q_k' Q}{2t^{2k-1} Q}. \tag{2}$$

To verify the identity, put $P(s) = \prod_i (1 + x_i s)$ and write the sum on the left as $L_k(s)$ after replacing t^2 by s. The elementary recurrences give

$$L_1 = \frac{P'}{P}, \qquad L_{k+1} = e_k \frac{P'}{P} - \frac{k e_k}{s} + \frac{L_k}{s}.$$

Induction therefore yields

$$s^{k-1} L_k(s) = \left(\sum_{j<k} e_j s^j \right) \frac{P'(s)}{P(s)} - \frac{d}{ds} \left(\sum_{j<k} e_j s^j \right),$$

and $s = t^2$ gives (2).

For a general coefficient vector define

$$F_k(e) = \frac{1}{2} \int_0^\infty t^{-k} \frac{Q_k Q' - Q_k' Q}{Q^{3/2}} \, dt, \qquad 1 \le k < n. \tag{3}$$

Equations (1) and (2) show that

$$F_k(e) = \frac{V_k(E)}{\kappa_k}, \qquad V_n(E) = \kappa_n \sqrt{e_n}. \tag{4}$$

Thus the top intrinsic volume already determines the leading coefficient, while the remaining volumes are the values of the transform with that coefficient fixed.

The advantage of (3) is that it makes sense without real roots. Fix $e_0, e_n > 0$ and allow $e_1, \ldots, e_{n-1} \ge 0$. Expanding the numerator cancels all terms on the same side of the cut at k, leaving

$$F_k(e) = \sum_{\ell < k \le j} (j - \ell) e_\ell e_j \int_0^\infty \frac{t^{2\ell + 2j - k - 1}}{Q(t)^{3/2}} \, dt. \tag{5}$$

Every summand is nonnegative and integrable; its power at zero is at least $k - 1$, and its power at infinity is at most $k - n - 3$. The lower bound $Q(t) \ge e_0 + e_n t^{2n}$ provides locally uniform estimates for coefficient derivatives of every order. These estimates remain valid on a neighborhood of each compact box in the closed orthant, since

$$\frac{\sum_{j=1}^{n-1} t^{2j}}{e_0 + e_n t^{2n}}$$

is bounded and small negative coefficient perturbations consequently preserve a fixed positive lower bound for $Q/(e_0 + e_n t^{2n})$.

We shall prove the following analytic statement, including its boundary and properness assertions.

Theorem 2.1. *For fixed $e_0, e_n > 0$, the map*

$$(e_1, \dots, e_{n-1}) \longmapsto (F_1(e), \dots, F_{n-1}(e))$$

is injective on $[0,\infty)^{n-1}$, is smooth on a neighborhood of each compact coefficient box, and has positive principal Jacobian minors everywhere on the closed orthant. Moreover,

$$F_k(e) > \sqrt{e_k} \qquad (1 \le k < n).$$

It is therefore a proper smooth embedding, as a manifold with corners, onto a closed image.

3. An ordered moment kernel

The positivity needed for the Jacobian is a property of moments of a positive measure. The measure below need not have finite total mass; only the specified moments will enter the argument.

Lemma 3.1. *Let μ be a nonzero positive σ-finite Borel measure on $(0,\infty)$, and write $a_r = \int t^r \, d\mu(t)$ whenever the integral is finite. For $0 < p_1 < \cdots < p_m$, suppose that a_{3p_i} and $a_{2p_j - p_i}$ are finite whenever $i \le j$. Then the symmetric matrix*

$$C_{ij} = \min(p_i, p_j)\, a_{3\min(p_i,p_j)}\, a_{2\max(p_i,p_j)-\min(p_i,p_j)} \tag{6}$$

is positive definite.

Proof. Normalize the power tilts to probability measures,

$$d\nu_i(t) = \frac{t^{3p_i}}{a_{3p_i}}\, d\mu(t), \qquad q_i(s) = \nu_i((0, 1/s)).$$

Increasing the exponent moves mass to the right, so

$$q_1(s) \ge q_2(s) \ge \cdots \ge q_m(s). \tag{7}$$

Indeed, for $p_i < p_j$, the numerator of $\nu_i((0,c)) - \nu_j((0,c))$ is

$$\int_{0<x<c\le y} \left(x^{3p_i} y^{3p_j} - x^{3p_j} y^{3p_i}\right) d\mu(x)\, d\mu(y) \ge 0.$$

The moment assumptions justify both products of integrals.

This ordering makes the mixed moment in (6) a Gram entry. On $(0,\infty)\times(0,1)$ with Lebesgue measure, set

$$f_i(s,u) = s^{p_i-1/2}\mathbf{1}_{\{0<u<q_i(s)\}}.$$

For $i \le j$, Tonelli's theorem and (7) give

$$\begin{aligned} G_{ij} := \langle f_i, f_j\rangle &= \int_0^\infty s^{p_i+p_j-1} q_j(s)\, ds \\ &= \frac{1}{p_i+p_j}\int t^{-p_i-p_j}\, d\nu_j(t) = \frac{a_{2p_j-p_i}}{(p_i+p_j)a_{3p_j}}. \end{aligned} \tag{8}$$

In particular, the functions are square-integrable and G is positive semidefinite with positive diagonal.

One further Gram matrix supplies strict positivity. Put

$$R_{ij} = \frac{\min(p_i,p_j)}{\max(p_i,p_j)} = \frac{1}{p_ip_j}\int_0^{\min(p_i,p_j)} 2r\, dr.$$

Its Gram functions $\sqrt{2r}\,\mathbf{1}_{(0,p_i)}(r)/p_i$ are linearly independent, as their support intervals are strictly nested; hence $R \succ 0$. If $d_i = p_i a_{3p_i}$, then (8) gives

$$\left[\frac{C_{ij}}{d_id_j}\right] = G + G\circ R,$$

where $\circ$ denotes entrywise product. Choose $\lambda > 0$ with $R \succeq \lambda \mathrm{Id}$. The Schur product theorem yields

$$G\circ R \succeq \lambda\,\mathrm{diag}(G_{11},\dots,G_{mm}) \succ 0.$$

Positive diagonal congruence now proves $C \succ 0$. □

4. **Positivity of the coefficient Jacobian**

Fix the endpoint coefficients and write $J_{kj} = \partial F_k/\partial e_j$ for $1 \le k, j < n$. The head and tail of Q give two complementary formulas for its derivatives. Define

$$A_k(t) = \sum_{\ell<k}(k-\ell)e_\ell t^{2\ell}, \qquad B_k(t) = \sum_{\ell>k}(\ell-k)e_\ell t^{2\ell}.$$

Lemma 4.1. *The entries of J are*

$$J_{kj} = \frac{1}{2}\int_0^\infty \frac{t^{2j-k-1}}{Q(t)^{3/2}}\begin{cases} A_k(t), & j \ge k, \\ B_k(t), & j < k \end{cases} dt. \tag{9}$$

On the diagonal, either A_k or B_k gives the same integral.

Proof. When $j \geq k$, the head Q_k is independent of e_j. Differentiate (3) and integrate the total derivative

$$\frac{d}{dt}\left(\frac{t^{2j-k}Q_k}{Q^{3/2}}\right).$$

The remaining numerator is $kQ_k - \frac{1}{2}tQ_k' = A_k$, giving the first line of (9). The boundary expression is $O(t^{2j-k})$ at zero and $O(t^{2j+k-2-3n})$ at infinity, so both boundary terms vanish.

When $j < k$, use the tail $T_k = Q - Q_k$ and write the numerator in (3) as $QT_k' - Q'T_k$. Now T_k is independent of e_j, and integration of the negative total derivative

$$-\frac{d}{dt}\left(\frac{t^{2j-k}T_k}{Q^{3/2}}\right)$$

leaves $\frac{1}{2}tT_k' - kT_k = B_k$. Its boundary expression is $O(t^{2j+k})$ at zero and $O(t^{2j-k-n})$ at infinity, again with vanishing limits. These differentiations use the convergent combined integrand, so no divergent head or tail integral is separated from its cancelling term.

Finally,

$$\int_0^\infty t^{k-1}(A_k - B_k)Q^{-3/2}\,dt = \left[\frac{t^k}{\sqrt{Q}}\right]_0^\infty = 0,$$

which proves the diagonal assertion. □

Introduce the finite positive Mellin moments

$$m_r = \int_0^\infty \frac{t^{r-1}}{Q(t)^{3/2}}\,dt, \qquad 0 < r < 3n,$$

and set $D = \operatorname{diag}(m_3, m_6, \ldots, m_{3(n-1)})$. The exponents $3k$ in this rescaling are chosen to put the two factors in each Jacobian entry into the ordered moment kernel.

Proposition 4.2. *The symmetric part* $\operatorname{sym}(DJ) = (DJ + J^T D)/2$ *is positive definite at every point of the closed coefficient orthant.*

Proof. Use $(A_k + B_k)/2$ on the diagonal of (9). Holding the coefficient vector fixed and grouping by e_ℓ gives

$$\operatorname{sym}(DJ) = \frac{1}{4}\sum_{\ell=0}^{n} e_\ell C^{(\ell)}. \tag{10}$$

The matrix $C^{(\ell)}$ has no entries between the index sets $k < \ell$ and $k > \ell$; its row and column at $k = \ell$, when present, are zero. We identify its two remaining blocks.

On the right, write $k = \ell + p$ and $j = \ell + q$, with $0 < p \le q$. Then

$$C^{(\ell)}_{kj} = p\, m_{3\ell+3p}\, m_{3\ell+2q-p}.$$

This is the kernel of Lemma 3.1 for the measure

$$d\mu(t) = \frac{t^{3\ell}}{tQ(t)^{3/2}}\, dt, \qquad a_r = m_{3\ell+r}.$$

The right block is therefore positive definite whenever nonempty. On the left, reverse the indices and write $k = \ell - p$, $j = \ell - q$, with $0 < p \le q$. The entries become

$$C^{(\ell)}_{kj} = p\, m_{3\ell-3p}\, m_{3\ell-2q+p},$$

the same kernel for the reciprocal pushforward of μ, whose moments are $a_r = m_{3\ell-r}$. Thus the left block is positive definite as well. All moment indices in these formulas lie strictly between 0 and $3n$; the possible infinite mass of the underlying measure at $\ell = 0$ or n causes no difficulty.

Every $C^{(\ell)}$ is consequently positive semidefinite. For $\ell = 0$ its right block contains all indices, so $C^{(0)} \succ 0$; since $e_0 > 0$, (10) is strictly positive definite. □

Proof of injectivity in Theorem 2.1. A real matrix is a *P-matrix* if every nonempty principal minor is positive. Proposition 4.2 implies that J is a P-matrix; for a nonempty index set I, the symmetric part of $D_I J_I$ is positive definite, and so are the symmetric parts of

$$(1-u)\mathrm{Id} + uD_I J_I, \qquad 0 \le u \le 1.$$

These matrices are nonsingular throughout the path, hence their determinants retain the positive sign of the identity. As D_I is positive diagonal, $\det J_I > 0$.

The Gale–Nikaido theorem [1, Theorem 4 and Remark 4.3] states that a differentiable map on a rectangle is injective if its Jacobian is a P-matrix everywhere there. Any two points of $[0,\infty)^{n-1}$ lie in a compact box on a neighborhood of which our transform is smooth. The preceding argument applies on that box, including its boundary, and proves injectivity. Smoothness was established after (5); the coefficient bound and properness will be proved below. □

Proof of Theorem 1.1. The case $n = 1$ is immediate. For $n \ge 2$, equal top intrinsic volumes give equal leading coefficients e_n, while $e_0 = 1$ for both ellipsoids. Equality of the remaining volumes gives equality of $F_1, \dots, F_{n-1}$ by (4), and hence equality of every coefficient by Theorem 2.1. The polynomials

$$\prod_{i=1}^{n}(1 + a_i^2 s), \qquad \prod_{i=1}^{n}(1 + b_i^2 s)$$

are therefore identical. Their roots, with multiplicities, are the negative reciprocal squared semiaxes, so the unordered semiaxis lists agree. □

5. **Bounds and stability of reconstruction**

The coefficient map remains regular at repeated semiaxes. We first give a bound that makes its properness explicit, then separate its smooth inversion from the root-matching step where the Hölder exponent appears.

5.1. **A coefficient box from the intrinsic volumes**

Fix $1 \le k < n$ and put $c = e_k$. Divide the polynomial at the kth term

$$Q(t) = t^{2k}\big(h(t) + c + u(t)\big),$$

where

$$h(t) = \sum_{\ell<k} e_\ell t^{2(\ell-k)}, \qquad u(t) = \sum_{\ell>k} e_\ell t^{2(\ell-k)}.$$

Because both endpoint coefficients are positive, h decreases strictly from ∞ to 0, whereas u increases strictly from 0 to ∞. We may therefore regard h as a positive decreasing function of u. Substitution in (3) cancels the powers of t and gives

$$\begin{aligned} F_k &= \frac{1}{2}\int \frac{h\,du - (c+u)\,dh}{(c+u+h)^{3/2}} \\ &= \sqrt{c} + \frac{1}{2}\int_0^\infty \left(\frac{1}{\sqrt{c+u}} - \frac{1}{\sqrt{c+u+h(u)}}\right) du. \end{aligned} \tag{11}$$

For the second equality, integrate

$$d\left(\frac{c+u}{\sqrt{c+u+h}}\right) - \frac{du}{2\sqrt{c+u+h}}$$

on finite intervals. The lower boundary tends to zero, while the upper boundary differs from $\sqrt{c+u}$ by at most $h(u)/(2\sqrt{c+u})$, which also tends to zero. When $c = 0$, the remaining kernel $u^{-1/2}$ is integrable at zero. The integrand in (11) is strictly positive, proving $F_k > \sqrt{e_k}$.

This completes Theorem 2.1. Indeed, the coefficient bound makes preimages of compact sets bounded; continuity makes them closed, hence compact. Injectivity and the nonsingular Jacobian give a smooth local inverse, including across each boundary face, and properness makes the image closed.

For ellipsoids, AM–GM applied to the $\binom{n}{k}$ products of k squared semiaxes gives a complementary lower bound. Thus the measured volumes determine the box

$$\binom{n}{k}\left(\frac{V_n}{\kappa_n}\right)^{2k/n} \le e_k < \left(\frac{V_k}{\kappa_k}\right)^2 \quad (1 \le k < n), \qquad e_n = \left(\frac{V_n}{\kappa_n}\right)^2.$$

5.2. **Smooth recovery of coefficients**

Allow e_n to vary and define, with $e_0 = 1$,

$$\mathcal{G}(e_1, \dots, e_n) = (\kappa_1 F_1, \dots, \kappa_{n-1} F_{n-1}, \kappa_n \sqrt{e_n}).$$

This is a smooth injective map on $U = (0, \infty)^n$, and

$$\det D\mathcal{G} = \frac{\prod_{k=1}^n \kappa_k}{2\sqrt{e_n}} \det J > 0.$$

For $n = 1$ the same assertions follow from the last coordinate alone. Hence its image Ω is open and its inverse is smooth.

If all semiaxes lie in $[a, b]$, their coefficient vectors lie in the compact box

$$K = \prod_{j=1}^n \left[\binom{n}{j} a^{2j}, \binom{n}{j} b^{2j}\right] \Subset U.$$

Choose $\rho > 0$ such that the closed ρ-neighborhood of $\mathcal{G}(K)$ lies in Ω. On that compact neighborhood $D\mathcal{G}^{-1}$ is bounded. For image points at distance at most ρ, integrate the derivative along their segment; for more distant pairs use $\operatorname{diam}(K)/\rho$. In the maximum norm this gives

$$\|e - f\|_\infty \le L \|\mathcal{G}(e) - \mathcal{G}(f)\|_\infty, \qquad e, f \in K, \tag{12}$$

with L depending only on n, a, b. This argument requires no convexity of Ω.

5.3. **Root matching and the optimal exponent**

We record the root-matching estimate with multiplicities. Let P, Q be monic polynomials of degree n, with all roots in $[A, B]$, where $0 < A \le B$, and let Δ be the maximum distance between their coefficients. Then

$$d_{\text{roots}}(P, Q) \le C_{n,A,B} \Delta^{1/n}. \tag{13}$$

To see this, put $H = n(1 + B)^{n-1}$ and first assume $0 < \Delta < 1/(2H)$. Take $r = (2H\Delta)^{1/n} < 1$ and choose $r' > r$, arbitrarily close to r, avoiding the finitely many

tangencies among the circles of radius r' centered at the roots of P. On the boundary of their disk union,

$$|P(z)| \geq (r')^n > 2H\Delta, \qquad |P(z) - Q(z)| \leq H\Delta,$$

since $|z| \leq B + 1$. Each component has a piecewise smooth Jordan boundary because the centers are real. Rouché's theorem gives equal root counts, with multiplicities, in each component; its diameter is at most $2nr'$, so matching within components and letting $r' \downarrow r$ proves the estimate. For $\Delta \geq 1/(2H)$ the trivial distance bound $B - A$ suffices.

Apply (13) to the monic polynomials with roots a_i^2 and b_i^2, whose coefficient distance is $\|e - f\|_\infty$. Taking square roots is $1/(2a)$-Lipschitz on $[a^2, b^2]$, so (12) proves the estimate in Theorem 1.2.

For optimality, choose a monic degree-n polynomial f with distinct real roots and a sufficiently small fixed $c \neq 0$ such that $g = f - c$ also has distinct real roots. Such a choice follows by preserving sign changes in disjoint intervals around the roots of f. Let their root lists be u_i and v_i, and fix $x_0 \in (a^2, b^2)$. For small $\varepsilon > 0$, take squared semiaxes

$$x_0 + \varepsilon u_i, \qquad x_0 + \varepsilon v_i.$$

Their monic root polynomials are

$$\varepsilon^n f\left(\frac{z - x_0}{\varepsilon}\right), \qquad \varepsilon^n g\left(\frac{z - x_0}{\varepsilon}\right),$$

which differ only in their constant term, by $c\varepsilon^n$. Bounded derivatives of $\mathcal{G}$ near the spherical coefficient vector therefore make the intrinsic-volume difference $O(\varepsilon^n)$, whereas the semiaxis matching distance is bounded below by a positive multiple of ε. No exponent greater than $1/n$ can hold uniformly in any neighborhood of that sphere.

When $n \geq 2$, rescale both ellipsoids to volume $\kappa_n x_0^{n/2}$. Their scale factors tend to one and differ by $O(\varepsilon^n)$, so the semiaxis distance remains of order ε, while homogeneity keeps every intrinsic-volume difference $O(\varepsilon^n)$. This proves the fixed-volume assertion and completes Theorem 1.2.

References

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