A Counterexample to Smilansky's Conjecture

Caius Folkerts

Abstract. We construct a rational cubical four-polytope with 144 vertices and 130 facets whose only parallel redrawings are homotheties and translations; in particular, it is Minkowski indecomposable. This disproves Smilansky's conjecture that every Minkowski-indecomposable polytope has a triangular two-face. Three copies of a 32-vertex polytope are joined around a quadrilateral, reducing the deformation space to dimension at most two. A final gluing, with a cyclic permutation of the coordinates of a cube, leaves only the homothety line. The construction makes the obstruction explicit; the deformations of the individual cubical facets are globally incompatible.

1. Introduction

Minkowski decomposability is governed by the compatibility of edge directions. If a polytope is a sum $P = A + B$, its vertices inherit positions in each summand, and the difference of those positions along an edge is parallel to that edge. Conversely, the space of such parallel redrawings records the infinitesimal freedom to change the summands [1]. The resulting local rigidity criteria leave open how much freedom in the facets can survive when they are assembled.

A combinatorial three-cube has a three-dimensional space of parallel redrawings modulo translations. In particular, each cubical facet admits nontrivial deformations. The following construction shows that these deformations can be globally incompatible.

Theorem 1. *There is a rational Minkowski indecomposable four-polytope K all of whose facets are combinatorial cubes. Its face vector is*

$$f(K) = (144, 404, 390, 130).$$

Every parallel redrawing of its edge graph is a homothety and a translation.

The proof separates the geometry of gluing from the choice of coordinates. Its essential ingredient is a cubical polytope with two adjacent facets that impose different restrictions on its redrawings; one determines the whole polytope, whereas the other receives at most a two-dimensional family of redrawings. Joining three copies cyclically

forces both properties onto the same interface. A further gluing along a cube then intersects two deformation spaces whose only common direction is uniform scaling.

Section 3 proves this gluing principle, and Section 4 constructs the required 32-vertex polytope. All convexity arguments are geometric; the coordinates exhibit one block with the stated incidences and seven specified noncoplanar configurations.

2. **Parallel redrawings and projective changes**

Let P have vertices $p_v \in \mathbb{R}^d$. A *parallel redrawing* assigns a vector $h_v \in \mathbb{R}^d$ to each vertex, subject to

$$h_w - h_v = \lambda_{vw}(p_w - p_v) \qquad (vw \in E(P)), \tag{1}$$

where the edge scalars are arbitrary real numbers. Write $\mathcal{D}(P)$ for this vector space modulo translations and $D(P) = \dim \mathcal{D}(P)$. The homothety $h_v = p_v$ gives a distinguished line in $\mathcal{D}(P)$.

If J is a face, or an interface whose edges remain on the boundary after gluing, we say that J *determines* P when restriction $\mathcal{D}(P) \to \mathcal{D}(J)$ is injective. Equivalently, a redrawing that vanishes on J vanishes everywhere.

Lemma 2. *If $D(P) = 1$, every Minkowski summand of P is a translate of a nonnegative scalar multiple of P.*

Proof. Suppose $P = A + B$. The unique decomposition $p_v = a_v + b_v$ of an exposed vertex assigns a vertex a_v of A to each vertex of P. Along an edge, the exposed faces of the summands are parallel segments or points, so

$$a_w - a_v = \lambda_{vw}(p_w - p_v), \qquad 0 \le \lambda_{vw} \le 1.$$

The hypothesis gives $a_v = \alpha p_v + a$ with one $\alpha \in [0, 1]$. Every vertex of A occurs among the a_v, as follows by choosing a generic direction in its normal cone. Hence $A = \alpha P + a$ and $B = (1 - \alpha)P - a$. □

Lemma 3. (i) *A four-cycle whose vertices are affinely independent has only homothetic redrawings, modulo translations.*

(ii) *If a redrawing vanishes at two neighbors of a vertex of a convex polytope, it vanishes at that vertex.*

(iii) *For a combinatorial three-cube J, one has $D(J) = 3$. Restriction to any quadrilateral facet T is surjective with one-dimensional kernel, and $D(T) = 2$.*

Proof. The four directed edge vectors of an affinely independent four-cycle have just one linear dependence, their sum being zero. Summing (1) around the cycle forces the four scalars to agree. For the second assertion, the remaining value lies on two distinct edge-direction lines; these cannot coincide at an extreme point, so their intersection is zero.

For the third assertion, vary the offsets of the six fixed facet planes. The three incident planes determine each vertex displacement, and adjacent displacements differ in the common edge direction. Conversely, the normal component of a redrawing is constant along a facet, so every redrawing arises this way. Six independent offsets, modulo three translations, give $D(J) = 3$. On a chosen facet, the four neighboring offsets prescribe its edge offsets freely; restriction is therefore onto and its kernel has dimension $3 - 2 = 1$. □

We use an explicit form of projective invariance to keep track of restrictions under gluing. The invariance of decomposability itself is due to Kallay [2].

Lemma 4. *Under a projectivity $T(p) = (Ap+b)/(c\cdot p+\delta)$ with positive denominator on P, the number $D(P)$, every face-restriction rank, and the property that a face determines P are unchanged.*

Proof. The pointwise linear transformation

$$h'(T(p)) = Ah(p) - (c \cdot h(p))T(p) \tag{2}$$

is an isomorphism of full redrawing spaces. Indeed, an edge scalar λ becomes $(c \cdot p + \delta)\lambda - c \cdot h(p)$, and the inverse projectivity gives the inverse map. This transformation commutes with restriction and preserves vanishing on a face. It also preserves translations and homotheties together. Dimensions of the full spaces and restriction images are unchanged; subtracting the translation dimension gives the assertions. □

3. A gluing principle for cubical polytopes

The following proposition isolates the properties needed of the auxiliary polytope. We prove it in Section 4, after showing how these properties give the desired construction.

Proposition 5. *There is a rational cubical four-polytope P with*

$$f(P) = (32, 80, 72, 24)$$

and facets F, H, G such that

(i) *$F \cap H = Q$ is a quadrilateral, and G is disjoint from $F \cup H$;*

(ii) *H and G each determine P;*

(iii) *the image of $\mathcal{D}(P) \to \mathcal{D}(F)$ has dimension at most two;*

(iv) *a projectivity with positive denominator on P makes F and H affinely equivalent by a map fixing Q pointwise;*

(v) *G is projectively equivalent to a parallelepiped.*

3.1. **Three copies around a quadrilateral**

Apply the normalization in Proposition 5(iv), choose $w \in \operatorname{relint} Q$, and translate w to the origin. Let L be the direction plane of Q. Since the affine correspondence $F \to H$ fixes L pointwise, complementary transverse vectors u, v can be chosen so that corresponding outer vertices have the form

$$r_j + z_j u, \qquad r_j + z_j v, \qquad r_j \in L, \quad z_j > 0. \tag{3}$$

For this splitting, choose a transverse vector in $\operatorname{lin} F$, take its image in $\operatorname{lin} H$, and complement L by their span. The whole polytope lies in the wedge between its supporting facets F, H.

In a new transverse plane take $u_0 = (1, 0)$, $u_1 = (0, 1)$ and $u_2 = (-1, -1)$. For $i \in \mathbb{Z}/3\mathbb{Z}$, make an affine copy P_i that fixes L and sends (u, v) to (u_i, u_{i+1}). Formula (3) makes the H facet of P_i coincide with the F facet of P_{i+1}. The three interfaces are cubes, all meeting in Q.

The pieces occupy three sectors covering the transverse plane. Their union need not yet be convex; a compatible compression makes it so without altering their common interfaces.

Lemma 6. *The pieces admit projective changes that agree on common interfaces, fix Q pointwise, and make their union a convex polytope R. Its facets are exactly their noninterface facets, with the original vertex sets.*

Proof. For transverse coordinates (s, t), put

$$\ell_0 = s + t, \qquad \ell_1 = -2s + t, \qquad \ell_2 = s - 2t.$$

Each form is one on the two bounding rays of its sector, and $\max_i \ell_i$ equals ℓ_j in sector j. If $n(p)$ is the transverse component, apply to P_i

$$S_i(p) = \frac{p}{1 + M\ell_i(n(p))},$$

with $M > 0$ large. The denominators are positive and the formulas agree on the interfaces.

A noninterface facet K of P_i does not contain w, so normalize its inequality as $f_K(p) \le 1$, with f_K linear. After compression it becomes $f_K(p) + M\ell_i(n(p)) \le 1$. A vertex p of another original piece P_j satisfies this transformed inequality exactly when

$$f_K(p) - 1 \le M(\ell_j - \ell_i)(n(p)). \tag{4}$$

The coefficient on the right is nonnegative. If it is zero, p lies on the common interface, where the inequality already holds; otherwise it is positive and sufficiently large M makes the inequality strict. One value works for all vertex–facet pairs.

The intersection of these noninterface halfspaces is the union of the compressed pieces; within each sector, its own piece inequalities and the sector inequalities define that piece. As the sectors cover the plane, the union is convex. Strictness away from interfaces prevents any facet merger or additional vertex on a retained facet. Every old edge and vertex belongs to a noninterface facet, so survives. Although the relative interior of Q is now internal, its edges and vertices remain on the boundary. □

Lemma 7. *One has $1 \le D(R) \le 2$. Every interface and every surviving copy of G determines R.*

Proof. If a redrawing vanishes on one interface, it vanishes on the adjacent piece in which that interface is H. It then vanishes on the next interface, and propagation around the cycle kills all three pieces. Thus restriction to an interface is injective. In its other adjacent piece the same interface is F, where the restriction image has dimension at most two. Lemma 4 therefore gives $D(R) \le 2$, while the homothety line gives the lower bound. A redrawing zero on a copy of G first vanishes on its whole piece, then on the other pieces by the same propagation. □

The pairwise intersections are cubes and the triple intersection is Q, whose edges survive. Hence

$$f_0(R) = 3 \cdot 32 - 3 \cdot 8 + 4 = 76, \qquad f_1(R) = 3 \cdot 80 - 3 \cdot 12 + 4 = 208.$$

There are $3(24 - 2) = 66$ facets and therefore $f_2(R) = 3f_3(R) = 198$.

3.2. **Removing the last deformation**

Choose a surviving copy of G. We can make it an affine unit cube by a projectivity valid on all of R. To check the denominator, choose the desired map on G, with

positive affine denominator l there, and first compress towards $w \in \operatorname{relint} G$ by

$$p \longmapsto w + \frac{p - w}{1 + sz(p)},$$

where $z \geq 0$ vanishes on G. The denominator then becomes

$$\frac{l(p) + sz(p)l(w)}{1 + sz(p)},$$

which is positive on every vertex for sufficiently large s. We may therefore assume $G = [0,1]^3 \times \{0\}$.

The three edge-direction scale factors identify $\mathcal{D}(G)$ with $\mathbb{R}^3$, with homothety line $\mathbb{R}(1,1,1)$. The restriction image W of $\mathcal{D}(R)$ contains this line and has dimension at most two. If σ cyclically permutes the cube coordinates, its action on $\mathbb{R}^3/\mathbb{R}(1,1,1)$ has no invariant real line; an eigenvalue on such a line would be real with cube one, hence one, whereas the quotient has no nonzero fixed vector. Consequently

$$W \cap \sigma W = \mathbb{R}(1,1,1). \tag{5}$$

Place a second copy of R on the other side of G, using σ on G and reversing the transverse coordinate. Compress the two pieces towards an interior point of G. The halfspace argument of Lemma 6, with two sectors, makes their union convex and preserves all noninterface facets; call it K.

These compressions preserve the restriction spaces on G. Indeed, anchor a redrawing at one vertex of G. Its values on G lie in the direction space of G, where the transverse functional in (2) vanishes. The two restriction images are therefore exactly W and σW.

A redrawing of K restricts to both copies of R, so its common restriction to G is homothetic by (5). After subtracting this homothety and a translation, it vanishes on G and hence on both copies, because G determines each one. Thus $D(K) = 1$, and Lemma 2 proves indecomposability. Finally,

$$f_0(K) = 2 \cdot 76 - 8 = 144, \qquad f_1(K) = 2 \cdot 208 - 12 = 404,$$
$$f_2(K) = 2 \cdot 198 - 6 = 390, \qquad f_3(K) = 2 \cdot 66 - 2 = 130.$$

All changes of basis and projectivities may be chosen rational. The compression parameters may be integers exceeding the finitely many rational bounds in (4), so K is rational. This proves Theorem 1 once Proposition 5 is established.

4. **The 32-vertex block**

Work in homogeneous coordinates (x, y, z, t, w), with $w = 1$ as affine chart, and put

$$A_0 = x, \quad A_1 = t, \quad A_2 = w - x - \tfrac{1}{2}t, \quad A_3 = 4w - x - 4t.$$

The rows of the following table define eight further linear forms

	x	y	z	t	w
U_0	1	0	0	$-1/2$	0
U_1	0	0	0	0	0
U_2	$-7/4$	0	0	$-4/3$	$7/4$
U_3	$-73/92$	0	0	$-361/92$	$315/92$
V_0	0	1	0	$1/2$	0
V_1	0	0	$-1/2$	$1/2$	$1/2$
V_2	0	-2	0	$89/102$	2
V_3	0	0	1	$1/2$	0

Set $C = \frac{3}{4}t - \frac{1}{4}w$, $W_j = V_j - \frac{115}{258}C$, and

$$B_0 = \frac{9W_0 + W_1}{10}, \quad B_1 = W_1, \quad B_2 = W_2, \quad B_3 = 14569W_3 + 459W_2.$$

Writing $M_{ij} = U_i + V_j$, define

$$P = \{(x, y, z, t) : A_i \geq 0,\ B_j \geq 0,\ M_{ij} \geq 0 \text{ for } 0 \leq i, j \leq 3\}. \tag{6}$$

4.1. **Vertices and supporting facets**

Label a vertex by five bits,

$$v = a + 2b + 4c + 8d + 16r, \qquad a, b, c, d, r \in \{0, 1\},$$

and define predicates I_i, J_j by

i or j	0	1	2	3
I_i	$b = 0$	$a = 1$	$b = 1$	$a = 0$
J_j	$c = 0$	$d = 1$	$c = 1$	$d = 0$

The facet incidences are

$$\begin{aligned} A_i &: r = 0 \text{ and } I_i, \\ B_j &: r = 1 \text{ and } J_j, \\ M_{ij} &: I_i \text{ and } J_j. \end{aligned} \tag{7}$$

Each facet has eight labels, with three bits free, and the graph is the five-cube graph.

The coordinates can be recovered from four linear equations. For a label v, let $i < i'$ satisfy I and $j < j'$ satisfy J. Its point is the unique solution with $w = 1$ of

$$\begin{cases} A_i = A_{i'} = M_{ij} = M_{ij'} = 0, & r = 0, \\ B_j = B_{j'} = M_{ij} = M_{i'j} = 0, & r = 1. \end{cases} \tag{8}$$

In all 32 cases these affine equations are independent.

Here is an exact check of the halfspaces. Replace each coordinate vector by its primitive integer homogeneous representative q_v with positive last entry, and each defining form by its primitive integer multiple of the same sign. Substitution in (8) gives zero precisely at the incidences in (7); the least positive evaluations at the remaining vertices are

i or j	0	1	2	3
A_i	1	1	1	3
B_j	575	115	1955	432055

M_{ij}	$j=0$	$j=1$	$j=2$	$j=3$
$i=0$	1	1	51	1
$i=1$	1	1	204	1
$i=2$	11	6	357	11
$i=3$	242	46	11132	92

These inequalities identify the whole polytope. Let P_0 be the convex hull of the 32 listed points. The four independent active planes at each point make it a vertex of P_0. Within each proposed facet, the six prescribed quadrilaterals are exposed by neighboring rows and form a cubical boundary. The facet is three-dimensional; a proper exposed face containing four extreme points cannot be a line, and a planar facet could not have such a proper face. The six quadrilaterals therefore form its entire boundary. Every ridge is paired with another proposed facet, so connectedness of the facet-adjacency graph of P_0 excludes any additional facet. Its complete facet inequalities are consequently (6), proving $P = P_0$ and $f(P) = (32, 80, 72, 24)$.

The accompanying file *verify_block.py* reconstructs all coordinates from (8) and checks these integer evaluations, the determinants below, and the two perspectivities by exact rational arithmetic.

Choose

$$F = B_1 = \{24, 25, 26, 27, 28, 29, 30, 31\},$$
$$H = M_{31} = \{8, 10, 12, 14, 24, 26, 28, 30\},$$
$$G = A_1 = \{1, 3, 5, 7, 9, 11, 13, 15\}.$$

Then $Q = F \cap H = \{24, 26, 28, 30\}$, and $G = [0,1]^3 \times \{0\}$ is disjoint from $F \cup H$.

4.2. **Two determining facets**

Only seven noncoplanarity facts are needed. The following determinants, using the primitive vectors q_v in the indicated order, are nonzero modulo 101

Vertex labels	Columns	Determinant modulo 101
$(0,1,3,2)$	(x,y,t,w)	100
$(4,5,7,6)$	(x,y,z,t)	67
$(8,9,11,10)$	(x,y,z,t)	1
$(12,13,15,14)$	(x,y,z,t)	51
$(16,20,28,24)$	(x,y,z,t)	28
$(18,22,30,26)$	(x,y,z,t)	12
$(0,16,12,28)$	(x,y,z,t)	64

The first six rows are graph four-cycles. The last says that the spoke lines p_0p_{16} and $p_{12}p_{28}$ are skew.

Suppose a redrawing vanishes on H. The cycles

$$(8,9,11,10), \quad (12,13,15,14), \quad (16,20,28,24), \quad (18,22,30,26)$$

each contain a zero edge, so Lemma 3 makes them zero. Thus all vertices $8, \dots, 15$ and all even vertices $16, \dots, 30$ are zero. The two-neighbor rule completes the propagation; use $v+8, v+16$ for $v = 0, 2, 4, 6$; then $v+8, v-1$ for $v = 1, 3, 5, 7$; finally $v-16, v-1$ for each odd $v \geq 17$. Hence H determines P.

If instead the redrawing vanishes on G, the first four cycles in the table make all vertices $0, \dots, 15$ zero. On $(16,20,28,24)$ it is a homothety $h_v = \alpha p_v + b$. The values at 16 and 28 lie on the direction lines of the two skew spokes. If $\alpha \neq 0$, the point $-b/\alpha$ would lie on both spoke lines, which is impossible; if $\alpha = 0$, the vector b is parallel to both nonparallel lines and must vanish. That cycle is therefore zero. Next $17, 21, 25, 29$ have zero neighbors $v-1, v-16$, and every remaining second-half vertex has zero neighbors $v-2, v-16$. Thus G determines P as well.

4.3. **The restriction on the third facet**

Consider

$$M = (8,9,11,10), \qquad T = (24,25,27,26).$$

The cycle M is nonplanar, whereas T is a quadrilateral facet of F. Their four corresponding spoke lines concur at

$$a_* = (21, -21, 95, 42)/53.$$

Indeed, the eight forms $U_i + V_j$, $j = 0, 1$, annihilate the homogeneous point $(21, -21, 95, 42, 53)$. The apex is outside the plane of T, since otherwise all vertices of M would be coplanar.

Subtract the homothety and translation of a redrawing on M. Its values on T then have the form $h(T_j) = \beta_j(T_j - a_*)$. Modulo the direction plane of T, the vectors $T_j - a_*$ have a common nonzero image, while adjacent redrawing differences have zero image. Thus the scalars β_j agree and the redrawing on T is homothetic. By Lemma 3(iii), the inverse image of this line in $\mathcal{D}(F)$ has dimension two, proving Proposition 5(iii).

4.4. **Matching the two interfaces**

The perspective center from F to H is

$$o = (-11, 11, -15, 51)/7.$$

The map fixes Q and matches

$$25 \mapsto 8, \quad 27 \mapsto 10, \quad 29 \mapsto 12, \quad 31 \mapsto 14.$$

For each pair (u, w), substitution in (8) gives

$$p_w = (1 - \tau)p_u + \tau o, \qquad \tau = \begin{cases} 7/73, & u = 25, 29, \\ 11/146, & u = 27, 31. \end{cases}$$

The projectivity

$$T_0(p) = \frac{p}{1 - (7/51)t(p)}$$

sends o to infinity and fixes G pointwise. Since $0 \le t \le 1$ on P, its denominator lies in $[44/51, 1]$. In the new chart the perspective map is affine, fixes Q, and maps the whole of F onto H because it matches all eight vertices. This proves the remaining assertions of Proposition 5 and completes the construction.

References

[1] M. Kallay, *Indecomposable polytopes*, Israel J. Math. **41** (1982), 235–243.

[2] M. Kallay, *Decomposability of polytopes is a projective invariant*, Ann. Discrete Math. **20** (1984), 191–196.