Face Numbers of Cubical Polytopes

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Abstract. Every cubical d-polytope with n vertices and e edges satisfies $2e \geq (d+1)n - 2^d$ for $d \geq 3$, proving Jockusch's 1993 lower bound conjecture. The resulting lower bounds for all face numbers are attained by stacked cubical polytopes. When $d \geq 4$, the right-hand side increases by $(d-1)$ times the imbalance between the graph's two color classes. The proof uses a generic linear system on the graph for which infinitesimal rigidity of the simplicial vertex figures makes one facet determine every solution throughout the boundary. The same argument gives the strengthened bound for cubical PL spheres.

1. Introduction

A convex d-polytope is *cubical* if every facet is combinatorially a $(d-1)$-cube. Starting with a cube and successively attaching cubes along facets gives the stacked cubical polytopes, whose face numbers satisfy linear relations in the number of vertices. Jockusch [4] conjectured that these relations give lower bounds for every cubical polytope. Klee [6] proved the edge bound for cubical 4-spheres in 2011, including the boundaries of cubical 5-polytopes. We establish the bounds in every dimension and strengthen them by a term measuring the imbalance in the graph's bipartition.

Write $f_i(P)$ for the number of i-faces. When $d \geq 3$, the graph of P is bipartite; its boundary sphere has zero first homology over $\mathbb{F}_2$, so every edge cycle is a sum of square boundaries and has even length. Let $B \sqcup W$ be its two color classes.

Theorem 1. *Every cubical convex d-polytope, $d \geq 3$, satisfies*

$$f_1(P) \geq \frac{d+1}{2} f_0(P) - 2^{d-1}.$$

For $d \geq 4$, the stronger inequality

$$f_1(P) \geq \frac{d+1}{2} f_0(P) - 2^{d-1} + \frac{d-1}{2}\big||B| - |W|\big|$$

holds. The first bound is attained by cubes and stacked cubical polytopes.

The edge inequality implies all the conjectured face-number bounds by the cubical McMullen–Perles–Walkup reduction; these are stated in Corollary 5. The proof of

the edge inequality uses the normalized $(1, d)$ system of bipartite rigidity [5], with parameters independent of the polytope's coordinates. Its solution space injects into the solution space on a single facet, whose dimension is 2^{d-1}, giving the edge bound by counting unknowns and equations. The imbalance term comes from the same count; exchanging the two colors gives a second inequality, and their maximum retains the difference between the color-class sizes.

The injection is a local-to-global consequence of rigidity. At each vertex, infinitesimal rigidity of the simplicial vertex figure forces vanishing on one incident facet to propagate to every neighbor. Connectedness of the facet adjacency graph then propagates vanishing through the whole boundary. Convexity enters only through vertex-figure rigidity; generic rigidity of simplicial PL spheres gives the same strengthened bound for cubical PL spheres.

2. **Local propagation**

Assume $d \geq 4$. Choose vectors $p_w \in \mathbb{R}^d$, $w \in W$, and consider the linear system

$$p_w \cdot u_b + s_w = 0 \qquad (bw \in E(P)), \tag{1}$$

with unknowns $u_b \in \mathbb{R}^d$ and $s_w \in \mathbb{R}$. Denote its solution space by $M(P, p)$.

The unequal numbers of unknowns at the two colors are deliberate. If one facet determines a solution, the number of equations must be at least $d|B| + |W|$ minus the dimension of the solution space on that facet. To prove such determination, we must transfer information between black vertices. Subtracting the two equations of a square at the white vertex opposite w eliminates the scalar unknown and gives an equation for the difference of two black vectors. With a suitable choice of the parameters, these are the rigidity equations of the vertex figure at w.

By the Dehn–Whiteley rigidity theorem, the vertices and edges of every convex simplicial k-polytope, $k \geq 3$, form an infinitesimally rigid framework in $\mathbb{R}^k$ [7, Theorems 3.1 and 8.6]. Thus any motion of its vertices has the form $v_a = t + Aq_a$, with $A^T = -A$. If the motion vanishes at the k vertices of one facet, then A annihilates their $(k-1)$-dimensional space of differences. Its rank is at most one and, being even, is zero. The whole motion consequently vanishes.

Lemma 2. *For generic p, the following holds for every white vertex w and every facet F containing it. If a solution of (1) has $u_b = 0$ at the neighbors of w in F, then $u_b = 0$ at every neighbor of w.*

Proof. Fix w and F. The vertex figure L at w is a simplicial $(d-1)$-polytope. Its vertices are indexed by the black neighbors of w, and the $d-1$ neighbors in F form a facet T of L. An edge ab of L corresponds to a square of P with cyclic order

$$w, a, z_{ab}, b, \qquad z_{ab} \in W.$$

Different edges have different opposite vertices z_{ab}. Otherwise two squares would intersect in a face containing their common opposite vertices w, z_{ab}, although no proper face of a square contains such a pair. The squares, and hence the edges of L, would coincide.

After pinning $u_b = 0$ on T, an edge equation at w gives $s_w = 0$. Together with the differences of the two equations at each opposite square vertex z_{ab}, the equations at w therefore imply

$$p_w \cdot u_b = 0 \quad (b \in V(L)), \qquad p_{z_{ab}} \cdot (u_a - u_b) = 0 \quad (ab \in E(L)).$$

To show that this pinned system generically has zero kernel, it suffices to exhibit one specialization of full column rank. Choose the vertex coordinates $q_b \in \mathbb{R}^{d-1}$ of L and specialize

$$p_w = e_d, \qquad p_{z_{ab}} = (q_a - q_b, 0).$$

These assignments are consistent because the labels z_{ab} are distinct and different from w. The first equations put all u_b in $\mathbb{R}^{d-1} \times \{0\}$; the second become the infinitesimal rigidity equations of L. The pinned-facet consequence of the rigidity theorem forces every u_b to vanish.

A full-column minor of this local matrix is therefore a nonzero polynomial in the coordinates of the p_w. Taking the product of one such minor for each of the finitely many pairs (w, F) gives a nonempty polynomial open set on which all the assertions hold simultaneously. The individual specializations need not agree, since it is their nonvanishing conditions that are being intersected. □

We also choose p so that the $d-1$ vectors indexed by the white neighbors of any black vertex within any facet are independent. These are further nonempty polynomial open conditions, compatible with the lemma.

3. **Determination by one facet**

For a facet F, let $M(F, p)$ be the solution space of the equations restricted to its vertices and edges.

Proposition 3. *For generic p and every facet F, restriction gives an injection*

$$M(P,p) \longrightarrow M(F,p), \qquad \dim M(F,p) = 2^{d-1}.$$

Proof. Suppose a solution vanishes on F. Let F' be an adjacent facet and $R = F \cap F'$ their common ridge. At every white vertex of R, all black neighbors in F have zero vectors. Lemma 2 makes all its black neighbors in P zero. Within the cube F', every vertex outside the facet R is adjacent to its matched vertex in R. Thus all black vectors on F' vanish, and its edge equations then make all white scalars vanish. The facet adjacency graph is connected, so this propagates through the boundary and proves injectivity.

Put $s = 2^{d-2}$. The facet has s vertices of each color and $(d-1)s$ edges, giving $(d+1)s$ scalar unknowns. Its edge equations are independent. Indeed, a linear relation among their rows, restricted to the vector columns of a black vertex b, reads

$$\sum_{w:\, bw \in E(F)} \lambda_{bw} p_w = 0.$$

The $d-1$ vectors are independent by our choice of p, so every coefficient at b is zero. Repeating this at all black vertices proves row independence. Hence

$$\dim M(F,p) = (d+1)s - (d-1)s = 2^{d-1}. \qquad \square$$

Proof of Theorem 1. Write $e = f_1(P)$. System (1) has $d|B| + |W|$ scalar unknowns and e equations. Proposition 3 gives

$$d|B| + |W| - e \le \dim M(P,p) \le 2^{d-1}.$$

Interchanging B and W gives the second asymmetric bound. Their maximum is

$$\max\{d|B| + |W|,\ |B| + d|W|\} = \frac{d+1}{2} f_0(P) + \frac{d-1}{2}\big||B| - |W|\big|.$$

This proves the stated inequality for $d \ge 4$. For $d = 3$, quadrilateral incidence and Euler's formula give $2f_1 = 4f_2$ and $f_0 - f_1 + f_2 = 2$, hence $f_1 = 2f_0 - 4$. $\square$

Corollary 4. *The strengthened edge bound in Theorem 1 holds for every finite cubical PL $(d-1)$-sphere, $d \ge 4$, whose cells intersect in common faces.*

Proof. The link of each vertex is a simplicial $(d-2)$-sphere. By Fogelsanger's rigidity theorem [3, Theorem 3.14], its graph has an infinitesimally rigid generic placement in $\mathbb{R}^{d-1}$. Use this placement for the coordinates q_b in Lemma 2; its facet vertices are affinely independent, so the same pinning argument applies. The common-face condition again makes the opposite square vertices distinct. The graph is bipartite

by the homology argument above, and the facet adjacency graph is connected. The rest of the proof is unchanged. □

4. **The remaining face numbers**

The edge bound controls the higher face numbers by summing the simplicial lower bounds over the vertex figures. This is the cubical McMullen–Perles–Walkup reduction observed by Jockusch and recalled in [1].

Corollary 5. *Let P be a cubical d-polytope, $d \geq 3$, with n vertices. Then*

$$f_i(P) \geq \frac{\binom{d}{i} + \binom{d-1}{i-1}}{2^i}\, n - 2^{d-i}\binom{d-1}{i-1} \qquad (1 \leq i \leq d-2),$$

$$f_{d-1}(P) \geq \frac{d-1}{2^{d-2}}\, n - 2(d-2).$$

All these bounds are attained by stacked cubical polytopes.

Proof. For $d \geq 4$, apply the simplicial lower bound theorem [2] to the $(d-1)$-dimensional vertex figures L_v and sum over vertices. Since an i-face has 2^i vertices, this gives, for $1 \leq i \leq d-2$,

$$2^i f_i(P) = \sum_v f_{i-1}(L_v) \geq 2\binom{d-1}{i-1} f_1(P) - (i-1)\binom{d}{i} n.$$

Substitute Theorem 1 and use Pascal's identity to obtain the first bound. For facets the corresponding simplicial bound is

$$f_{d-2}(L_v) \geq (d-2)\deg(v) - d(d-3).$$

Its sum gives $2^{d-1} f_{d-1}(P) \geq (2d-2)n - (d-2)2^d$. The dimension-three case follows from $f_1 = 2n-4$ and $f_2 = n-2$.

A d-cube attains every bound. Attaching another cube along a facet adds 2^{d-1} vertices, adds $2^{d-1-i}\big(\binom{d}{i} + \binom{d-1}{i-1}\big)$ i-faces for $1 \leq i \leq d-2$, and replaces one facet by $2d-1$ others. These are exactly the increments in the displayed lower bounds. □

For $d \geq 4$, the same substitution adds $2^{-i}(d-1)\binom{d-1}{i-1}\Delta$ to the i-face bound and $2^{1-d}(d-1)(d-2)\Delta$ to the facet bound, where $\Delta = \big||B| - |W|\big|$.

References

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