The sharp vertex bound for conditional decomposability

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Abstract. We prove that the least number of vertices of a conditionally Minkowski decomposable d-polytope is $4d-4$ for every $d \geq 3$. Equivalently, below this threshold decomposability is determined by the face lattice. The proof uses a small cover of the edge graph by rigid subgraphs and studies the homotheties induced on those pieces. A dimension count in their parameter space forces either a summand of smaller dimension or a facet satisfying Shephard's criterion; the remaining three-homothety case is controlled by concurrence and a forest count. These mechanisms give a uniform proof without a classification of the combinatorial types.

1. Introduction

A Minkowski decomposition depends on the geometry of a polytope, yet many sufficient conditions for decomposability depend only on its face lattice. A combinatorial type is *conditionally decomposable*, or *ambiguous*, if it has both decomposable and indecomposable realizations. The smallest such three-dimensional types have eight vertices. In arbitrary dimension, Wang and Yost constructed conditionally decomposable d-polytopes with $4d-4$ vertices, proved the lower bound $3d-3$, and asked whether their upper bound was sharp [5]. They also established sharpness when the decomposable realization has a segment summand.

Theorem 1. *For every $d \geq 3$, the least number of vertices of a conditionally decomposable d-polytope is $4d-4$.*

The lower bound follows from a stronger assertion. If one realization of a d-polytope with at most $4d-5$ vertices is decomposable, then every realization of its face lattice is decomposable. Wang and Yost's wedge construction supplies equality.

The numerical threshold comes from the interaction between the graph and the geometry of parallel maps. Since such a map is homothetic on each rigid part, a cover of the vertex set by a few rigid subgraphs encodes it by points in the $(d+1)$-dimensional space of homothety parameters. Differences of these parameters locate the edges joining different parts, giving a dimension bound on the number of vertices in each part; equality then produces actual faces of the polytope.

Two outcomes recur. If the parameter directions miss the polytope, a projective change makes the image of the parallel map lower dimensional; below the vertex threshold, this image is a genuine weak summand, whose decomposition can be reconstructed in every realization. If the parameter directions meet the polytope, the small number of remaining vertices forces a facet satisfying Shephard’s criterion. With three parameters, the same alternative is obtained by studying their three centers of concurrence. The graph reduction at the end of the proof shows that these cases exhaust the possibilities.

Throughout, $N = f_0(P)$ and P is full dimensional. A polytope is *universally decomposable* if every realization of its combinatorial type is decomposable. A *weak summand* of P is a polytope A for which $P = \varepsilon A + B$ for some $\varepsilon > 0$ and some polytope B.

2. Parallel maps and combinatorial forcing

A map $f : \mathrm{vert}(P) \to \mathbb{R}^d$ is *parallel* if

$$f(y) - f(x) \in \mathbb{R}(y - x)$$

on every edge xy. It is a *homothety* if $f(x) = \alpha x + b$ on all vertices. Kallay’s criterion says that P is decomposable exactly when it admits a parallel map that is not a homothety [3]. One way to see the converse is to form the continuous function that is linear on each cone of the normal fan and has gradient $f(v)$ on the cone at v. Adding a sufficiently large multiple of the support function of P makes it convex; after rescaling, it is the support function of a summand.

Call a geometric subgraph *rigid* when all its parallel maps are homotheties. Triangles are rigid, a rigid subgraph absorbs a vertex with two neighbors in it, and two rigid subgraphs with two common vertices merge. We shall also use rigidity of a $K_{2,3}$ consisting of actual polytope edges. If one of its four-cycles is nonplanar, that cycle is rigid and absorbs the fifth vertex. If all four-cycles are planar, the five vertices lie in one plane, which would put three edge rays at a vertex in a pointed two-dimensional cone, contradicting extremality of the middle ray.

Lemma 2 (Dense rigidity and concurrence)**.** *A geometric graph on $n \le 2k - 1$ polytope vertices, with minimum degree at least k, is rigid. The same numerical condition forces concurrence of a matching of edge lines if its cofaciality graph has minimum degree k and every witnessing two-face contains exactly two matching edges.*

Proof. The graph has a triangle, since its minimum degree exceeds $n/2$. Close that triangle under two-neighbor absorption, and suppose the resulting set has size $s < n$. Every outside vertex has at most one neighbor in it. The degree bound then gives

$$3 \le s \le k-1, \qquad s(k-s+2) \le n \le 2k-1,$$

where the middle expression counts the set and the distinct outside neighbors required by its vertices. But $s(k-s+2) \ge 3k-3 > 2k-1$ for $k \ge 3$. The case $k = 2$ is a triangle.

For the line version, three pairwise coplanar matching lines are concurrent unless all lie in one plane. In the latter case their witnessing faces would coincide and contain all three matching edges, contrary to the hypothesis. Once two lines through a common point have been obtained, any line cofacial with both must pass through that point; otherwise the same planar contradiction occurs. The preceding two-neighbor ordering therefore propagates concurrence to the entire family. □

In particular, a decomposable r-polytope has at least $2r$ vertices. A pyramid is rigid, since its apex and every base edge form a triangle, and these triangles are joined along edges by connectedness of the base graph.

A facet F has *Shephard's property* if every vertex of F has exactly one neighbor outside F. Shephard's criterion states that such a facet makes P decomposable whenever at least two vertices lie outside it [4]. It can be seen by moving its supporting plane slightly while retaining all other supporting planes; the facet vertices move along their unique exterior edges, giving a nontrivial parallel map. Thus every simple nonsimplex is universally decomposable.

Lemma 3. *If a facet has a rigid complement in a decomposable polytope, it has Shephard's property.*

Proof. Subtract the homothety on the complement from a nontrivial parallel map. If a facet vertex had two exterior neighbors, its value would be zero. The normal component of the map is constant on the facet, so it would then be zero everywhere on that facet. Every facet vertex has an exterior neighbor; its value is parallel to that exterior edge and has zero normal component, hence vanishes. This would make the whole map trivial. □

Parallel maps are projectively invariant. For

$$T(x) = \frac{Ax + b_0}{c \cdot x + \delta},$$

with positive denominator on P, the transported map is

$$f'(T(x)) = Af(x) - (c \cdot f(x))T(x). \tag{1}$$

A homothety (α, b) is sent to $(\delta\alpha - c \cdot b, Ab - \alpha b_0)$; thus its center transforms as a projective point, including a center at infinity.

We shall use Balinski's theorem that the graph of a d-polytope is d-connected [1], and two numerical facts. The excess

$$\xi(P) = \sum_{v \in \mathrm{vert}(P)} (\deg v - d)$$

of a nonsimple d-polytope is at least $d - 2$, and at equality the polytope has a facet with Shephard's property [4, Theorems 3.3 and 4.10]. A simplicial four-polytope on m vertices has at least $4m - 10$ edges [2].

3. **Proper images and persistent summands**

A low-dimensional image of a parallel map need not be convex in the right sense to define a summand. The vertex bound removes this difficulty; the quotient fibers must carry the entire image, and one of them is too small to support a nonhomothetic map.

Lemma 4. *Suppose $N < 4d - 4$ and a nonconstant parallel map has image of affine dimension $r < d$. After a translation, a change of sign, and a positive scaling, it is a weak summand map.*

Proof. Translate the image into an r-space L. The map is constant on every edge not parallel to L. Choose a functional vanishing on L and exposing a vertex of the quotient P/L. A strictly increasing edge path from any vertex ends in that exposed fiber, preserving f along the way. Thus each quotient-vertex fiber carries the entire image. A parallel map on a connected face takes values in a translate of its direction space, so each such fiber has dimension exactly r.

There are at least $d - r + 1$ disjoint vertex fibers. If all their restrictions were nonhomothetic, Lemma 2 would give

$$N \geq 2r(d - r + 1) \geq 4d - 4 \qquad (2 \leq r \leq d - 1),$$

a contradiction. On one fiber F the map is therefore $\alpha x + b$ with $\alpha \neq 0$; the same conclusion is automatic when $r = 1$.

To recover convexity from this one fiber, define $\psi(u) = \langle u, f(v) \rangle$ on the normal cone of each vertex v. Parallelism makes the pieces agree on their common walls.

The gradients lie in L, so ψ factors through L. Choose $w \in L^{\perp}$ exposing F. For large λ, a vertex exposed by $u + \lambda w$ lies in F and maximizes u there. Evaluating ψ at this direction therefore gives

$$\psi(u) = \psi(u + \lambda w) = \alpha h_F(u) + \langle u, b \rangle.$$

Choose the sign so that this factor is convex. A sufficiently small positive multiple can then be subtracted from h_P without losing convexity, giving the required weak summand. □

Lemma 5 (Two homotheties)**.** *If $N \leq 4d-5$ and a nontrivial parallel map is covered by two homotheties, P is universally decomposable. In particular, the conclusion holds when P has a segment weak summand.*

Proof. The lines of edges joining the two classes concur at the center of their difference. Neither class can be a singleton, since two-neighbor absorption would then make the map homothetic. A center in P would make the crossing edges a star or a single edge, contradicting d-connectivity. The center is therefore exterior and the crossing edges form a matching of size at most $2d - 3$.

Every two-face incident to a crossing edge contains exactly two crossing edges. Distinct two-faces give distinct partners, so the cofaciality graph has minimum degree $d - 1$. Lemma 2 forces a common projective center in every realization. It remains exterior because two disjoint actual edges cannot meet inside the polytope. Choose two distinct homotheties with this difference center; they define a nontrivial parallel map on the two classes in the new realization.

For a segment summand, the summand map takes two values and is already covered by two constant homotheties. □

We next reconstruct a simplex summand from its edge centers. The role of two disjoint simplex faces is to make the compatibility among these centers combinatorial.

Lemma 6 (Simplex reconstruction)**.** *Suppose the vertices are colored by the vertices of an r-simplex, every color occurs at least twice, and every crossing family ij has a combinatorially forced projective center z_{ij}. If there are two disjoint simplex faces with one vertex of each color, every realization admits a nontrivial parallel map that is homothetic on each color.*

Proof. Corresponding triangles in the two simplex faces force z_{ij}, z_{ik}, z_{jk} to be distinct and collinear. In one simplex, with homogeneous vertices $X_0, \ldots, X_r$, the centers z_{0i} span a hyperplane avoiding all the X_i, and the triangle collinearities put every z_{ij} in it. If ℓ defines that hyperplane, set $Y_i = X_i/\ell(X_i)$. Then $Y_i - Y_j$ represents z_{ij}. The linear

identification $(x, s) \mapsto (s, -x)$ turns the Y_i into compatible homothety parameters. Distinct parameters and at least two vertices in each class ensure nontriviality. □

Proposition 7. *Suppose $N \leq 4d - 5$ and P has a weak summand of dimension r with $2 \leq r \leq d - 2$. Then P is universally decomposable. The same holds for a two-dimensional weak summand when $d = 3$.*

Proof. Rescale the summand and write $P = A + B$, $\dim A = r$, $q = d - r$. Quotient by the direction space of A. Each quotient-vertex fiber contains A as a summand and therefore has at least $f_0(A)$ vertices. A vertex of P receives the label of its vertex in A.

Codimension at least three. Suppose $q \geq 3$. Since

$$(r + 2)(q + 1) - (4d - 5) = (r - 2)(q - 3) + 1 > 0,$$

A must be a simplex. At most r edges at a vertex change its label, one for each incident simplex edge, so every color has at least $q + 1$ vertices. For a fixed simplex edge, its m parallel copies form a matching, and

$$2m \leq N - (r - 1)(q + 1) \leq 4q - 1.$$

An edge belongs to at least $d - 1$ two-faces. At most $r - 1$ have a triangular image in A, one for each possible third label; every other two-face contains exactly two members of the fixed family. Its cofaciality graph has minimum degree q, so Lemma 2 forces its center in every realization.

A quotient-vertex fiber is either a simplex or has at least $2r$ vertices. If at most one were a simplex, then

$$N \geq (r + 1) + 2rq = (4d - 5) + (r - 2)(2q - 3).$$

For $r \geq 3$ this is impossible, and two simplex fibers give the conclusion by Lemma 6. When $r = 2$, the only remaining equality consists of one triangular and q quadrilateral fibers over a quotient simplex, with no additional vertices. The facet opposite the triangle has Shephard's property; any edge changing quotient position preserves the label, and the triangle has exactly one vertex of each label.

Codimension two. Now $d = r + 2$. The quotient is a triangle, since four vertex fibers would require $4(r + 1) > 4r + 3$ vertices. Also

$$3f_0(A) \leq 4r + 3 < 6r,$$

so A is indecomposable. Every vertex fiber is homothetic to A or has at least $2r$ vertices, and at least one is homothetic. If these three fibers contain all vertices,

the facet opposite a homothetic fiber has Shephard's property by the same label argument.

Suppose additional vertices occur and one fiber is nonhomothetic. Counting forces two simplex fibers, a $2r$-vertex fiber, and one extra vertex x. An edge through x cannot change its label, since that would require a second vertex at the same quotient point. At most two vertices of its label occur in the simplex fibers and at most r in the remaining fiber. Hence x is simple. The hyperplane through its neighbors cuts off only x, so deletion has a combinatorially determined type with a new simplex facet. It is the preceding Shephard type. In every realization a nontrivial parallel map on the deletion is homothetic on that simplex facet and extends by the same homothety to the restored vertex. Thus restoration preserves universal decomposability.

It remains that all three vertex fibers are homothetic and extra vertices occur. Their convex hull has fibers $\lambda(q)A + t(q)$, where $\lambda > 0$ and t are affine barycentric interpolants. An extra vertex violates a side inequality

$$u \cdot x \leq \lambda(q)h_A(u) + u \cdot t(q),$$

with u exposing a facet of A. The parallel exposed face of P beyond this inequality contains only extra vertices and retains that facet as a summand, so it needs at least r vertices. Thus

$$N \geq 3f_0(A) + r \geq 4r + 3.$$

Equality forces A to be a simplex and all extra vertices to form one $(r-1)$-simplex X, parallel to a facet of A, over one nonvertex quotient point. Its labels are precisely those of that facet.

The previous two-face count gives each edge family cofaciality degree at least two. A family involving the missing label has exactly three members, so its lines are forced concurrent. Every other family has four members, including one edge of X. A two-face through that edge with triangular image must lie in X, because a label-changing edge preserves the quotient position. There are at most $r-2$ such faces. The extra edge therefore has cofaciality degree at least $(r+1)-(r-2) = 3$; the other degrees are at least two. This graph contains K_4 minus at most one edge, so two-neighbor propagation forces concurrence. The simplex reconstruction lemma applies.

The three-dimensional endpoint. A planar summand of a three-polytope on at most seven vertices must be a triangle, since its two endpoint fibers already use twice its number of vertices. If one endpoint fiber is not homothetic to the triangle, it has at least four vertices, leaving no others; the opposite facet is Shephard. Otherwise both fibers are homothetic. Any additional vertex would violate a side inequality

of their hull, and the exposed face beyond it retains a triangle edge as a summand, requiring two additional vertices. This is impossible. Either endpoint facet is therefore Shephard. □

4. **The geometry of homothety parameters**

Fix a nontrivial parallel map f and a cover of its vertices by homotheties

$$H_i(x) = \alpha_i x + b_i, \qquad p_i = (\alpha_i, b_i) \in \mathbb{R}^{d+1}.$$

Enlarge each class to the full set $S_i = \{v : f(v) = H_i(v)\}$ and delete redundant classes. Every remaining class then has a private vertex. Put

$$X_v = (1, -v), \qquad D = \operatorname{span}\{p_i - p_j\}, \qquad t = \dim D, \qquad r = d + 1 - t.$$

The use of homogeneous coordinates identifies P with a subset of $\mathbb{P}(\mathbb{R}^{d+1})$. A vertex is *central* if $X_v \in D$; a noncentral vertex belongs to exactly one class, since two homotheties agreeing at v have difference proportional to X_v.

For an edge xy, with $x \in S_i$, $y \in S_j$ and edge scalar μ, parallelism gives the identity

$$p_j - p_i = (\alpha_j - \mu)X_y - (\alpha_i - \mu)X_x. \tag{2}$$

This puts a crossing edge and its homothety difference in the same projective line. It is the common source of the dimension counts and the concurrence arguments below.

Lemma 8 (The class bound)**.** *Every essential class has at least r noncentral vertices.*

Proof. At a private vertex $x \in S_i$, an edge to $y \notin S_i$ has $\alpha_j - \mu \neq 0$ in (2); otherwise x would also belong to S_j. Thus $X_y \in D + \mathbb{R}X_x$. The edge star at x spans the ambient space, so

$$\operatorname{span} X(S_i) + D = \mathbb{R}^{d+1}.$$

Central vertices contribute nothing modulo D, and $\mathbb{R}^{d+1}/D$ has dimension r. □

Let s be the number of central vertices, and write the number of noncentral vertices of S_i as $r + e_i$. The class bound gives $e_i \geq 0$. With $E = \sum_i e_i$, a cover by m essential classes therefore satisfies

$$N = mr + E + s.$$

These counts are defined before we know whether the central vertices form a face.

Equality produces a simplex of quotient directions and, more significantly, faces in P.

Lemma 9 (Fiber faces)**.** *Suppose a class has exactly $r \geq 2$ noncentral vertices $u_1, \ldots, u_r$. Their classes form a basis modulo D, they form a clique, and*

$$R_a = P \cap \mathbb{P}(D + \mathbb{R}X_{u_a})$$

is a t-face for each a. These faces meet in the face $F = P \cap \mathbb{P}(D)$. The corresponding quotient coordinates β_a are nonnegative on P, and every equation $\beta_a = 0$ defines a facet.

Proof. Every neighbor of u_a other than its classmates has quotient proportional to $[X_{u_a}]$, by (2). The star spans the quotient, so every classmate must be a neighbor. Normalize the quotient basis by $\beta_b(X_{u_a}) = \delta_{ab}$. For $b \neq a$, this functional is nonnegative on all edge rays at u_a and hence on P; varying a proves nonnegativity of every β_b.

The sum $\sum_{b \neq a} \beta_b$ exposes R_a. At u_a, only the $r-1$ classmate edges leave this face, so its dimension is at least $d - r + 1 = t$, and the defining projective subspace gives the reverse inequality. At a different u_b, only the edge to u_a can leave $\beta_a = 0$, making that face a facet. Their common zero set is F. □

If F is empty, the positive denominator $\sum_a \beta_a$ sends D to infinity. Formula (1) then makes all homothety scales equal. After subtracting their common homothety, the image has dimension at most the number of classes minus one. This is precisely the situation handled by Section 3.

For the other case, suppose the $t+1$ parameters are affinely independent and a minimum class exists, so Lemma 9 applies. Now $s = f_0(F)$, and a central vertex belongs to at most two classes. Let $I(z)$ be the set of classes containing a central vertex z. Central-to-noncentral edges can only join vertices in the same class, while crossing edges between noncentral vertices preserve their quotient direction. Each of the r faces R_a supplies at least $t - \deg_F(z)$ neighbors of z outside F. Counting these distinct neighbors gives

$$\sum_{i \in I(z)} e_i \geq r\big(t - \deg_F(z) - |I(z)|\big). \tag{3}$$

This inequality measures the cost of a central vertex not having enough neighbors in the central face.

Lemma 10 (A common central class)**.** *Suppose $r \geq 2$, every central vertex belongs to one class S_h, and*

$$\sum_{i \neq h} e_i \leq t - 2.$$

Then P has Shephard's property.

Proof. Among the other t classes, choose one with $e_i = 0$. Its vertices are simple; there are at most $r-1$ neighbors within the class and at most one towards each other parameter. At one such vertex take the facet omitting its edge towards S_h. This facet contains the whole minimum class and all difference centers between the other classes. Propagating along their crossing matchings puts at least r vertices of every class S_i, $i \neq h$, into the facet.

An omitted vertex of such a class has at most $r + e_i - 1$ neighbors in its own class, at most $\sum_{j\neq h,i} e_j$ in the other noncentral classes, and at most one towards S_h. Its degree is therefore at most

$$(r + e_i - 1) + \sum_{j\neq h,i} e_j + 1 = r + \sum_{j\neq h} e_j < d,$$

which is impossible. All classes other than S_h lie in the facet, and each of their vertices has its unique S_h neighbor outside. Facet connectivity then excludes any S_h vertex inside the facet. This is Shephard's property. □

5. **Three homotheties**

The class bound alone does not settle a cover by three homotheties. Here their difference directions form one projective line, and two-faces control the three possible crossing families.

Proposition 11. *If $d \geq 3$, $N \leq 4d-5$, and a nontrivial parallel map is covered by at most three homotheties, then P is universally decomposable.*

Proof. Pass to essential full classes and assign each shared vertex to one of its classes, obtaining a disjoint coloring. Two colors are covered by Lemma 5. If a crossing family is absent, say 01, subtract H_2 and split the resulting map into its restrictions supported on colors 0 and 1. Both are parallel and at least one is nontrivial, giving a two-class map. If the three parameters are collinear, replace them by two values on their common line; every crossing edge still respects the same center, so this also reduces to two classes.

We may therefore assume that the parameters are affinely independent, that all three crossing families are nonempty, and that their distinct centers lie on the line $\Pi = \mathbb{P}(D)$. Lemma 8 gives at least $r = d-1$ noncentral vertices of each color. If Π misses P, a positive projectivity sends it to infinity and makes the image at most two dimensional. Lemma 4, Lemma 5, and Proposition 7 apply. Henceforth assume $\Pi \cap P \neq \varnothing$.

Forcing an exterior center. An exterior center makes its crossing family a matching. For an edge off Π, at most one incident two-face has plane containing Π. Every other incident two-face uses just this family; another family would put two distinct centers in its plane. Such a face contains exactly two members of the matching. Distinct faces give distinct partner edges, so the off-line cofaciality graph has minimum degree $d-2$.

Once Π contains $s \geq 1$ vertices, an off-line family with m edges uses only noncentral vertices, whence

$$2m + r + s \leq N.$$

For $d \geq 4$, this gives $m \leq \lfloor (3d-5)/2 \rfloor \leq 2d-5$, and Lemma 2 forces its center in every realization. For $d = 3$, $m \leq 2$ and minimum degree one gives two cofacial lines, so they too have a forced intersection. The new center remains exterior, since two disjoint actual edges cannot meet in P. We use this argument only after showing that Π contains a vertex.

A center is a vertex. There cannot be two vertex centers; their owner pairs share a color, and deleting those two vertices disconnects that color's noncentral vertices from the others, contrary to Balinski's theorem. Suppose the center of 01 is a vertex u, assigned to color 0. If it has no color-0 neighbor, assign it to color 1 instead; the family 01 disappears and we are done. Otherwise u has neighbors of both colors 0 and 1.

It has no color-2 neighbor. Such a neighbor v would lie on Π, by (2), and would be unique. Delete its vertex from the vertex figure at u. The remaining graph is connected and contains both colors 0 and 1, so has an edge joining them. Its corresponding two-face contains one 01 edge. Parity around its boundary then forces an edge of each other crossing family, so its plane contains Π and hence the edge uv. But the two edges of this face at u were chosen to avoid uv, a contradiction.

The other centers must be exterior. Indeed, a center in the interior of an edge has that edge as its sole crossing edge. Parity in two incident two-faces puts Π on its line. The vertex u would then be an endpoint of that edge, giving the forbidden color-2 neighbor. Thus both nonstar families lie off Π, and the preceding matching argument forces their centers.

The vertex-figure graph is two-connected, so there are at least two edges between its color-0 and color-1 parts. The corresponding two-face planes each contain u and an edge of both nonstar families. In every realization their intersection therefore makes u and the two reconstructed centers collinear. Three distinct collinear centers give compatible homothety differences. If the two exterior centers coincide, merge colors 0 and 1. Either way a nontrivial parallel map is reconstructed.

No center is a vertex. If a center lies in the interior of an edge e, that family consists only of e. Every two-face through it uses the other two families by parity, and two such planes show that Π is the line of e. The other centers are exterior, since the same actual edge cannot belong to two families.

The same configuration is forced if all centers are exterior. The graph of crossing edges is a forest. To prove this, a cycle has one constant nonzero quotient direction modulo D, so lies in a plane containing Π. As a cycle of actual polytope edges, it bounds the section of P by that plane. Yet every supporting edge line meets Π outside P, which is incompatible with Π meeting that section. Thus there is no cycle.

Each crossing component spans at most one quotient direction, so there are at least r components. Any family containing an off-line edge has at least r edges, by its cofaciality degree. If all three did, the forest would have at least $3r + r = 4d - 4$ vertices. Hence some family consists of the unique edge e on Π.

In both cases, Π contains the endpoints of e, and the matching argument forces the other two centers in every realization. Each two-face through e contains an edge of both other families, so two such planes put their centers on the line of e. Choose the two homothety differences with these centers. Their third difference is also centered on e, and therefore respects the sole remaining crossing edge. Coincident centers give a two-color map. All constructed maps are nontrivial because each color has at least $r \geq 2$ vertices and the homotheties do not all agree. □

6. **The uniform parameter count**

With four or more classes, the vertex budget is strong enough to force full affine independence of the parameters. Equation (3) then leaves only two possible sizes for the central face.

Proposition 12. *Suppose $N \leq 4d - 5$ and a nontrivial parallel map has an essential cover by m homotheties, where $4 \leq m \leq d - 1$. Then either a positive projectivity gives an image of dimension at most $m - 1$, or P has Shephard's property.*

Proof. If $t \leq m - 2$, Lemma 8 gives

$$N \geq m(d - m + 3) \geq 4d - 4,$$

since the concave quadratic has value $4d - 4$ at both endpoints of $4 \leq m \leq d - 1$. Thus $t = m - 1$, $r \geq 3$, and

$$E + s \leq t, \qquad mr + t - (4d - 5) = (t - 3)(r - 3) \geq 0.$$

Some class has no excess, so the fiber faces exist. If $F = \varnothing$, the projective conclusion follows from Section 4.

Assume $F \neq \varnothing$. If $s \leq t-2$, inequality (3), with $\deg_F(z) \leq s-1$ and $|I(z)| \leq 2$, gives

$$E \geq r(t-s-1) > t-s \geq E,$$

a contradiction. Hence $s = t-1$ or $s = t$.

The case $s = t-1$. Now $E \leq 1$, and (3) forces every central vertex to be shared and to have degree $t-2$ in F. Thus F is a clique and $f|_F$ is a homothety H. If H were not a given parameter, the ownership sets at different vertices would be disjoint, and at most one vertex could be shared; two lines joining disjoint pairs of affinely independent parameters cannot intersect at H. Since $s \geq 2$, this is impossible. Therefore H is a common central class and Lemma 10 applies.

The case $s = t$. Here $E = 0$ and $\delta(F) \geq t-2$. Lemma 2 makes F rigid for $t \geq 5$, and the case $t = 3$ is also rigid. For $t = 4$, any triangle absorbs the remaining vertex, leaving only a central four-cycle as an exception.

First suppose $f|_F = H$. If H is a given parameter, use Lemma 10. Otherwise the ownership sets are disjoint and at most one vertex is shared. A singly owned vertex has degree $t-1$ by (3); a missing edge would therefore have two shared endpoints. It follows that F is complete.

If no vertex is shared, one class A owns no central vertex. Its r vertices supply the fiber faces, each with one noncentral vertex of every color. In each fiber face the A vertex is universal, and the other t vertices have minimum degree $t-2$, hence form a connected graph. The cone over a spanning tree is rigid. The noncentral vertices of that fiber form the complement of the facet $\beta_a = 0$, so Lemma 3 makes that facet Shephard.

If a central vertex is shared by classes a, b, all other central vertices are singly owned. The minimum a- and b-fiber systems coincide, since their matchings exhaust any singly owned class. Their two vertices in each fiber share the central vertex and the other $t-1$ row vertices. This contains a rigid $K_{2,3}$, forcing $p_a = p_b$, contrary to essentiality.

It remains that $t = 4$ and F is a four-cycle. Every central vertex is shared by (3). Along the connected graph of F, identity (2) puts its owner parameters in one translate of $D_F = \operatorname{span} X(F)$, of dimension three. Thus at most four classes occur. Their owner pairs are distinct and triangle-free, since a triangle of pairs would give three collinear polytope vertices. They form the cycle $12, 23, 34, 41$, leaving one class A unowned. Disjoint owner pairs cannot be adjacent in F, because the edge

homothety would meet two skew parameter lines; hence this is also the physical cyclic order.

All noncentral vertices are simple. At an A vertex take the facet omitting its edge towards class 1. Its homogeneous span is

$$W = \operatorname{span}\big(X(U_A), p_2 - p_A, p_3 - p_A, p_4 - p_A\big), \tag{4}$$

where $U_i = S_i \setminus F$. This facet contains exactly

$$U_A \cup U_2 \cup U_3 \cup U_4 \cup \{z_{23}, z_{34}\}.$$

Vertices in A and 3 have their unique exterior neighbor in class 1; those in 2 and z_{23} have it at z_{12}; those in 4 and z_{34} have it at z_{41}. The facet is therefore Shephard. □

7. **The two boundary covers**

The range $m \le d - 1$ leaves two small configurations that can occur in dimensions four and five. Both have the same seven-vertex residue. Let Σ consist of two triangles C, D meeting at a vertex z, and a disjoint edge J.

A seven-vertex graph of minimum degree three with no rigid set of four vertices has such a cover. Its degrees are three or four, and odd order forces a degree-four vertex z. Equality in the neighborhood count used below makes its neighbor graph $2K_2$. Its two nonneighbors are adjacent and each meets one endpoint of each K_2, giving C, D, J.

Lemma 13. *Suppose a nontrivial parallel map is covered, in dimension four, by a rigid four-set A and Σ, or, in dimension five, by two disjoint rigid four-sets A, B and Σ. Then P is universally decomposable.*

Proof. Fewer essential parameters are covered by Propositions 11 and 12, so assume that all four or five parameters are essential and distinct. The shared vertex z is central. If $t \le d - 2$, the class bound would require at least $3d$ noncentral vertices in addition to z, exceeding $4d - 5$. Hence $t = d - 1$ and $r = 2$.

The noncentral parts of C, D, J have exactly two vertices, so all excess lies in A, or in A, B. Lemma 9 applies and the central face contains z.

For $d = 4$, the budget is $E + s = 3$. At z, inequality (3) gives $\deg_F(z) \ge 1$, so $s \ge 2$. If $s = 2$, the homothety on that edge is a common central class; otherwise its other endpoint would be singly owned and require excess at least two, whereas $E = 1$. The common class is C or D, and Lemma 10 applies. If $s = 3$, the other two central vertices belong to A; the homothety on the central triangle would be H_A, giving three owners at z, which is impossible.

For $d = 5$, we have $E + s = 5$ and $\deg_F(z) \geq 2$, so $s \geq 3$. Suppose $s = 3$. The central triangle has a homothety H. If H is not a given parameter, its three ownership sets are disjoint. The vertex z uses C, D, and the other two central vertices belong respectively to A, B. Each original four-set has only one central vertex, so $e_A, e_B \geq 1$. At most one of those two central vertices can also belong to J, leaving a singly owned vertex. Inequality (3) requires excess at least two in its class, whence $E \geq 3$, contrary to $E = 2$. Thus H is a common central class, and Lemma 10 applies.

Suppose $s \geq 4$. If F were homothetic, one of A, B would own two of its vertices and hence its homothety, again giving three owners at z. Therefore F is nonrigid. A face of dimension at least three with at most five vertices is rigid by Lemma 2, so F is a quadrilateral or pentagon.

Both A and B own central vertices; each original four-set has at least two noncentral vertices, and neither can account for all $s - 1 \geq 3$ centers other than z. Together with C, D, these are four owner colors. The three-dimensional space D_F allows no fifth, leaving J unowned. Since $\deg_F(v) = 2$ and $E \leq 1$, equation (3) makes every central vertex shared. Their distinct owner pairs form a triangle-free graph on four colors, forcing $s = 4$ and the four-cycle from the proof of Proposition 12.

The sole excess vertex belongs to A or B; call this color 1 and use J as the unowned class. The hyperplane (4) again gives a Shephard facet, containing the other three classes and the two central vertices z_{23}, z_{34}. Its two ordinary color-1 vertices lie outside. If the possible third color-1 vertex were inside, it would have no neighbor in the facet; its classmates and incident centers are outside, and every other crossing edge has center outside the hyperplane. Facet connectivity excludes it. This completes both boundary cases. □

8. **Rigid covers and completion of the proof**

We finally show that the graph always supplies one of the covers already treated. Choose a largest rigid set in each remaining induced graph. Each removal lowers the minimum degree by at most one, because an outside vertex with two neighbors in the removed set would have been absorbed into it.

Lemma 14. *A graph of minimum degree three on at most seven polytope vertices is covered by three rigid sets; on at most six vertices, two suffice.*

Proof. There is a rigid set with at least three vertices. Otherwise the graph would be triangle-free and $K_{2,3}$-free. At a vertex of degree $D \geq 3$, its independent neighbors require at least $2D$ outgoing edges, whereas each other vertex can meet at most two

of them. Thus $D \le (n-1)/2$. For $n \le 7$ this forces a three-regular graph on seven vertices, impossible by parity.

Enlarge a rigid set maximally. Its complement has minimum degree two and at most four vertices. A three-vertex complement is a triangle, and a four-vertex complement has a perfect matching and is covered by two rigid edges. This proves the claims. □

The counting estimate used in the remaining argument is elementary but effective. If all rigid sets have size at most three, a degree-D vertex in a graph of minimum degree k satisfies

$$D(k-1) - 2\lfloor D/2 \rfloor \le 2(n-D-1). \tag{5}$$

Indeed, its neighbor graph has components of size at most two, while every other vertex meets at most two of those neighbors, by rigidity of $K_{2,3}$. Thus $n \ge 1 + k^2/2$. If rigid sets have size at most four, the same argument, with neighbor components of size at most three, gives

$$D(k-1) \le 2(n-1). \tag{6}$$

8.1. **The four-regular tail**

A graph of minimum degree four on at most ten remaining vertices is covered by three rigid sets, except for a configuration that cannot occur here. If it has a rigid four-set, remove it and apply Lemma 14. Otherwise (5) makes it four-regular and forces at least nine vertices. On nine vertices, equality gives a triangle; removing it leaves six vertices of minimum degree three and again gives a three-set cover.

Suppose the tail has ten vertices. Every vertex lies in one or two triangles. Some vertex lies in exactly one, since twenty triangle incidences cannot be divisible by three. Equality in (5) then makes each of its five nonneighbors adjacent to two vertices of its star. All four-cycles are planar, since a nonplanar one would be rigid. Thus the entire tail lies in the four-flat spanned by this star.

At this stage every tail vertex is simple in the original polytope; it has four tail neighbors and at most one neighbor in each of the $d-4$ removed sets. At the chosen vertex, the four tail edges therefore determine a four-face in that flat. All tail vertices lie in this face, and their four tail edges exhaust their possible edges within it. Face connectivity excludes any earlier vertex. We have obtained a simple four-polytope with ten vertices.

No such polytope exists. Its simplicial dual has ten facets, so if it has m vertices, Dehn–Sommerville gives $f_1 = m + 10$. Barnette's bound gives $m + 10 \ge 4m - 10$, hence $m \le 6$, contrary to $m + 10 \le \binom{m}{2}$. This excludes the ten-vertex tail.

8.2. **Dimensions at least six**

For a nonsimple d-polytope with $d \geq 6$ and $N \leq 4d-5$, a largest rigid set has at least five vertices. Otherwise (6) at a nonsimple vertex gives

$$d^2 - 1 \leq 8d - 12,$$

which is impossible for $d \geq 7$. For $d = 6$ it forces degree seven; the neighbor partition $3+3+1$ then requires at least $35-12 = 23$ outgoing edges, while the other vertices have capacity only 22.

After removing this first set, the residual graph has minimum degree $k = d-1$ and at most $4k-6$ vertices. For $k \geq 6$, inequality (5) forces a rigid set of size at least four. At $k = 5$, failure would force a five-regular graph on fourteen vertices, each neighborhood consisting of $2K_2$ and one isolated vertex. Every vertex would lie in two triangles, giving twenty-eight triangle incidences, again impossible. Continue removing at least four vertices until $k = 4$.

At most ten vertices now remain, and the preceding tail argument covers them by three rigid sets. In total we have at most $(d-4)+3 = d-1$ sets. A nontrivial parallel map is homothetic on each, so Propositions 11 and 12 apply. In the proper-image alternative, its dimension is at most $m-1 \leq d-2$, and Section 3 gives universal decomposability. In the other alternative the Shephard facet does so; the pyramid exception is excluded by the nontrivial map. Simple types were already settled by Shephard's criterion.

8.3. **Dimensions four and five**

For $d = 4$, inequality (5) gives a rigid set R of size at least four in every nonsimple polytope on at most eleven vertices. If $|R| \geq 5$, Lemma 14 covers its complement by two sets. If $|R| = 4$, the same holds unless the complement has seven vertices and no rigid four-set. That is precisely the configuration Σ from Section 7. Thus either Proposition 11 or Lemma 13 applies.

For $d = 5$, the same inequality gives a rigid set R of size at least four. If $|R| \geq 5$, its complement has at most ten vertices and minimum degree four; the tail argument covers it by three sets. The excluded four-regular tail is globally simple after this single removal, so the same four-face contradiction applies.

Suppose $|R| = 4$. If all residual rigid sets have size at most three, (5) makes the residual graph four-regular, so all its vertices are simple in P. All excess lies in R, and

$$\xi(P) \leq 12 + (N-4) - 20 \leq 3.$$

The excess theorem gives Shephard's property; a pyramid is again impossible. Otherwise remove a largest residual rigid set S of size at least four. If $|S| \geq 5$, the remaining at most six vertices need two further sets. If $|S| = 4$, the remaining at most seven vertices need two sets unless they form Σ. We obtain either four classes, handled by Proposition 12, or the second boundary cover of Lemma 13.

Proof of Theorem 1. In dimension three, Lemma 14 supplies at most three rigid pieces, and Proposition 11 applies. The preceding reductions settle every larger dimension. Thus below $4d - 4$ vertices, any decomposable realization forces decomposability of the whole combinatorial type. Wang and Yost's construction [5] gives a conditionally decomposable type with exactly $4d - 4$ vertices in every dimension $d \geq 3$. □

References

[1] M. L. Balinski, *On the graph structure of convex polyhedra in n-space*, Pacific J. Math. **11** (1961), 431–434.

[2] D. Barnette, *A proof of the lower bound conjecture for convex polytopes*, Pacific J. Math. **46** (1973), 349–354.

[3] M. Kallay, *Indecomposable polytopes*, Israel J. Math. **41** (1982), 235–243.

[4] G. Pineda-Villavicencio, J. Ugon and D. Yost, *The excess degree of a polytope*, SIAM J. Discrete Math. **32** (2018), 2011–2046.

[5] J. Wang and D. Yost, *Lower bound results for conditionally decomposable polytopes*, Adv. Geom. (2026), DOI 10.1515/advgeom-2026-0006.