Compatibility and realization of coordinate projection areas

Caius Folkerts

Abstract. Four-coordinate inequalities characterize the coordinate projection areas of convex bodies in dimension four. In every higher dimension, we show that every positive hollow matrix of rank three lies outside the closure of the realization locus, although each of its four-coordinate restrictions is realizable. The obstruction comes from incompatible equality cases on overlapping coordinate subspaces. We also characterize limits produced by bodies of unbounded diameter through weighted triangle inequalities, and realize their strict part by centrally symmetric polytopes with at most $d(d-1)+2$ vertices. Together these results determine both exact and limiting realizability for every nonnegative hollow matrix of rank at most three.

1. Introduction

Let $\phi_{ij}\colon \mathbb{R}^d \to \mathbb{R}^2$ retain coordinates i, j. For a nonempty compact convex set $K \subset \mathbb{R}^d$, write

$$\mathcal{A}(K)_{ij} = \mathrm{Area}(\phi_{ij}K), \qquad \mathcal{R}_d = \{\mathcal{A}(K) \mid K \subset \mathbb{R}^d \text{ compact and convex, } K \neq \varnothing\}.$$

We identify an array with its symmetric matrix, setting the diagonal entries equal to zero; such a matrix is called *hollow*. A positive array has strictly positive off-diagonal entries. Convex sets of lower affine dimension are allowed throughout.

In dimension four, Huang, Huh, Michałek, Wang and Wang [2] proved that an array is realizable exactly when

$$\sqrt{q_{ij}q_{k\ell}} \le \sqrt{q_{ik}q_{j\ell}} + \sqrt{q_{i\ell}q_{jk}} \qquad (i,j,k,\ell \text{ distinct}). \tag{1}$$

Equivalently, its hollow matrix has at most one positive eigenvalue. Every four-coordinate projection of a higher-dimensional body obeys the same criterion, and Huh [3] asked whether these conditions suffice in dimension five. The difficulty is that individually realizable four-dimensional images must also agree where their coordinate spaces overlap.

Equality in (1) makes this compatibility visible. When the entries are positive and

$$\sqrt{q_{ab}q_{cd}} = \sqrt{q_{ac}q_{bd}} + \sqrt{q_{ad}q_{bc}},$$

the quartet carries the partition $\{a,b\} \mid \{c,d\}$. We prove that the partitions attached to two such quartets must agree on their common triple.

Theorem 1.1. *Let $d \geq 5$. Every positive hollow symmetric $d \times d$ matrix of rank three lies outside $\overline{\mathcal{R}_d}$, although all of its four-coordinate restrictions belong to $\mathcal{R}_4$.*

Compatibility permits at most three boundary quartets on five labels, whereas a positive rank-three matrix places every quartet on the boundary. The theorem excludes approximation as well as exact realization. For example,

$$Q = \begin{pmatrix} 0 & 1 & 3 & 4 & 3 \\ 1 & 0 & 1 & 3 & 4 \\ 3 & 1 & 0 & 1 & 3 \\ 4 & 3 & 1 & 0 & 1 \\ 3 & 4 & 3 & 1 & 0 \end{pmatrix}$$

is the squared-distance matrix of five consecutive vertices of a regular hexagon of circumradius one. With $\mathbf{1} = (1,1,1,1,1)^T$, $b = (2,1,-1,-2,-1)^T$ and $c = (0,1,1,0,-1)^T$, it has the factorization

$$Q = 2\mathbf{1}\mathbf{1}^T - \tfrac{1}{2} b b^T - \tfrac{3}{2} c c^T.$$

The three vectors are independent, so Q has rank three and signature $(1,2,2)$ and lies outside $\overline{\mathcal{R}_5}$. Small perturbations give a nonempty open set of nonrealizable positive arrays for which every four-coordinate matrix has the strict signature $(1,3)$. Thus the failure of the four-coordinate criterion persists away from its equality cases.

Passing from exact realization to its closure requires understanding bodies whose diameters tend to infinity while their projection areas remain bounded. Those bodies become long and thin in a single direction. Their limiting areas are governed by weighted triangle inequalities

$$w_k q_{ij} \leq w_j q_{ik} + w_i q_{jk} \qquad (i,j,k \text{ distinct}), \qquad w \in \mathbb{R}^d_{\geq 0} \setminus \{0\}. \tag{2}$$

Let $\mathcal{E}_d$ be the set of nonnegative arrays admitting such a vector w.

Theorem 1.2. *An array belongs to $\mathcal{E}_d$ if and only if it is the limit of the projection-area arrays of convex bodies whose diameters tend to infinity. In particular,*

$$\overline{\mathcal{R}_d} = \mathcal{R}_d \cup \mathcal{E}_d.$$

If all weights are positive and all inequalities in (2) are strict, the array itself is realized by centrally symmetric polytopes with at most $d(d-1)+2$ vertices and arbitrarily large diameter.

For positive weights, division by $w_i w_j w_k$ turns (2) into the ordinary triangle inequalities for $q_{ij}/(w_i w_j)$. The metric interpretation gives a construction in which distance potentials determine transverse vertices, while a long common axis makes every coordinate projection a diamond of the prescribed area. The same inequalities describe the only possible loss of compactness. At rank three they would make the rescaled entries squared distances of points on a circle with no obtuse triangle, which is impossible on five labels.

Zero entries admit an exact reduction. After deleting zero rows, coordinates with zero mutual area form classes of proportional rows. Collapsing each class preserves both rank and realizability. Combining this reduction with Theorem 1.1 gives a complete criterion at rank at most three. An array is realizable, equivalently lies in the closure, precisely when there are at most four classes.

The projections considered here retain two coordinates. The complementary projections that delete two coordinates have a different realization problem, even in dimension five; their rank-three obstruction is treated in [1].

2. **Compatibility of equality cases**

A *boundary quartet* is a positive four-coordinate array for which one of the inequalities (1) is an equality. Its partition is the pair of opposite edges giving the largest product. The first step is to ensure that the equality theorem for mixed volumes can be applied to its realizing body.

Lemma 2.1. *Every realization of a boundary quartet in $\mathbb{R}^4$ has nonempty interior.*

Proof. Positive coordinate dilations make the areas on opposite pairs equal. Explicitly, dilation of coordinate i by

$$\lambda_i = \left(\frac{\prod_{j<k,\ j,k\neq i} q_{jk}}{\prod_{j\neq i} q_{ij}} \right)^{1/4}$$

changes q_{ij} to $\sqrt{q_{ij}q_{k\ell}}$, where $\{i,j,k,\ell\} = \{1,2,3,4\}$. Relabeling therefore reduces the data to

$$q_{12} = q_{34} = a + b, \qquad q_{13} = q_{24} = a, \qquad q_{14} = q_{23} = b, \qquad a, b > 0.$$

Suppose the realizing set has dimension at most three. It is a linear image, up to translation, of a full-dimensional convex body $L \subset \mathbb{R}^3$. Denote its four coordinate functionals by $a_1, \dots, a_4 \in \mathbb{R}^3$. The brightness norm of L,

$$N(v) = \frac{1}{2} \int_{S^2} |\langle u, v \rangle| \, dS_L(u),$$

satisfies $q_{ij} = N(a_i \times a_j)$, by the Cauchy projection formula [4].

For any relation $\sum_i c_i a_i = 0$, taking cross products and applying the triangle inequality gives

$$(a+b)|c_2| \le a|c_3| + b|c_4|, \quad (a+b)|c_1| \le b|c_3| + a|c_4|,$$
$$(a+b)|c_4| \le a|c_1| + b|c_2|, \quad (a+b)|c_3| \le b|c_1| + a|c_2|.$$

Their sums agree, so each inequality is an equality. The largest of the four absolute coefficients is a strict weighted average of two others; propagation shows that all four absolute coefficients are equal. Thus every nonzero relation has all coefficients nonzero, the relation space is one-dimensional, and every triple of the a_i is independent. Changing coordinate signs, which leaves all areas unchanged, we may assume $a_1 + a_2 + a_3 + a_4 = 0$.

Put $x = a_1 \times a_3$, $y = a_2 \times a_3$ and $z = a_1 \times a_2$. These vectors are independent, and the six areas give

$$N(x) = a, \quad N(y) = b, \quad N(z) = a + b,$$
$$N(z+x) = b, \quad N(z-y) = a, \quad N(x+y) = a + b.$$

An integral of absolute linear forms satisfies Hlawka's inequality

$$N(r+s) + N(s+t) + N(t+r) \le N(r) + N(s) + N(t) + N(r+s+t),$$

which follows by integrating the scalar absolute-value inequality. Apply it to $r = z+x$, $s = y - z$, $t = -x$. The displayed values cancel, leaving $N(x - y + z) \le 0$, contrary to independence and the fact that N is a norm. □

We use mixed area measures with the normalization

$$\int h_T \, dS^{(n)}_{K_1,\dots,K_{n-1}} = nV_n(T, K_1, \dots, K_{n-1}).$$

They are positive and multilinear, and their first moment is zero. These facts, and continuity under Hausdorff convergence, hold without interior assumptions [5, Section 4.1].

Lemma 2.2. *Fix $K \subset \mathbb{R}^n$ and put $q(P,Q) = V_n(K[n-2], P, Q)$. If $q(P,P) = q(P,Q) = q(Q,Q) > 0$, then*

$$S^{(n)}_{K[n-2],P} = S^{(n)}_{K[n-2],Q}.$$

For the coordinate segment $E_i = [0, e_i]$, with π_i deleting coordinate i, one also has

$$S^{(n)}_{K[n-2],E_i} = \frac{1}{n-1} \, \iota_* S^{(n-1)}_{(\pi_i K)[n-2]}, \tag{3}$$

where ι embeds the unit sphere of $e_i^\perp$ in S^{n-1}.

Proof. Let the common mixed volume be $a > 0$. Alexandrov–Fenchel applied to $P + sT, Q$ gives

$$(a + sq(Q,T))^2 \geq a\{a + 2sq(P,T) + s^2 q(T,T)\}.$$

Divide by $s > 0$ and let $s \downarrow 0$ to obtain $q(Q,T) \geq q(P,T)$. Interchanging P, Q proves equality. The defining integral then identifies the measures, since differences of support functions contain all C^2 functions on the sphere [5, Lemma 4.5].

For (3), extrusion along E_i gives

$$nV_n(K[n-2], T, E_i) = V_{n-1}((\pi_i K)[n-2], \pi_i T).$$

Apply the defining integral in dimension $n-1$ and use $h_{\pi_i T} = h_T$ on $e_i^\perp$. □

A nonzero normal direction u is *1-extreme* for K if it cannot be written as the sum of three linearly independent normals at a common boundary point. The full-dimensional equality theorem of Shenfeld and van Handel [5, Theorem 2.2] says that, when K has interior and $q(Q,Q) > 0$, equality in

$$q(P,Q)^2 \geq q(P,P)q(Q,Q)$$

holds exactly when $h_P - \lambda h_Q$ is linear on all 1-extreme normals of K, for some $\lambda \geq 0$. The comparison bodies P, Q need not have interior. If P is not a point, then $\lambda > 0$, since otherwise P would agree with a point on every regular supporting normal of K, and those normals positively span $\mathbb{R}^n$.

Proposition 2.3. *For the positive coordinate projection areas of one convex set, the partitions of two boundary quartets sharing three labels agree on those labels.*

Proof. Let the shared triple be S, and write the quartets as $S \cup \{a\}$ and $S \cup \{b\}$. Their projections K_a, K_b have interior by Lemma 2.1; their common projection L onto $\mathbb{R}^S$ is therefore three-dimensional.

For either quartet I, form the complementary matrix

$$M_{ii} = 0, \qquad M_{ij} = q_{k\ell} \quad \text{when } \{i,j,k,\ell\} = I.$$

It is the matrix of $12V_4(K_I, K_I, E_i, E_j)$. Its kernel has a vector c whose signs are the two blocks of the boundary partition. To see the signs, use the positive diagonal normalization in Lemma 2.1. The normalized matrix has kernel vector $(1, 1, -1, -1)$ when its largest pair is $12 \mid 34$, and undoing the normalization preserves signs.

The rectangles $P = \sum_{c_i>0} c_i E_i$ and $Q = \sum_{c_i<0} (-c_i) E_i$ have equal positive mixed self-volume and mixed volume. Lemma 2.2 consequently gives

$$\sum_{c_i>0} c_i \mu_i = \sum_{c_i<0} (-c_i) \mu_i, \qquad \mu_i = S^{(4)}_{K_I, K_I, E_i}.$$

Orient c so the omitted coordinate, a or b, has positive sign. The two negative coordinates then form the two-element block of the restricted partition on S. Each μ_i is supported on $H_i = \{u_i = 0\}$, while the measure of the omitted segment is $S_L/3$ by (3). Positivity of the measures therefore confines S_L to the union of the two hyperplanes indexed by that block.

If the restricted partitions differ, their two-element blocks have a single common label j. Intersecting the resulting support restrictions inside $\mathbb{R}^S$ gives

$$\operatorname{supp} S_L \subset (H_j \cap S^2) \cup \{e_j, -e_j\}.$$

Write $S \setminus \{j\} = \{r, s\}$. The Cauchy projection formula and centeredness now imply

$$S_L(\{e_j\}) = S_L(\{-e_j\}) = q_{rs} > 0.$$

Hence L has two-dimensional exposed faces in both directions $\pm e_j$. Projection commutes with the exposed face in any direction belonging to the projection space, so the corresponding faces of K_a also have dimension at least two. Both normals are 1-extreme, since three normals at a common point whose sum is $\pm e_j$ would all annihilate the two-dimensional direction space of that face, and could not be independent in $\mathbb{R}^4$.

Apply the equality theorem to the rectangles associated with K_a. The label j belongs to its negative block, so P has zero width in direction e_j whereas Q has positive width. Adding the support-function equalities at e_j and $-e_j$ cancels the translation and equates these widths up to the positive factor λ, a contradiction. □

Corollary 2.4. *Among five coordinate labels with positive projection areas, at most three quartets are boundary quartets. If at least two are boundary quartets, their partitions extend uniquely to a partition of the five labels into blocks of sizes two and three, and every boundary quartet omits a label from the larger block. The bound three is attained.*

Proof. Suppose two boundary quartets omit m, n. Their common triple has a partition $\{a, b\} \mid \{r\}$, so their partitions are $\{a, b\} \mid \{r, n\}$ and $\{a, b\} \mid \{r, m\}$. Both extend to $\{a, b\} \mid \{r, m, n\}$. A boundary quartet omitting a would have to induce both $\{b, m\} \mid \{r, n\}$ and $\{b, n\} \mid \{r, m\}$ by compatibility, which is impossible. The same holds for omission of b, leaving at most the other three quartets.

For equality, take the convex hull of the square $[-1,1]^2$ on the first block and the cube $[-1/2,1/2]^3$ on the second. Its within-block areas are 4 and 1, respectively, and every cross-block projection is a diamond of area 1. Exactly the three quartets omitting a coordinate of the second block are boundary quartets. □

Corollary 2.5. *A positive hollow symmetric matrix of rank three and order at least five is not realizable as a coordinate projection-area matrix.*

Proof. Every three-label principal determinant is $2q_{ij}q_{ik}q_{jk} > 0$, so that principal matrix has signature $(1,2)$. Interlacing and rank three give the full signature $(1,2,d-3)$ and signature $(1,2,1)$ on every quartet. The four-dimensional criterion then places every quartet on its boundary, contradicting Corollary 2.4. □

3. **Long convex bodies and weighted metrics**

Bounded projection areas permit a body to become arbitrarily long only if its transverse dimensions shrink. The next lemma records the resulting limit, without imposing interior assumptions on the bodies.

Lemma 3.1. *Suppose $\mathcal{A}(K_m) \to q$ and $L_m = \operatorname{diam} K_m \to \infty$. There are a unit vector $v \in \mathbb{R}^d$ and a seminorm N on $\mathbb{R}^d$ such that*

$$q_{ij} = N(v_i e_j - v_j e_i).$$

Consequently $q \in \mathcal{E}_d$, with weights $w_i = |v_i|$.

Proof. Translate the midpoint of a diameter chord to the origin and write its endpoints as $\pm L_m v_m/2$, with $|v_m| = 1$. For every $z \in K_m$, the triangle with these endpoints and z projects to a triangle of area

$$\frac{L_m}{2}|v_{m,i}z_j - v_{m,j}z_i| \le \mathcal{A}(K_m)_{ij}.$$

The identity $|v_m \wedge z|^2 = \sum_{i<j}(v_{m,i}z_j - v_{m,j}z_i)^2$ bounds the transverse component of z by B/L_m, for a fixed B. The diameter condition also gives $|\langle z, v_m\rangle| \le L_m/2$.

Write $z = L_m t v_m + L_m^{-1} y$, where $y \perp v_m$, and let C_m be the set of points $tv_m + y$ so obtained. These bodies lie in a fixed compact cylinder, with $|t| \le 1/2$ and $|y| \le B$, and contain the segment $[-v_m/2, v_m/2]$. After passage to a subsequence,

$$v_m \longrightarrow v, \qquad C_m \longrightarrow C$$

in Euclidean and Hausdorff distance. For $-1/2 \le t \le 1/2$, put $C_t = \{y \in v^\perp \mid tv + y \in C\}$ and define

$$N(\xi) = \int_{-1/2}^{1/2} \mathrm{width}_{C_t}(\xi)\, dt.$$

Each fiber is nonempty, and widths are even and subadditive, so N is a seminorm.

Fix i, j with $\rho = (v_i^2 + v_j^2)^{1/2} > 0$. Rotate the coordinate plane to align its first axis with $(v_{m,i}, v_{m,j})$, then apply the area-preserving map $\mathrm{diag}(L_m^{-1}, L_m)$. The resulting image of K_m is the image of C_m under a linear map converging to

$$tv + y \longmapsto \left(\rho t, \frac{v_i y_j - v_j y_i}{\rho}\right).$$

Continuity of planar area under Hausdorff convergence and integration over the first coordinate give

$$q_{ij} = \int_{-1/2}^{1/2} \mathrm{width}_{C_t}(v_i e_j - v_j e_i)\, dt.$$

If $\rho = 0$, use the same rotation when $\rho_m > 0$ and any planar axis when $\rho_m = 0$. The transformed projection lies in a rectangle with side lengths at most $\rho_m + 2B/L_m^2$ and $2B$, where $\rho_m = (v_{m,i}^2 + v_{m,j}^2)^{1/2}$; its area tends to zero. Thus the formula holds for every pair.

Finally,

$$v_k(v_i e_j - v_j e_i) = v_j(v_i e_k - v_k e_i) - v_i(v_j e_k - v_k e_j).$$

Applying N gives (2) with $w_i = |v_i|$, and these weights are not all zero. □

The converse is an exact construction. For a compact convex set B, denote its width functional by

$$W_B(\xi) = \max_{x \in B} \langle \xi, x \rangle - \min_{x \in B} \langle \xi, x \rangle.$$

Planar extrusion gives, for every $L > 0$,

$$\mathcal{A}\big([-Lw/2, Lw/2] + L^{-1}B\big)_{ij} = W_B(w_i e_j - w_j e_i) + L^{-2}\mathcal{A}(B)_{ij}. \tag{4}$$

Indeed, adding a segment to a planar convex set increases each nonempty fiber parallel to that segment by its length; integrating the fibers gives the width term. This remains true for degenerate planar sets.

Proposition 3.2. *If $q \in \mathcal{E}_d$ and w is any witness in* (2)*, there is a centrally symmetric polytope B containing the origin such that*

$$W_B(w_i e_j - w_j e_i) = q_{ij} \qquad (i < j).$$

Thus $q + L^{-2}\mathcal{A}(B)$ is realizable for every $L > 0$, by bodies of diameter at least $L|w|$.

Proof. Let $I = \{i \mid w_i > 0\}$, $J = \{j \mid w_j = 0\}$, and choose $a \in I$. The weighted inequalities imply

$$q_{jk} = 0 \quad (j,k \in J), \qquad q_{ij} = w_i\rho_j \quad (i \in I,\ j \in J), \qquad \rho_j = q_{aj}/w_a.$$

For $i,k \in I$, the numbers $d_{ik} = q_{ik}/(w_iw_k)$ form a pseudometric. Consider its anchored Lipschitz polytope

$$C_I = \{f \in \mathbb{R}^I \mid f_a = 0,\ |f_i - f_k| \le d_{ik} \text{ for } i,k \in I\}.$$

It is compact and centrally symmetric. For any fixed i, the distance potential $f_\ell = d_{i\ell} - d_{ia}$ belongs to C_I and satisfies $f_k - f_i = d_{ik}$. Hence the range of $f_k - f_i$ on C_I is exactly $[-d_{ik}, d_{ik}]$.

In $\mathbb{R}^d = \mathbb{R}^I \times \mathbb{R}^J$, take

$$B = \tfrac{1}{2}\operatorname{diag}(w_I)C_I \ \times \prod_{j\in J}[-\rho_j/2, \rho_j/2].$$

For $i,k \in I$, the functional $w_ix_k - w_kx_i$ becomes $\frac{1}{2}w_iw_k(f_k - f_i)$, of width q_{ik}. For $i \in I$, $j \in J$, its width is $w_i\rho_j = q_{ij}$, and for two labels in J both sides vanish. Equation (4) proves the assertion about realizations and diameters. □

Theorem 3.3. *The set $\mathcal{E}_d$ is closed and consists exactly of the limits of projection-area arrays along sequences with diverging diameter. Moreover,*

$$\overline{\mathcal{R}_d} = \mathcal{R}_d \cup \mathcal{E}_d.$$

The approximating bodies for every point of $\mathcal{E}_d$ may be chosen to be full-dimensional centrally symmetric polytopes.

Proof. Normalize witnesses for (2) by $\sum_i w_i = 1$. Compactness of the simplex shows that the inequalities persist under limits, proving closedness. Lemma 3.1 and Proposition 3.2 establish the characterization by diverging diameter. Adding a sufficiently small cube to each approximating polytope gives full-dimensional centrally symmetric approximants with the same limit.

For an arbitrary convergent sequence of realizable arrays, either the realizing bodies have a subsequence of bounded diameter or they have a subsequence whose diameters tend to infinity. In the former case, translation and Blaschke selection give a compact convex limit realizing the array; in the latter, the limit belongs to $\mathcal{E}_d$. The converse inclusion follows from Proposition 3.2. □

4. **Exact realizations from strict metrics**

A limiting prism becomes an exact realization when its transverse vertices can be assigned independently to the coordinate planes. Strict triangle inequalities provide precisely the separation needed for this construction.

Theorem 4.1. *Suppose $q_{ij} > 0$ and $w_i > 0$ satisfy*

$$w_k q_{ij} < w_j q_{ik} + w_i q_{jk} \qquad (i, j, k \text{ distinct}).$$

Then q is realized by centrally symmetric polytopes with at most $d(d-1)+2$ vertices and arbitrarily large diameter.

Proof. Put $d_{ij} = q_{ij}/(w_i w_j)$ and $d_{ii} = 0$. For each pair $r = (i, j)$, define

$$f_k^r = \frac{d_{ki} - d_{kj}}{2}, \qquad b_k^r = \frac{w_k f_k^r}{2}.$$

The distance potential f^r attains the difference $f_j^r - f_i^r = d_{ij}$ on its designated pair. On every other pair (k, ℓ), it satisfies the strict inequality

$$|f_\ell^r - f_k^r| < d_{k\ell}.$$

Indeed, twice the difference is $(d_{\ell i} - d_{ki}) - (d_{\ell j} - d_{kj})$. Each term has absolute value at most $d_{k\ell}$, and at least one of i, j is outside $\{k, \ell\}$, making that bound strict.

We seek a polytope with long diagonal $[-Tw, Tw]$ whose projection in plane $p = (k, \ell)$ is the diamond with transverse vertices $\pm\phi_{k\ell} b^p/T$. Set $v_p = (w_k, w_\ell)$ and $c_p = \phi_{k\ell} b^p$. Then

$$\det(v_p, c_p) = \tfrac{1}{2} w_k w_\ell (f_\ell^p - f_k^p) = q_{k\ell}/2 > 0.$$

Every other projected transverse vector has a unique expression $\phi_{k\ell} b^r = \alpha_{pr} v_p + \beta_{pr} c_p$, and taking determinants gives

$$|\beta_{pr}| = \frac{|f_\ell^r - f_k^r|}{d_{k\ell}} < 1.$$

Since there are only finitely many pairs, for all sufficiently large T one has $|\alpha_{pr}|/T^2 + |\beta_{pr}| \le 1$ simultaneously. This is exactly the condition that $\phi_{k\ell} b^r/T$ lie in the desired diamond.

Consequently

$$P_T = \operatorname{conv}\big(\{Tw, -Tw\} \cup \{b^r/T, -b^r/T \mid r = (i, j),\ i < j\}\big)$$

has that diamond as every coordinate-plane projection, of area $2|\det(Tv_p, c_p/T)| = q_{k\ell}$. There are at most $2 + 2\binom{d}{2} = d(d-1) + 2$ vertices, and $\operatorname{diam} P_T \ge 2T|w|$. □

The strict realizations are dense in $\mathcal{E}_d$. To include zero weights, first take the polytope B from Proposition 3.2, replace w by $w + \varepsilon\mathbf{1}$, and use the arrays

$$q_{ij}^{\varepsilon} = W_B\big((w_i + \varepsilon)e_j - (w_j + \varepsilon)e_i\big).$$

They satisfy the weighted inequalities with positive weights and converge to q. For positive weights the associated pseudometric becomes a strict metric by adding $\delta > 0$ to every off-diagonal distance. The resulting exact realizations approach the original array. Thus the metric construction describes the whole escape set through the closure of an explicit family of polytopes.

5. **Rank three and the failure of approximation**

The compatibility obstruction excludes exact rank-three realizations. To exclude approximation, it remains to show that rank-three arrays cannot satisfy the weighted triangle inequalities.

Lemma 5.1. *A positive hollow symmetric matrix of rank three satisfying the triangle inequalities is the squared-distance matrix of at most four points on a circle. If there are four points, they form a rectangle.*

Proof. We first use an elementary property of four-point metrics. If $H = (h_{ij})$ is such a metric, then

$$z^T H z \le 0 \qquad \text{whenever } \sum_i z_i = 0.$$

To see this, relabel so that $h_{12} + h_{34}$ is the largest of the three sums on opposite pairs. For a subset S, let δ_S be the matrix equal to one across the cut $S \mid S^c$ and zero elsewhere. Then

$$H = \sum_{i=1}^{4} x_i \delta_{\{i\}} + y_{13}\delta_{\{1,3\}} + y_{14}\delta_{\{1,4\}},$$

where

$$\begin{aligned} 2x_1 &= h_{13} + h_{14} - h_{34}, & 2x_2 &= h_{23} + h_{24} - h_{34},\\ 2x_3 &= h_{13} + h_{23} - h_{12}, & 2x_4 &= h_{14} + h_{24} - h_{12},\\ 2y_{13} &= h_{12} + h_{34} - h_{13} - h_{24}, & 2y_{14} &= h_{12} + h_{34} - h_{14} - h_{23}. \end{aligned}$$

The coefficients are nonnegative, and $z^T \delta_S z = -2(\sum_{i\in S} z_i)^2$ when $\sum_i z_i = 0$.

Now let D have rank three and satisfy the hypotheses. Its principal matrix A on the first three labels is invertible, since $\det A = 2D_{12}D_{13}D_{23} > 0$. Set $\alpha_k = A^{-1}(D_{1k}, D_{2k}, D_{3k})^T$ for $k \geq 4$. The rank condition gives

$$D = M^T A M, \qquad M = (e_1, e_2, e_3, \alpha_4, \ldots, \alpha_n).$$

For each $k \geq 4$, the principal matrix on $1, 2, 3, k$ has kernel vector $(-\alpha_k, 1)$. Its coordinate sum must be zero. Otherwise the all-ones vector could be shifted by a multiple of this kernel vector to have coordinate sum zero without changing its strictly positive quadratic value, contrary to the four-point inequality. Thus every column of M has coordinate sum one.

Choose a nondegenerate planar triangle with squared side lengths A_{ij}, and place its circumcenter at the origin. Such a triangle exists because the triangle inequalities for the positive numbers A_{ij} imply strict triangle inequalities for their square roots. Write its vertices as p_1, p_2, p_3, with $|p_i| = r$, and put $p_k = \sum_{i=1}^{3} \alpha_{k,i} p_i$. The sum-one condition gives

$$D_{k\ell} = 2r^2 - 2\langle p_k, p_\ell \rangle.$$

The zero diagonal therefore puts every p_k on the same circle, and the displayed identity becomes $D_{k\ell} = |p_k - p_\ell|^2$. Positive off-diagonal entries make the points distinct.

The metric inequalities now say that every triangle of these points is nonobtuse. For five points in cyclic order, some two consecutive arc gaps sum to less than π, so their three endpoints form an obtuse triangle. Hence there are at most four points. If there are four, the opposite angles of their inscribed quadrilateral sum to π and are all at most $\pi/2$, so the quadrilateral is a rectangle. □

Corollary 5.2. *A positive hollow symmetric matrix of rank three on five labels admits no nonzero nonnegative weights satisfying* (2).

Proof. Suppose such weights exist. Two of them cannot vanish, since an inequality with those weights on the right and a positive weight on the left would fail. If exactly one weight vanishes, replace it by a sufficiently small positive number. Inequalities with that weight on the right are only relaxed, while those with it on the left still hold because their right sides are strictly positive. We may therefore assume all weights are positive.

Division by $w_i w_j w_k$ shows that $D_{ij} = q_{ij}/(w_i w_j)$ satisfies the ordinary triangle inequalities. Positive diagonal congruence preserves rank, so Lemma 5.1 would realize D by five points on a circle with no obtuse triangle, a contradiction. □

Proof of Theorem 1.1. For five labels, Corollary 2.5 excludes $\mathcal{R}_5$ and Corollary 5.2 excludes $\mathcal{E}_5$. Theorem 3.3 therefore excludes their union $\overline{\mathcal{R}_5}$. For $d > 5$, any five-label principal matrix still has rank three, since its three-label principal determinants are positive. Projection onto those five coordinates takes $\overline{\mathcal{R}_d}$ into $\overline{\mathcal{R}_5}$, giving the same obstruction. Every four-coordinate restriction is realizable by (1). □

Corollary 5.3. *For every $d \geq 5$, there is a nonempty open set of positive arrays disjoint from $\overline{\mathcal{R}_d}$ on which every four-coordinate principal matrix has signature $(1, 3)$.*

Proof. Choose any positive hollow rank-three matrix Q of order d. Its signature gives a factorization $Q = aa^T - bb^T - cc^T$. The zero diagonal gives $a_i^2 = b_i^2 + c_i^2$, and positive off-diagonal entries force all a_i to have the same nonzero sign. Taking $a_i > 0$, and putting $D = \operatorname{diag}(a_i)$, $u = D^{-1}b$, $v = D^{-1}c$, we obtain

$$Q = D(\mathbf{1}\mathbf{1}^T - uu^T - vv^T)D, \qquad u_i^2 + v_i^2 = 1.$$

Set $Q_\varepsilon = Q + \varepsilon D(\mathbf{1}\mathbf{1}^T - I)D$. The inner form is strictly negative on $\mathbf{1}^\perp$, where it equals $-(u^Tx)^2 - (v^Tx)^2 - \varepsilon|x|^2$, and has a positive value on $\mathbf{1}$. Thus Q_ε has signature $(1, d-1)$, and the same argument on any four labels gives signature $(1, 3)$. Theorem 1.1 excludes Q from the closed realization locus, so all sufficiently small positive perturbations remain outside it. Positivity and these strict signatures persist on an open neighborhood. □

6. **Zero areas and a complete rank-three criterion**

The local four-coordinate inequalities determine how zero entries can occur. They correspond to coordinate replication, so no additional geometric obstruction is concealed in the zero pattern.

Proposition 6.1. *Let Q be a nonnegative hollow symmetric matrix whose four-label principal matrices have at most one positive eigenvalue. After deleting zero rows and columns, the relation*

$$i \sim j \quad \Longleftrightarrow \quad q_{ij} = 0$$

is an equivalence relation, and its classes are precisely the positive proportionality classes of rows. Choose one representative from each class, and let Q^* be the resulting principal matrix. Then Q^* is positive, $\operatorname{rank} Q^* = \operatorname{rank} Q$*, and Q is realizable if and only if Q^* is realizable. The same equivalence holds for membership in the closures of the respective realization loci.*

Proof. If $q_{ij} = 0$, the determinant on labels i, j, k, ℓ is

$$(q_{ik}q_{j\ell} - q_{i\ell}q_{jk})^2.$$

The spectral hypothesis makes this determinant nonpositive, so it vanishes. If rows i, j are nonzero, this identity first gives a common positive column and then their positive proportionality. Conversely, proportional rows have $q_{ij} = 0$, by examining column j. This proves the equivalence relation.

There are positive factors r_i, equal to one at representatives, such that $q_{ij} = r_i r_j q^*_{[i][j]}$. Define $R\colon \mathbb{R}^m \to \mathbb{R}^d$ by $(Rx)_i = r_i x_{[i]}$ on nonzero rows and $(Rx)_i = 0$ on removed rows. Then $Q = RQ^*R^T$, while Q^* is a principal matrix, proving equality of ranks. A body realizing Q projects to one realizing Q^*; conversely, the image under R of a body realizing Q^* has the prescribed areas. These fixed linear maps also preserve limits of area arrays, proving the closure assertion. □

Theorem 6.2. *Let Q be a nonnegative hollow symmetric matrix of rank at most three. Its area array is realizable if and only if it lies in the closure of the realization locus, and these conditions hold exactly when its quotient in Proposition 6.1 has at most four classes. The all-zero matrix is realized by a point.*

Proof. The matrix has at most one positive eigenvalue. For rank at most two this follows from zero trace. If rank is three and two eigenvalues were $a, b > 0$, the third would be $-a - b$, giving

$$\operatorname{tr}(Q^3) = -3ab(a + b) < 0,$$

whereas entrywise nonnegativity gives $\operatorname{tr}(Q^3) = 6\sum_{i<j<k} q_{ij}q_{ik}q_{jk} \geq 0$. Thus Proposition 6.1 applies. A quotient on at most four labels is realizable by the four-dimensional theorem, padding with zero coordinates if necessary. A quotient on at least five labels is positive and has rank three, so Theorem 1.1 excludes even limiting realization. □

References

[1] Caius Folkerts, *Rank-three obstructions to coordinate projection volumes*, manuscript, 2026.

[2] Daoji Huang, June Huh, Mateusz Michałek, Botong Wang and Shouda Wang, *Realizations of homology classes and projection areas*, arXiv 2505.08881, 2025.

[3] June Huh, *Volume polynomials*, arXiv 2601.13249, 2026.

[4] Rolf Schneider, *Convex Bodies: The Brunn–Minkowski Theory*, second expanded edition, Encyclopedia of Mathematics and its Applications 151, Cambridge University Press, 2014.

[5] Yair Shenfeld and Ramon van Handel, *The extremals of Minkowski's quadratic inequality*, Duke Mathematical Journal **171** (2022), no. 4, 957–1027.