Six-connected K_4-covered graphs are rigid in three dimensions

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Abstract. Every six-connected graph whose edges lie in copies of K_4 is generically rigid in three dimensions, and the connectivity threshold is sharp. The result extends to graphs whose edges are covered by arbitrary generically rigid subgraphs of order at least four. In a minimal counterexample, integrability of Euclidean body motions forces a rank loss of at least seven when a vertex shared by three or more rigid pieces is deleted; a global count of transmitted stresses then rules out the remaining flex. An exact defect formula for generic pair-pin frameworks supplies the independence needed in this count.

1. Introduction

A graph assembled from tetrahedra can still flex at the intersections of its rigid pieces. We determine exactly how much vertex connectivity eliminates this freedom. A graph is K_4-*covered* if each edge lies in a complete subgraph on four vertices. Jackson, Jordán and Villányi [1] proved that seven-connectivity forces such a graph to be generically rigid in $\mathbb{R}^3$ and exhibited flexible five-connected examples. We prove their conjecture that the threshold is six.

Theorem 1.1. *Every six-connected K_4-covered graph is generically rigid in $\mathbb{R}^3$.*

Since completing a rigid subgraph adds only edges in its rigidity closure, the same threshold applies whenever the edges are covered by arbitrary rigid pieces.

Corollary 1.2. *Let G be six-connected. If every edge of G lies in a generically rigid subgraph of order at least four, then G is generically rigid in $\mathbb{R}^3$. More generally, it is enough that the union of the cliques obtained by completing a family of rigid clusters of order at least four is a six-connected spanning graph.*

Proof from Theorem 1.1. The completed spanning graph is K_4-covered and has rigidity rank at most that of G. Theorem 1.1 makes it rigid, and hence makes G rigid. □

The proof begins with an edge-maximal counterexample of least order. Its maximal rigid cliques of order at least four, called *bodies*, meet in at most two vertices, and

a rank-contribution argument reduces all flexibility to a single mode. At a vertex shared by three or more bodies, the compatible infinitesimal rotations must arise from actual Euclidean motions. Their Maurer–Cartan equations impose the integrability conditions that force a rank loss of at least seven when this vertex is deleted.

At intersections of two bodies, the remaining force transfers resolve into hinge forces and scalar forces directed toward virtual centers. After separating these transfers from internal body stresses, the edges of tetrahedral bodies identify the scalars in a linear code. A covering inequality bounds its dimension by the information visible at larger bodies and multiply shared vertices; the corresponding local rank contributions then contradict the rank of a flexible graph. The independence needed to control these scalar forces comes from an exact defect formula for pair-pin frameworks. Section 5 derives that formula, including rigid triangles, from Zheng's sparse Pfaffian codimension theorem [2].

For the rank calculations, write $r(G)$ for the rank of the generic three-dimensional rigidity matrix of $G = (V, E)$. At a generic placement p, its stress space is

$$\mathcal{S}(G) = \left\{ \omega \in \mathbb{K}^E : \sum_{w : vw \in E} \omega_{vw}(p_v - p_w) = 0 \quad (v \in V) \right\},$$

so $r(G) = |E| - \dim \mathcal{S}(G)$. We may take $\mathbb{K}$ to be the rational function field in the coordinates of p. The differential arguments use a real generic placement and a neighborhood on which the relevant ranks are constant. All connectivity statements mean vertex connectivity.

2. Rank contributions and supported stresses

Inserting the vertices of G in a uniformly random order splits its rigidity rank into expected increments. The increment when v enters is its *rank contribution*, denoted $r_c(v)$, and these contributions sum to $r(G)$. To estimate an increment, we bound the stresses that can first appear at that vertex. The next lemma also applies to proper subspaces of the stress space; its marker represents the arrival of v, and its coordinates represent the incident edges, whose nonzero stress support must lie among the preceding neighbors.

Lemma 2.1. *Let $s \geq 1$ be an integer and let $W \subseteq \mathbb{K}^d$ have dimension r. Suppose every nonzero vector of W has at least s nonzero coordinates. Give the coordinates and one extra marker independent uniform priorities, and let A consist of the coordinates*

preceding the marker. Then

$$\mathbb{E}\dim(W \cap \mathbb{K}^A) \le \Phi_s(r) := \frac{r(r+1)}{2(r+s)}.$$

The bound is sharp over an infinite field.

Proof. For $r = 0$ there is equality, so assume $r > 0$. The coordinate functionals on W form a rank-r representable matroid M of cogirth at least s. The supported subspace $W \cap \mathbb{K}^A$ is the kernel of projection onto the coordinates outside A, so its dimension is $r - \operatorname{rank}_M(A^c)$. Conditional on the marker, A^c is an independent random subset with a uniform inclusion probability $p \in [0,1]$. Give the elements of M independent rate-one exponential clocks, and let $k(t)$ be the rank still missing at time t. The clock subset has inclusion probability $p = 1 - e^{-t}$; changing variables in the integral over p therefore gives

$$F(M) = \mathbb{E}\int_0^\infty e^{-t}k(t)\,dt.$$

If g is the number of nonloops, then $g \ge r + s - 1$; extending $r - 1$ independent elements to a hyperplane gives a cocircuit of size at most $g - r + 1$. The first nonloop clock has rate g, and its index is uniform. Before that clock rings, the missing rank is r; afterwards the remaining process is the one for the contraction by its element. Memorylessness therefore gives

$$F(M) = \frac{r + g\,\mathbb{E}_e F(M/e)}{g+1}.$$

Contraction preserves the lower bound on cogirth. Induction on r now applies, and the last expression decreases with g, since $\Phi_s(r-1) < r$. Consequently

$$F(M) \le \frac{r + (r+s-1)\Phi_s(r-1)}{r+s} = \Phi_s(r).$$

All inequalities are equalities for the uniform matroid $U_{r,r+s-1}$, represented over an infinite field by an MDS code. □

Corollary 2.2. *Let a fixed subspace of $\mathcal{S}(G)$ have star image of dimension R at v. Its expected induced-prefix dimension increase at v is at most $\Phi_4(R)$. In particular, if $d(v) = \deg_G(v)$ and $\delta(v) = r(G) - r(G - v)$, then*

$$r_c(v) \ge \frac{d(v)}{2} - \Phi_4\big(d(v) - \delta(v)\big).$$

Proof. The new supported stresses modulo the old ones inject into the global star image, with support among the preceding neighbors. A nonzero equilibrium stress at

a generic vertex has at least four nonzero star coordinates, so Lemma 2.1 applies. For the full stress space, the star image has dimension $d(v) - \delta(v)$, while the expected number of newly inserted edges is $d(v)/2$. □

3. **A minimal flexible graph**

Suppose that G is a flexible six-connected K_4-covered graph with as few vertices as possible and, subject to this, as many edges as possible. Set

$$\delta(v) = r(G) - r(G - v), \qquad f = 3|V(G)| - 6 - r(G) > 0.$$

A *body* is a maximal rigid clique of order at least four. We use the standard gluing, coning and planar rigidity facts in [1, Section 2].

Lemma 3.1. *Every rigid cluster of G of order at least four induces a clique. The graph has no simplicial vertex, and for every v the graph*

$$H_v = G - v + K(N_G(v))$$

is rigid. Moreover, $\delta(v) \geq 5$, every vertex belongs to at least two bodies, and distinct bodies intersect in at most two vertices.

Proof. Completing a rigid cluster preserves rank, six-connectivity and K_4 coverage. Edge maximality gives the first assertion. Every vertex has degree at least six. If v were simplicial, then $G - v$ would remain six-connected; after deleting at most five other vertices, the surviving neighbors of v form a nonempty clique, so adding v cannot join different components. The graph $G - v$ is also K_4-covered, since the neighbor clique replaces any covering tetrahedron that used v. Vertex minimality would make $G - v$ rigid, and three generic bars from v would make G rigid.

The same separator argument shows that H_v is six-connected. Completing the neighbor clique preserves K_4 coverage, including for new edges. Thus H_v is rigid by minimality. The nonsimplicial case of [1, Lemma 8.23] gives $\delta(v) \geq 5$. Every neighbor of v lies with v in a body. A vertex in just one body would therefore be simplicial. Finally, two rigid bodies sharing three generic vertices have rigid union, so maximality bounds their intersection by two. □

We shall repeatedly use the following planar observation.

Lemma 3.2. *For any $b > 0$ bodies through v, put*

$$I = \sum_C |C \setminus \{v\}|, \qquad N = \left| \bigcup_C (C \setminus \{v\}) \right|.$$

Their planar rigidity bases in $G[N_G(v)]$ have independent union, so $2(I - N) \leq 3b - 3$.

Proof. Each $C \setminus \{v\}$ is a maximal planar rigid component of the neighborhood; a larger one would cone with v to a larger rigid cluster. A circuit in the union of the chosen bases is planar rigid. Whenever it meets a basis in an edge, it glues rigidly to that component along the edge's endpoints. It cannot lie in one basis, and meeting two would merge distinct maximal components. The union is therefore independent, whence $2I - 3b \leq 2N - 3$. □

Lemma 3.3. *A degree-six vertex belongs to exactly two bodies. They are either two K_4's meeting only at that vertex* (type A)*, or a K_5 and a K_4 sharing an edge through it* (type B)*.*

Proof. The neighborhood is not planar rigid, since its cone would make v simplicial. Its planar rank is thus at most eight. By Lemma 3.2, each incident body contributes at least three independent neighborhood edges, so there are at most two bodies. There are at least two by Lemma 3.1. Their neighbor cliques cover six vertices and meet in at most one. Their sizes are therefore $3, 3$ with empty intersection, or $4, 3$ with one common vertex. □

3.1. **Alternating forms**

Represent the infinitesimal rigid motion $x \mapsto \Omega x + t$ by

$$S = \begin{pmatrix} \Omega & t \\ -t^{\mathsf{T}} & 0 \end{pmatrix}.$$

For $p = (x, 1)$, one has $Sp = (\Omega x + t, -x \cdot t)$. Thus two body motions agree at a pin p precisely when their difference annihilates p. The space of alternating forms in four variables has dimension six. Its Pfaffian is a nondegenerate quadratic form of Witt index three; a nonzero singular form has rank two and a projective kernel line. For a projective point a, let L_a be the three-dimensional space of forms annihilating a.

Lemma 3.4. *Let W be a space of alternating forms in four variables. If $\dim W \geq 3$ and contraction $S \mapsto Sp$ has rank at most two for generic $p \in \mathbb{P}^3$, then $W = L_a$ for a unique projective point a. A space of dimension at most two has no nonzero member annihilating a generic p.*

Proof. Write $d = \dim W$. If the Pfaffian is not identically zero on W, its zero locus in $\mathbb{P}(W)$ has dimension at most $d - 2$. Since each singular form has a kernel line, the incidence of pairs $([S], p)$ with $Sp = 0$ has dimension at most $d - 1$. The rank

hypothesis instead gives fibers of dimension at least $d-3$ over generic $p \in \mathbb{P}^3$, hence dimension at least d.

Thus W is totally isotropic, so $d = 3$. Such a space of decomposable two-forms either consists of forms containing a fixed covector, or is $\Lambda^2 U$ for a three-dimensional covector space U. Indeed, start with $\ell_1 \wedge \ell_2$ and $\ell_1 \wedge \ell_3$. A form wedging to zero with both either contains ℓ_1 or belongs to $\Lambda^2\langle \ell_1, \ell_2, \ell_3\rangle$; if one does not contain ℓ_1, all belong to this exterior square. The first alternative has all kernel points in the hyperplane $\ell_1 = 0$ and cannot account for generic p. The second is L_a, with a the annihilator of U.

For $\dim W \leq 2$, the projective singular forms have dimension at most one. Their kernel lines fill a set of dimension at most two. □

Proposition 3.5. *The minimal counterexample has $f = 1$.*

Proof. At a type-A vertex v, work over the coordinate field with p_v omitted. A motion of $G - v$ restricts to rigid motions on its two incident body triangles. Subtract the first and retain the relative motion of the second. This map, modulo common rigid motions, is injective; if both triangles move together, all neighbors move rigidly, and the rigidity of H_v makes the entire motion common. Call its image W.

If the star of v has a nonzero stress image, then $\delta(v) = 5$. Consequently $\dim W = f + 2$, and reintroducing the generic pin p_v imposes $\delta(v) - 3 = 2$ conditions on W. Lemma 3.4 gives $\dim W = 3$ and $f = 1$. At a type-B vertex, the relative motion in G is a rotation about the common hinge. The same injectivity gives $f \leq 1$.

Suppose $f \geq 2$. There are no type-B vertices, and each degree-six vertex has $\delta(v) = 6$. Its incident edges are rigidity-matroid coloops, so $r_c(v) = 3$. If $d(v) = d \geq 7$, Corollary 2.2 and $\delta(v) \geq 5$ give

$$r_c(v) \geq \frac{d}{2} - \Phi_4(d-5) = 4 - \frac{6}{d-1} \geq 3.$$

Summing gives $r(G) \geq 3|V(G)|$, a contradiction. □

3.2. **Point pins and virtual centers**

A vertex in exactly two bodies meeting only there is a *point pin*. It is *soft* if $\delta(v) = 5$. A vertex in exactly two bodies meeting in an edge is a *hinge pin*. A vertex in at least three bodies is a *hypervertex*.

Lemma 3.6. *For a soft point pin v, the relative motions of its two bodies in $G - v$ form L_{a_v}, where a_v is defined over the coordinate field omitting p_v. The flex of G is*

a rotation about $a_v p_v$. The point a_v is not another actual vertex of either incident body. In a finite-center chart, a force at v annihilating all these relative motions is a scalar multiple of $p_v - a_v$.

Proof. As in Proposition 3.5, the relative-motion map is injective and has dimension $f + \delta(v) - 3 = 3$. Generic evaluation has rank two, so Lemma 3.4 applies before p_v is chosen. If a_v were another actual vertex of either body, that vertex would be fixed relative to the other body in every motion. It would enlarge a maximal rigid cluster, contrary to closure. Finally, $F \cdot (\omega \times (p_v - a_v)) = 0$ for every ω precisely when F is parallel to $p_v - a_v$. □

Lemma 3.7. *At a type-A vertex with nonzero star-stress image, that image is one-dimensional and each nonzero vector has at least five nonzero coordinates.*

Proof. Its dimension is $6 - \delta(v) \leq 1$. Generic equilibrium excludes support at most three. If a stress uses exactly four edges at v, retain those edges and all of $G - v$. This extension has rank increment three. By [1, Lemma 8.3], its four neighbors form a rigid cluster in $G - v$. Adding v gives a rigid K_5 after closure, contrary to type A. □

4. The rank lost at a hypervertex

Proposition 4.1. *Every hypervertex u satisfies $\delta(u) \geq 7$.*

Closure already gives $\delta(u) \geq 5$, so it remains to exclude five and six. Deleting u in these cases gives configuration spaces of dimensions three and four, whose tangent vectors are the compatible infinitesimal body motions. The pin equations at a single configuration still allow different bodies to have proportional rotations or rotate about different centers. Varying the configuration imposes a further condition; twists of actual body motions satisfy the Maurer–Cartan equations. The next lemmas extract the restrictions this imposes in each of the two possible dimensions.

Ground a triangle in one body after deleting a vertex u. The generic bar-length fiber of $G - u$ is locally a smooth manifold of dimension

$$f + \delta(u) - 3 = \delta(u) - 2.$$

Relative body configurations are smooth maps on this fiber. All generic rank and common-new-pin identities are rational identities in the coordinates of $G - u$. In particular, the coordinates of p_u are algebraically independent of those remaining after deletion. A rank condition obtained by putting u back is therefore an identity

for a generic new pin p, rather than a condition at one special point. These identities continue to hold on a neighborhood in the fiber.

We may make the finitely many common kernel points under consideration finite. Indeed, a fixed rational projective change $p' = Tp$ sends compatible alternating matrices to $S' = T^{-\mathsf{T}} S T^{-1}$ and preserves their pin equations. Choose T so that those points are finite at the initial placement. The transformed placement is still generic; we use its own Euclidean length fiber in what follows.

4.1. **Integrability of body motions**

We record the differential restrictions needed for fibers of dimension three and four. A body configuration is $x \mapsto Rx + t$, and its Maurer–Cartan form records the spatial infinitesimal twist.

Lemma 4.2. *If a three-dimensional submanifold of* $\mathrm{SE}(3)$ *has tangent twists equal to all rotations about a finite point a, then a is locally fixed in both the spatial and body frames.*

Proof. Put $b = R^{-1}(a - t)$. The angular projection is an isomorphism, so R gives local coordinates. Infinitesimal fixation of a says

$$dt = -dR\, b, \qquad da = R\, db.$$

Differentiating gives $(R^{-1}dR) \wedge db = 0$. Write $db = A\omega$ in angular Maurer–Cartan directions. Then

$$\omega \times A\nu = \nu \times A\omega \qquad (\omega, \nu \in \mathbb{R}^3).$$

The three pairs of coordinate vectors force the off-diagonal entries of A to vanish and its diagonal entries to be pairwise negatives. Thus $A = 0$, and $db = da = 0$. □

Proportional infinitesimal rotations have the same axes but may have different speeds at one configuration. Actual body motions restrict this freedom; rescaling a Maurer–Cartan form scales its exterior derivative linearly and its bracket term quadratically. The following two lemmas show that a nonzero proportionality factor must be one.

Lemma 4.3. *Let M be a four-dimensional configuration manifold with body Maurer–Cartan forms θ_1, θ_2. Suppose a rank-three distribution $D \subset TM$ maps isomorphically under θ_1 to the rotations about a finite point, and $\theta_2 = \lambda\theta_1$ on D, with $\lambda \neq 0$. Then $\lambda = 1$. The center may vary and D need not be integrable. The same conclusion holds on a three-dimensional manifold with $D = TM$.*

Proof. Locally write $D = \ker \beta$ and $\theta_2 - \lambda\theta_1 = Z\beta$. The angular Maurer–Cartan equations, restricted to $X, Y \in D$ with angular components ω, ν, give

$$(\lambda^2 - \lambda)(\omega \times \nu) - d\lambda(X)\nu + d\lambda(Y)\omega = \operatorname{ang}(Z)\, d\beta(X, Y),$$

up to a common sign convention. Write $d\lambda|_D = g \cdot \omega$. As an operator on $\omega \times \nu$, the left side is $(\lambda^2 - \lambda)I + \operatorname{cross}(g)$, of rank at most one. Its symmetric part forces $\lambda^2 - \lambda = 0$, since a rank-one matrix and its transpose have sum of rank at most two. A nonzero skew matrix of order three has rank two, so also $g = 0$. The three-dimensional case has zero right-hand side. □

Lemma 4.4. *If two relative-body maps on a four-dimensional manifold are immersions and their full Maurer–Cartan forms satisfy $\theta_2 = \lambda\theta_1$, with $\lambda \neq 0$, then $\lambda = 1$.*

Proof. If $\lambda \notin \{0, 1\}$, the Maurer–Cartan equations imply, on the four-dimensional twist image W, an identity

$$[X, Y] = \alpha(X)Y - \alpha(Y)X$$

for a linear functional α. An abelian subspace of $\mathfrak{se}(3)$ has angular projection of dimension at most one. If that projection is nonzero, its translational kernel lies on the rotation axis, so the subspace has dimension at most two. Hence the only three-dimensional abelian subspace is the translation space. This excludes $\alpha = 0$. If $\alpha \neq 0$, its three-dimensional kernel must be the translation space. Choosing X with $\alpha(X) \neq 0$ would make $\operatorname{cross}(\operatorname{ang} X)$ act on all translations as the nonzero scalar $\alpha(X)$, which is impossible. □

Lemma 4.5. *Two bodies rotating about distinct fixed centers cannot have a four-dimensional relative-configuration manifold such that, for a generic new pin p, some tangent motion with nonzero projection to each body fixes p in both bodies.*

Proof. Let the centers be a, b. Both angular projections of the tangent image $W \subseteq \mathbb{R}^3 \oplus \mathbb{R}^3$ are surjective, since they contain nonzero vectors parallel to $p - a$ and $p - b$ for generic p. If the combined image had dimension three, its projections would be isomorphisms; a linear map would send $p - a$ to a vector parallel to $p - b$ identically in p. Comparing quadratic and linear terms makes that map scalar and gives $a = b$. Thus the image has dimension four.

Write $W = \ker[A\ B]$, where A, B are 2×3 matrices of rank two. The new-pin condition gives

$$\det\big(A(p - a), B(p - b)\big) = 0.$$

The entries of $A(p-a)$ are relatively prime affine linear polynomials. It follows that $B(p-b) = cA(p-a)$ for a nonzero scalar c. Hence $B = cA$ and $A(b-a) = 0$. Writing $d = b - a \neq 0$, every tangent space has the form

$$W_c = \{(\omega_1, \omega_2) : \omega_1 + c\omega_2 \in \mathbb{R}d\}.$$

The fixed ambient rotation fields $(d, 0)$ and $(0, d)$ are tangent everywhere, so their flows preserve the manifold locally. Let Q be a small nontrivial rotation about d, and flow in the first field. In spatial angular coordinates, the differential sends (ω_1, ω_2) to $(Q\omega_1, \omega_2)$. The relation $\omega_1 = -c\omega_2$ on $d^\perp$ is consequently transported to $\omega_1 = -cQ\omega_2$. At the new point it must have the form $\omega_1 = -c'\omega_2$. A small nontrivial planar rotation is not a scalar operator, giving the contradiction. □

4.2. **The deletion bound**

Proof of Proposition 4.1. Ground one incident body after deleting u. No other incident body has zero relative motion in the surviving flex of G, since it would merge with the base body. Each relative-body projection therefore has dimension at least three; by Lemma 3.4, a smaller space cannot have a nonzero member annihilating generic p_u.

If $\delta(u) = 5$, the grounded fiber has dimension three. Every nonbase projection is an isomorphism onto some L_{a_i}. Make the centers finite. For two such projections, the common-pin kernel gives an invertible angular map T with

$$T(p - a_1) \;\parallel\; p - a_2$$

identically in p. Its quadratic terms make T scalar and its linear terms give $a_1 = a_2$; after writing $q = p - a_1$, the identity is $(Tq) \times (q + a_1 - a_2) = 0$, whose homogeneous quadratic part forces every vector to be an eigenvector of T. The full twists are therefore proportional. Lemma 4.3 makes them equal, contradicting maximality of the bodies.

Suppose now that $\delta(u) = 6$. The grounded fiber M has dimension four, and each nonbase projection has rank three or four.

A rank-three projection exists. Its image is L_a by Lemma 3.4, and its local configuration image has fixed finite center a by Lemma 4.2. Some second projection is nonzero on its one-dimensional kernel; otherwise all neighbor bodies would be fixed there, contrary to rigidity of H_u. The combined projection to these two bodies is consequently injective.

If the second projection also has rank three, its center b is fixed. Lemma 4.5 excludes $a \neq b$. For $a = b$, the same two-by-two determinant calculation gives

$$W = \{(\omega_1, \omega_2) : \omega_1 + c\omega_2 \in \mathbb{R}d\}, \qquad c \neq 0, \quad d \neq 0.$$

The rank-three distribution $\omega_1 + c\omega_2 = 0$ has proportional spherical twists. The generic-new-pin kernel lies in it, since both angular vectors are parallel to $p - a$ whereas d is fixed as p varies. Lemma 4.3 makes the two twists equal in the flex of G, again merging the bodies.

If the second projection has rank four, identify TM with its alternating-form image W_2, and let $A : W_2 \to \mathbb{R}^3$ be the first angular projection. Generic contraction on W_2 has rank three, so there is a unique projective kernel form $S(p)$. The common-pin condition says that $A(S(p))$ is nonzero and parallel to $p - a$. The image closure of $p \mapsto [S(p)]$ has dimension two, with fibers the kernel lines of its forms. For generic S in that image, all generic points of its kernel line yield the same $S(p)$. Since $A(S)$ is fixed, that line passes through a. The image is therefore the whole plane $\mathbb{P}(L_a)$, and $L_a \subset W_2$. On L_a, the map A is a nonzero scalar multiple of the identity on angular vectors. Lemma 4.3 applies to its rank-three preimage in TM, with the same contradiction.

Every nonbase projection has rank four. Choose two, with images W_1, W_2 and matching isomorphism $T : W_1 \to W_2$. Generic contraction on W_1 again has a unique projective kernel form $S(p)$. Its image closure Z is an irreducible surface, since a fiber is an open part of a kernel line. The Pfaffian does not vanish identically on the four-dimensional space W_1, so Z is a component of its quadric section in $\mathbb{P}(W_1)$. For generic $S \in Z$, every generic point of $\ker S$ also belongs to $\ker T(S)$. Rank-two alternating forms with the same kernel are proportional; hence $T(S) \parallel S$ on Z.

If Z spans $\mathbb{P}(W_1)$, then $T(W_1) \subseteq W_1$. Irreducibility places Z in one of the finitely many eigenspaces of T, so $T = \lambda I$ on W_1. Lemma 4.4 gives $\lambda = 1$. Otherwise Z is a plane of decomposable forms. The classification in Lemma 3.4 excludes the plane whose kernel points lie in a fixed hyperplane; therefore $Z = \mathbb{P}(L_a)$ and $T|_{L_a} = \lambda I$. Make a finite. Its rank-three preimage in TM satisfies Lemma 4.3, even if the center varies and the distribution is not integrable. Again $\lambda = 1$. In both cases the surviving flex makes the two bodies identical, contrary to maximality. This excludes $\delta(u) = 6$. □

5. **The defect of a pair-pin framework**

Let J be a finite loopless multigraph on a nonempty set of bodies, with each edge representing one generic pin shared by its two endpoints. A body has at least three

noncollinear vertices; adding private vertices does not change its compatible rigid motions. Put

$$\ell(0) = 0, \quad \ell(1) = 3, \quad \ell(2) = 5, \quad \ell(a) = 6 \quad (a \ge 3),$$

and, over partitions $\mathcal{P}$ of the body set, define

$$D(J) = \max_{\mathcal{P}} \left\{ 6(|\mathcal{P}| - 1) - \sum_{A<B\in\mathcal{P}} \ell\big(m_J(A,B)\big) \right\}. \tag{1}$$

The expression in this maximum measures the motions obtained by requiring all bodies in each block to move together. There are six degrees of freedom per block, less six common motions; one, two or three pins between two blocks remove at most three, five or six of them. The theorem says that these blockwise counts account for every generic dependence. Its final assertion will rule out a scalar force system that is supported on small bodies and has no adequate boundary.

Theorem 5.1. *The generic dimension of compatible body motions modulo common rigid motions is $D(J)$. If no two distinct bodies belong to one rigid cluster, the singleton partition is optimal. Under this additional hypothesis, if J is simple, all $3|E(J)|$ pin-constraint rows are independent.*

The upper bound rests on the following consequence of Zheng's null-difference height theorem [2, Theorem 1.3]. In the six-dimensional space of alternating forms in four variables, let $X_1, \dots, X_t$ be pairwise distinct. For a simple $(2,2)$-sparse graph F, the equations

$$\operatorname{Pf}(X_i - X_j) = 0 \qquad (ij \in E(F))$$

have codimension at least $|E(F)|$ on the distinct-point locus. Here $(2,2)$-sparse means that every nonempty vertex set spans at most twice its size minus two edges.

Proof. One, two and three generic pins impose respectively three, five and six conditions on a relative alternating form. For any partition $\mathcal{P}$, motions constant on its blocks therefore have dimension at least the expression in (1). This gives the lower bound.

For the upper bound, stratify twists by their equality partition, with $t \ge 2$ distinct blocks, and ground one block. A stratum dominating generic pin space has no pair of blocks sharing three pins; a nonzero singular alternating difference has only a projective kernel line. Hence crossing multiplicities are one or two. Let J_0 be the simple support graph on the blocks, let M be the number of crossing pins, and put

$$R = \max\{0, 6(t-1) - 2M - D(J)\}.$$

Each crossing pin will cost two dimensions in the incidence count, because it must lie on a kernel line in three-dimensional pin space. The number R is the additional codimension needed in the twist variables. We obtain it from independent Pfaffian equations. Since the crossing multiplicities are one or two, their total constraint count is $2M + |E(J_0)|$; this is the relation between the defect formula and the following sparse-graph calculation. We claim that J_0 contains a $(2,2)$-sparse set of R edges. The two-forest matroid-union formula is

$$r_2(J_0) = \min_{\mathcal{Q}} \{2(t - |\mathcal{Q}|) + s_{\text{cross}}(\mathcal{Q})\}.$$

For a partition $\mathcal{Q}$ into q parts, write s_{int} and s_{cross} for the numbers of internal and crossing support edges. If $s_{\text{int}} \le 2(t-q)$, the defect bound at the original equality partition gives

$$2M + s_{\text{cross}} \ge 6(t-1) - D(J) - s_{\text{int}} \ge 4t + 2q - 6 - D(J).$$

If $s_{\text{int}} \ge 2(t-q)$, then $2M_{\text{int}} \ge 4(t-q)$, while subadditivity of ℓ at the coarsened partition gives

$$2M_{\text{cross}} + s_{\text{cross}} \ge \sum \ell(\text{coarse multiplicity}) \ge 6(q-1) - D(J).$$

Adding yields the same inequality. Every term of the matroid-union formula is thus at least $6(t-1) - 2M - D(J)$, proving the claim.

Apply the codimension theorem to this R-edge set. The grounded distinct-twist space has dimension at most $6(t-1) - R$. Each crossing pin has affine fiber dimension at most one, and each noncrossing pin has three free coordinates. If there are m pins in total, the incidence stratum has dimension at most

$$6(t-1) - R + M + 3(m - M) \le 3m + D(J).$$

There are finitely many equality partitions; the all-equal stratum has grounded dimension zero. The generic compatible-motion space therefore has dimension at most $D(J)$.

For an optimal partition, its block-constant motion subspace already has the full dimension $D(J)$, so it is the entire motion space. Bodies in one block then have identical twists in every motion and form a rigid cluster. If no bodies merge, the partition is singleton. For simple J, its value is $6(|V(J)| - 1) - 3|E(J)|$, proving the last assertion. □

6. **Body loads and scalar forces**

Return to the minimal counterexample, with body family $\mathcal{B}$. Its stresses combine internal stresses of individual bodies with balanced forces transmitted between bodies. Shared edges make this separation nonunique; opposite coefficients on two copies of an edge cancel in the graph while transferring a load between its bodies. We identify this ambiguity in a way that respects every induced subgraph, as required by the final rank count in a random vertex order.

Lemma 6.1. *Assign coefficients $a_{C,w}$ to the copies of edges vw in bodies through v. If*

$$\sum_{C\ni v,w} a_{C,w} = 0 \quad \textit{for every } w, \qquad \sum_{w} a_{C,w}(p_w - p_v) = 0 \quad \textit{for every } C,$$

then all coefficients vanish.

Proof. Select the b bodies with a nonzero coefficient. For an active neighbor w, let $a_w \geq 2$ count its nonzero copies and t_w its total multiplicity among these bodies. The number J of active incidences satisfies

$$J = \sum_w a_w \leq 2\sum_w (a_w - 1) \leq 2\sum_w (t_w - 1) = 2(I - N) \leq 3b - 3,$$

by Lemma 3.2. But each active body needs at least four generic directions for equilibrium, so $J \geq 4b$. □

For $T \subseteq V(G)$, define the *compatible body-load space* $\mathcal{C}(T)$ to consist of loads $F_{C,v} \in \mathbb{K}^3$ on the copies of $v \in C \cap T$ such that

$$\sum_{C\ni v} F_{C,v} = 0, \qquad \sum_{v\in C\cap T} F_{C,v} = 0, \qquad \sum_{v\in C\cap T} p_v \times F_{C,v} = 0.$$

Thus loads cancel at actual vertices, and each body fragment has zero total force and moment. A *duplicate-edge mode* assigns opposite coefficients to two copies of the same edge. Its induced body loads span a subspace $\mathcal{D}(T) \subseteq \mathcal{C}(T)$.

Proposition 6.2. *The local clique stress spaces have direct sum, and for every T,*

$$\left(\bigoplus_{C\in\mathcal{B}} \mathcal{S}(K_C)\right) \cap \mathcal{S}(G[T]) = \bigoplus_{C\in\mathcal{B}} \mathcal{S}(K_{C\cap T}).$$

Moreover,

$$\mathcal{S}(G[T])\Big/\bigoplus_{C\in\mathcal{B}}\mathcal{S}(K_{C\cap T})\cong\mathcal{C}(T)/\mathcal{D}(T), \tag{2}$$

$$r(G[T])=\sum_{C\in\mathcal{B}}r(K_{C\cap T})-\dim\mathcal{C}(T), \tag{3}$$

$$\mathcal{D}(V(G))\cap\mathcal{C}(T)=\mathcal{D}(T). \tag{4}$$

All maps are compatible with extension by zero. If t_v is the number of bodies through v, the projection of $\mathcal{C}(V(G))$ onto the load copies at v has dimension $3t_v-\delta(v)$.

Proof. A dependence among local stresses satisfies Lemma 6.1 at each vertex. If a sum of local stresses is supported on T, the same lemma at omitted vertices shows that each summand is supported on its fragment $C\cap T$.

For the remaining assertions, take edge coefficients separately on the complete fragments. Coefficients whose induced loads cancel at each actual vertex map onto $\mathcal{S}(G[T])$ by summing edge copies; the kernel consists of duplicate-edge coefficients. The same space maps onto $\mathcal{C}(T)$ by taking body loads; its kernel is the sum of the fragment stress spaces. Surjectivity holds because the kernel of a complete fragment's rigidity matrix consists of restrictions of rigid motions, including for fragments of size at most two. These two exact sequences give (2) and (3).

If duplicate loads vanish at an omitted vertex, their copied edge coefficients satisfy Lemma 6.1 there and hence vanish. This proves (4). Finally, deleting v lowers the rank of each incident complete body by three. Subtracting (3) before and after deletion gives the projection dimension. □

Lemma 6.3. *Every compatible body load can be adjusted by duplicate-edge modes so that it vanishes at all two-body hinge pins. The adjustment leaves all soft point-pin forces unchanged.*

Proof. At a hinge pin v with partner w, the load image has dimension $6-\delta(v)\le 1$ and contains the nonzero duplicate direction of vw. Thus it is exactly this direction and $\delta(v)=5$. If w is also a two-body vertex, consider loads vanishing at v. They form $\mathcal{C}(V\setminus\{v\})$ over the field omitting p_v. Their projection at w is nevertheless constrained to the line p_w-p_v. Algebraic independence of p_v forces this projection to vanish. Killing the load at v therefore kills both endpoints. If w is a hypervertex, only v needs to be killed. Hinge edges have disjoint two-body endpoints, so the adjustments are simultaneous. A soft point pin lies on no shared edge. □

6.1. **The scalar coordinate code**

Choose a generic chart in which all virtual centers are finite, as above. Let H be the hypervertices and P the soft point pins. At $v \in P$, Lemma 3.6 identifies the body force with a scalar λ_v times $p_v - a_v$. This gives a canonical nonzero functional on the actual stress space $\mathcal{S} = \mathcal{S}(G)$; there is no duplicated edge at v, and local clique stresses have zero body force. Put

$$U = \operatorname{span}\{\lambda_v : v \in P\} \subseteq \mathcal{S}^*, \qquad r_U = \dim U.$$

For $h \in H$, let $L_h \subseteq \mathcal{S}^*$ be the span of its actual edge-coordinate functionals, and set $a_h = \dim(U \cap L_h)$. For a body C of order at least five, put

$$r_C = \dim \operatorname{span}\{\lambda_v : v \in C \cap P\}.$$

A tetrahedron has no internal stress, so its loads determine its edge forces; a nonzero edge coefficient therefore identifies the scalars at its soft endpoints up to a fixed factor. Grouping these identifications gives coordinates that can be read at hypervertex stars or in larger bodies. We call these the *observers* of a coordinate. The resulting coordinates may still satisfy linear relations, which is why the next lemma is formulated in terms of their code rather than their number.

Lemma 6.4. *There is a scalar coordinate code of dimension r_U, with observers the hypervertices and bodies of order at least five, such that every coordinate has an observer. A coordinate with just one observer is observed by a hypervertex and consists entirely of type-A pins.*

Proof. In a generic K_4, balanced loads have a unique six-edge coefficient vector. At a soft pin, its three incident coefficients are fixed multiples of λ_v, with at least two nonzero. Indeed, one coefficient would direct the force toward an actual neighbor. Since a_v is independent of p_v, this would force a_v to be that neighbor, contrary to Lemma 3.6.

Join soft pins by the K_4 edges with nonzero coefficient. Each such edge identifies their functionals up to nonzero scale. Choose a coordinate x_S for each connected component S. Their image on $\mathcal{S}$ is a code of dimension r_U; the coordinates may be dependent. A large body observes each component meeting it. A hypervertex observes a component joined to it by a nonzero K_4 edge. Such an edge belongs to a unique body, so its coefficient is an actual bar coordinate and the captured scalar lies in $U \cap L_h$.

There are no other nonzero boundary edges from soft pins. A nonsoft two-body point pin has zero load, by Proposition 6.2. At a hinge pin the load is along its shared

edge; in a K_4, its other two coefficients vanish. Hence a nonzero soft-boundary edge ends at a hypervertex.

Suppose S has no hypervertex observer. Each K_4 meeting S contains at least three of its vertices, since every soft vertex has at least two nonzero coefficients. If exactly three occur, retain their rigid triangle and discard the zero-load fourth corner. If there is no large-body observer, the scalar x_S alone gives a nonzero compatible pin-load stress of this pair-pin framework. If there is just one large-body observer C, include C too. Summing force and moment balance over the small bodies shows that the boundary load on C is balanced, so the same conclusion holds.

Every retained pin is shared by exactly two bodies, and each body pair shares at most one pin. No retained bodies can merge rigidly; adjoining a discarded fourth corner to its triangle would extend a merger to the original bodies. The last assertion of Theorem 5.1 excludes this nonzero pin-load stress. Thus every coordinate has an observer, and a sole observer is a hypervertex. Such a component has no soft port in a large body, so all its pins are type A. □

Lemma 6.5. *If a scalar component S has sole observer h, there is a nonzero stress s_S supported on $S \cup \{h\}$ with*

$$\mathcal{S}(G) = \mathbb{K}s_S \oplus \mathcal{S}(G - S).$$

On every induced prefix the first summand is present exactly when its entire vertex support is present. Distinct such components give independent modes with disjoint edge supports. Each has at least five nonzero star edges at h.

Proof. Take a stress with $x_S \neq 0$ and retain its edges incident with S. Every retained nonzero edge stays in $S \cup \{h\}$. Vertices of S remain in equilibrium, and the remaining force at h is zero because the total force of any bar-coefficient vector is zero. This gives s_S. All star coefficients at vertices of S are fixed multiples of x_S; subtracting the corresponding multiple of s_S kills them. Every vertex of S, as well as h, occurs in its support. These facts give the splitting on every prefix and the independence for distinct components.

Generic equilibrium at h needs at least four star edges. Four would, by [1, Lemma 8.3], make those neighbors a rigid cluster and produce a rigid K_5 containing a type-A vertex. Thus at least five occur. Also Lemma 3.7 shows that the vertex support has at least six vertices. □

Proposition 6.6. *Let z_h count scalar components with sole observer h. Then*

$$2r_U \leq \sum_{h \in H} a_h + \sum_{\substack{C \in \mathcal{B} \\ |C| \geq 5}} r_C + \sum_{h \in H} z_h. \tag{5}$$

The exceptional coordinates are independent actual stress modes.

Proof. The large-body observer has coordinate rank r_C, and the coordinates captured at h have rank at most a_h. Every nonexceptional coordinate has at least two observers. Add one singleton observer for each exceptional coordinate. These sets form a two-cover, so linear Shearer's inequality gives (5). For completeness, order the coordinates and write rank as successive increments. In a subset, conditioning only on earlier coordinates of that subset can only increase an increment. The two-cover therefore counts each full-code increment at least twice. Independence of the exceptional axes follows from Lemma 6.5. □

7. **The rank estimate**

We now prove Theorem 1.1. The minimal counterexample satisfies

$$r(G) = 3n - 7, \qquad \delta(v) \geq 5, \qquad \delta(h) \geq 7 \quad (h \in H).$$

Set $K = \operatorname{ann}_{\mathcal{S}}(U)$ and $K(T) = K \cap \mathcal{S}(G[T])$. In a uniform vertex order, let $\Delta_v \dim K(T)$ denote its dimension increase when v enters. Since these increases sum to $\dim K = \dim \mathcal{S} - r_U$,

$$r(G) = \sum_{v \in V(G)} \left(\frac{d(v)}{2} - \mathbb{E}\Delta_v \dim K(T) \right) - r_U. \tag{6}$$

We shall show that the total excess over $3n$ inside this sum is at least r_U. By (5), it is enough to bound that excess below by

$$\frac{1}{2} \sum_{h \in H} a_h + \frac{1}{2} \sum_{\substack{C \in \mathcal{B} \\ |C| \geq 5}} r_C + \frac{1}{2} \sum_{h \in H} z_h.$$

Thus the local estimates must cover the scalar information visible at hypervertices and in large bodies, together with the extra contribution of singly observed coordinates. Two-body vertices will supply the body terms. The stronger deletion bound at hypervertices supplies the rest, after the local stresses of certain K_5 bodies are counted separately.

7.1. **The contributions of two-body vertices**

Lemma 7.1. *If $v \notin H$ belongs to bodies of orders a, b, and ε_v is zero for a point pin and one for a hinge pin, then*

$$\frac{d(v)}{2} - \mathbb{E}\Delta_v \dim K(T) = \left(3 - \frac{6}{a}\right) + \left(3 - \frac{6}{b}\right) - \frac{\varepsilon_v}{2}.$$

Proof. At a point pin, both body forces vanish on K; for a nonsoft point pin they vanish already on the full stress space. At a hinge pin whose partner is in the prefix, adjust the duplicate edge to kill both body forces. If its partner w is absent, the compatible prefix load space is defined over the coordinate field omitting p_w, but its force at v lies on the line $p_w - p_v$. That force must therefore vanish.

A balanced star in a complete fragment extends to a local stress; the opposite loads on the other vertices have zero force and moment and can be resolved by their complete graph. Subtract these local stresses to kill the star at v. Their star images are direct by Lemma 6.1. Hence the prefix birth is exactly the sum of the two local clique births. In K_m, the expected rank contribution is

$$\frac{1}{m}\sum_{j=0}^{m-1}\min(j,3) = 3 - \frac{6}{m}.$$

At a hinge the common edge has been counted twice in the two clique degrees; correcting this subtracts $1/2$. □

Put $q(m) = 3/2 - 6/m$. Allocate the excess over three from Lemma 7.1 to its two body incidences as follows

incidence in a body of order m	allocation
point pin	$q(m)$
hinge with a K_4 on the other side	$q(m) - 1/2$
hinge with both bodies of order at least 5	$q(m) - 1/4$.

The K_4 side of the second row receives zero. Two K_4's cannot meet at a two-body hinge, since its degree would be five. Thus a body of order at least six receives at least $1/2$ per soft pin and nonnegative amounts from all other two-body vertices. A K_5 receives $3/10$ per soft pin, $-1/5$ per type-B vertex, and at least $1/20$ per other two-body vertex.

The hinge partner of a type-B vertex is a hypervertex in its K_5. Indeed, if both hinge endpoints belonged only to that K_5 and K_4, they would have five common external neighbors. Deleting those five would isolate their edge from another vertex. Such another vertex exists because a flexible six-connected graph has at least eight vertices.

7.2. **Local K_5 modes and hypervertex contributions**

The K_5 bodies are the only large bodies whose two-body allocations can be negative. They also have exactly one internal stress, whose appearance in a random prefix can

be counted exactly. Separating this mode before applying the general support bound supplies the additional 3/10 per incident hypervertex used below.

Select each K_5 body containing a soft point pin or a type-B vertex. Its unique local stress is denoted s_C. On K it has a canonical coefficient ρ_C; at a soft private port the body force vanishes, so its core star is a multiple of the local K_5 star. The other body's star is separated by Lemma 6.1. At a type-B port its full actual star image is already this one local mode. Coefficients read at different private ports agree on their common core edges.

These functionals are independent, as witnessed by the stresses s_C. On a prefix, $\rho_C \neq 0$ is possible only when $C \subseteq T$, since all four core-star coefficients at a private port are nonzero. Subtracting $\rho_C s_C$ thus preserves prefix support. With $K' = K \cap \bigcap_C \ker \rho_C$, we have

$$K(T) = \bigoplus_{\substack{C \text{ selected} \\ C \subseteq T}} \mathbb{K} s_C \;\oplus\; K'(T). \tag{7}$$

Let g_h be the number of selected cores through $h \in H$. Each selected coefficient is read from an actual edge hp to a private port; that edge is unique to the core when p is soft, and at a type-B port the entire star is the core mode. The g_h coefficients therefore remove exactly g_h independent coordinates from the h-star image. Nonincident selected modes have zero h-star. Since the star image of K has dimension $d(h) - \delta(h) - a_h$, that of K' has dimension

$$R_h = d(h) - \delta(h) - a_h - g_h \geq 0.$$

Each local core mode is born at h with probability 1/5. By (7) and Corollary 2.2, the excess over three at h is at least

$$\begin{aligned} \frac{d(h)}{2} - \frac{g_h}{5} - \Phi_4(R_h) - 3 &= \frac{a_h}{2} + \frac{3g_h}{10} + \frac{\delta(h) - 3}{2} - \frac{6}{R_h + 4} \\ &\geq \frac{a_h}{2} + \frac{3g_h}{10} + e_h, \qquad e_h = \frac{\delta(h) - 6}{2} \geq \frac{1}{2}. \end{aligned} \tag{8}$$

Reserve $a_h/2$ for scalar observation, 3/10 for each selected core, and e_h for the remaining charges.

7.3. **The body budgets**

A body of order at least six pays $r_C/2$, since it receives at least $1/2$ for each of its soft ports and r_C is at most their number. For a selected K_5, let p, b, j, z_0 count respectively its soft point pins, type-B vertices, hypervertices, and other two-body vertices. Their sum is five. Its two-body allocations and hypervertex credits supply at least

$$B_C = \frac{3p}{10} - \frac{b}{5} + \frac{z_0}{20} + \frac{3j}{10} = \frac{p+j}{2} + \frac{z_0}{4} - 1.$$

Use Lemma 6.3 to kill all two-body hinge loads. Only soft and hypervertex forces remain. If $j = 1$ and $p > 0$, moments about the sole hypervertex give a nonzero equation on the soft coefficients; a soft force axis cannot pass through another actual body vertex. Hence $r_C \le p - 1$. If $j = 0$, then $b = 0$. Let k be the rank of the soft force-wrench columns. Their axes are pairwise distinct, since coincident axes at two generic soft pins would put a virtual center at the other pin. Three decomposable forms in a two-dimensional span lie on a Klein line, whose physical lines form a planar pencil. Four generic body points cannot lie in that plane. Consequently

$$k \ge \min(p, 2) + \mathbf{1}_{\{p \ge 4\}}, \qquad r_C \le p - k.$$

These bounds pay the observer cost

case	bound on r_C	$B_C - r_C/2$ is at least
$j \ge 2$	p	$j/2 - 1 + z_0/4$
$j = 1,\ p > 0$	$p - 1$	$z_0/4$
$j = 0$	$p - k$	$k/2 - p/4 + 1/4.$

All entries in the last column are nonnegative. The only remaining case is $j = 1, p = 0$, whose deficit is

$$\max\{0, b/4 - 1/2\}.$$

Thus only cores with three or four type-B vertices need further credit, respectively $1/4$ and $1/2$. Unselected bodies have no observer cost and no negative allocation.

7.4. **The remaining hypervertex charges**

Fix $h \in H$, and put $a = z_h$. Let q_b, for $b = 2, 3, 4$, count cores whose only hypervertex is h, which have no soft point ports and exactly b type-B vertices. The amount still charged to h is

$$\frac{a}{2} + \frac{q_3}{4} + \frac{q_4}{2}. \tag{9}$$

All these type-B vertices have hinge partner h. Let k be the number of nonzero h-star edges in the a exceptional type-A modes. Their supports are disjoint and $k \geq 5a$. Put

$$g = q_2 + q_3 + q_4, \quad B = 2q_2 + 3q_3 + 4q_4, \quad w = B - g = q_2 + 2q_3 + 3q_4,$$

and $D = d(h) - k - B$. The a exceptional modes and g core modes are independent on their specified star supports. Subtracting them kills the $k + B$ specified edges. A residual equilibrium star on D generic directions has dimension at most $\max(0, D - 3)$. Therefore

$$\delta(h) \geq k + B - a - g + \min(3, D) \geq 4a + w + \min(3, D), \tag{10}$$

in addition to $\delta(h) \geq 7$. We check that $e_h = (\delta(h) - 6)/2$ pays (9). Subtracting the required amount from (10) gives the useful form

$$e_h - \frac{a}{2} - \frac{q_3}{4} - \frac{q_4}{2} \geq \frac{3a}{2} - 3 + \frac{q_2}{2} + \frac{3q_3}{4} + q_4 + \frac{\min(3, D)}{2}. \tag{11}$$

Only the cases with few exceptional modes and few core ports can leave a deficit in this bound. In those cases the neighborhood rigidity estimate supplies the missing directions counted by D.

If $a = 0$ and $g > 0$, the g core K_4's in $N(h)$ supply five independent planar edges each. The outer triangles from the B type-B vertices supply at least B more; each triangle contains at most three of those vertices. Lemma 3.2 and the nonrigidity of the neighborhood give

$$5g + B \leq 2d(h) - 4, \qquad D = d(h) - B \geq \frac{5g - B + 4}{2} \geq \frac{g + 4}{2} > 2.$$

Hence

$$e_h \geq \max\left\{\frac{1}{2}, \frac{w - 3}{2}\right\} \geq \frac{q_3}{4} + \frac{q_4}{2}.$$

The last inequality is immediate for $q_3 + q_4 \leq 2$; otherwise the second term has surplus at least $3q_3/4 + q_4 - 3/2$. The case $g = 0$ has no charge.

If $a \geq 2$, the right side of (11) is already nonnegative.

It remains that $a = 1$. The baseline $e_h \geq 1/2$ suffices if $q_3 = q_4 = 0$. The bound (10) with $D \geq 0$ suffices if $q_3 + q_4 \geq 2$, if $q_3 = 1, q_2 \geq 2$, or if $q_4 = 1, q_2 \geq 1$.

When $q_2 = q_3 = 1, q_4 = 0$, an inactive vertex of the three-port core lies outside the type-A component and the selected type-B set. Thus $D \geq 1$, supplying the remaining credit.

For the two remaining cases, observe that distinct type-B vertices in one core, with hinge partner h, have outer K_4's with disjoint external pairs. Otherwise their neighborhood triangles would be vxa and wxb for a shared x. The core edge vw forms a rigid triangle vwx; gluing along vx and wx, then along v, w to the core, merges distinct maximal planar rigid components.

For a single three-port core, its six external neighbors and one inactive core vertex give $k + D \geq 7$. The first inequality in (10) says $\delta(h) \geq k + 1 + \min(3, D) \geq 8$, since $k \geq 5$. Thus $e_h \geq 1$ pays the required $3/4$. For a single four-port core, its eight external neighbors give $k + D \geq 8$, and $\delta(h) \geq k + 2 + \min(3, D) \geq 10$. This pays the required one unit.

The two-body allocations and (8) pay all body observer costs, all negative type-B allocations, and all exceptional charges. Their total is at least

$$\frac{1}{2}\sum_{h\in H} a_h + \frac{1}{2}\sum_{\substack{C\in\mathcal{B}\\|C|\geq 5}} r_C + \frac{1}{2}\sum_{h\in H} z_h \geq r_U,$$

by Proposition 6.6. Equation (6) now gives $r(G) \geq 3n$, contradicting $r(G) = 3n - 7$ and proving Theorem 1.1.

References

[1] B. Jackson, T. Jordán and S. Villányi, *Rank contributions of vertices in rigidity matroids of clique covered graphs*, arXiv 2607.26266 (2026).

[2] D. Zheng, *Stress degeneracy of direction complexes of* $(2, 2)$*-sparse graphs and three-dimensional body-pin rigidity*, preprint (2026), DOI 10.13140/RG.2.2.17830.28485.