Higher Specht bases for Springer and Δ-Springer rings

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Abstract. We construct higher Specht bases for every Springer and Δ-Springer ring by a uniform triangular pairing that separates copies of the same Specht module in the same degree. The pairing also gives simultaneous harmonic decompositions, making sums and intersections of the defining ideals visible on their tableau index sets. Removing the power relations produces explicit free bases over the elementary-symmetric parameters and a finite flat family whose fibers split into tensor products of Springer and coinvariant rings. The underlying parameter modules are free integrally, while the prescribed unnormalized higher Specht families are bases over a field exactly in characteristic zero or characteristic greater than n, independently of the exponent bound.

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1. Introduction

The graded character of a Springer ring specifies how often each Specht module occurs in each degree. A higher Specht basis realizes these multiplicities by explicit polynomials indexed by standard tableaux [5]; proving that the proposed polynomials form a basis requires distinguishing copies with the same shape and degree. We construct a triangular pairing that separates these multiplicity vectors for every Springer and Δ-Springer ring, and that makes their harmonic copies compatible as the defining ideal varies.

The three-row and hook cases for Springer rings, together with a general nonvanishing theory, were established in [1]. Our pairing resolves the remaining independence problem and proves the higher Specht basis conjectures of Chou and Hanada for every partition. The additional symmetric factors for Δ-Springer rings are introduced in Section 4.

Fix $\mu \vdash n$. Pad μ' with zeros to length n, and use the Springer presentation [4]

$$p_m^n(\mu) = \sum_{j=n-m+1}^{n} \mu'_j, \qquad R_\mu = \mathbb{C}[x_1, \dots, x_n]/I_\mu,$$

$$I_\mu = \big(e_d(Y) : 1 \le d \le |Y|,\ d > |Y| - p^n_{|Y|}(\mu)\big).$$

For standard tableaux S, T of a common shape, let $\mathrm{ind}(S)$ be the cocharge filling, let $\mathrm{cc}(S)$ be the sum of its labels, and let $\mathrm{ctype}(S)$ be its catabolizability type, with the conventions of Section 2.1. Put

$$F_T^S = \left(\sum_{\sigma \in C(T),\, \rho \in R(T)} \mathrm{sgn}(\sigma)\sigma\rho \right) x_T^{\mathrm{ind}(S)}.$$

Here $R(T)$ and $C(T)$ are the row and column stabilizers, and the label in each box of T specifies the variable receiving that box's exponent. The group acts by permuting variables.

Theorem 1.1. *For every partition $\mu \vdash n$, the set*

$$\mathcal{B}_\mu = \{F_T^S \bmod I_\mu : S, T \in \mathrm{SYT}(\lambda),\ \lambda \vdash n,\ \mathrm{ctype}(S) \trianglerighteq \mu\}$$

is a basis of R_μ. For each fixed S, its T-indexed span is isomorphic to the irreducible symmetric-group module of shape $\mathrm{sh}(S)$*, with the usual standard-polytabloid action.*

The indexing realizes the catabolizability formula for the graded Springer character; the multiplicity of shape λ is

$$\sum_{\substack{S\in \mathrm{SYT}(\lambda)\\ \mathrm{ctype}(S)\trianglerighteq\mu}} q^{\mathrm{cc}(S)}.$$

This character formula is recorded in [3, (1)]. To turn it into a polynomial basis, we construct an apolar witness for each indexing tableau and order the pairings triangularly.

Blasiak's catabolism insertion produces a tableau of shape $\mathrm{ctype}(S)$ from the cocharge labels of a standard word inserting to S [3, Theorem 3.4], and the product of its column Vandermondes annihilates I_μ when $\mathrm{ctype}(S) \trianglerighteq \mu$. Choosing a column reading of the original tableau gives the witness exponents two monotonicities; squared norm orders the pairings, and initial-column sums resolve the ties. The resulting matrix is triangular with positive diagonal, proving independence even among copies of equal shape and degree.

The elementary-symmetric multipliers for Δ-Springer rings preserve the extremal exponent layer on which this order depends. The family, previously established for two-row partitions at the minimal exponent bound [2], is therefore a basis for every $\lambda \vdash k \leq n$ and every $s \geq \ell(\lambda)$ (Theorem 4.1). A compatible choice of witnesses makes the harmonic Specht copies simultaneous for all Griffin ideals with fixed n, s, so that finite sums and intersections are governed by unions and intersections of tableau index sets.

Removing the power bounds leaves the same finite family of higher Specht polynomials, now a free basis over $\mathbb{C}[e_1, \ldots, e_{n-|\lambda|}]$, answering Griffin's question for the untruncated rings [8]. This freeness has a geometric interpretation; the parameters define a finite flat family with a Springer ring at the origin, and their characteristic polynomial separates every other fiber into Springer and coinvariant factors. The parameter freeness holds integrally. For the specified unnormalized higher Specht polynomials themselves, the coefficient pairing gives characteristic zero or characteristic greater than n as the exact range of fields, independently of the exponent bound.

2. Column witnesses

2.1. Cocharge conventions

All tableaux use French diagrams. Rows increase from left to right; columns increase from bottom to top. The row word reads rows from top to bottom, each from left to

right. Cocharge labels start at zero on entry 1; the label increases by one exactly when the next entry is to the left of the preceding entry in that word. The catabolizability type is computed by the cyclic insertion algorithm recalled in Section 2.3, applied to the cocharge labels of the row word.

The tableau $a = \mathrm{ind}(S)$ is weakly increasing along rows and strictly increasing up columns. The row assertion follows because the label is nondecreasing as the entry increases. For two entries in one column, the larger entry lies earlier in the row word. Between the two entry values there must therefore be an inverse descent, giving a strict increase in cocharge.

Moreover $\mathrm{ind}(S)$ determines S. For each cocharge label, its entries in S form a consecutive interval, and occur in increasing order in the row word. Fill the zero-labelled positions in row-word order by $1, 2, \ldots$, then the one-labelled positions, and so on. This recovers S uniquely.

2.2. **Annihilation**

A product of column Vandermondes records the column lengths of a tableau. Elementary-symmetric differential operators detect how many of those columns lie wholly in a given set of variables. This gives the annihilation statement below, including the outside powers needed for Griffin rings. It is also the membership assertion in the harmonic-generator theorem of Rhoades–Yu–Zhao [7, Definition 9, Lemma 11 and Theorem 15].

Lemma 2.1 (Injective-tableau witnesses)**.** *Let $\nu \vdash k \le n$, let $s \ge \ell(\nu)$ be a positive integer, and let U be an injective filling of the French diagram of ν with labels in $[n]$. Write K for its set of labels. Let g_U be the product of the Vandermonde polynomials in its column label sets, with either choice of overall sign, and put*

$$G_{U,s} = g_U \prod_{i \notin K} x_i^{s-1}.$$

Then

$$\partial_i^s G_{U,s} = 0 \quad (i \in [n]), \qquad e_d(\partial_Y) G_{U,s} = 0 \quad \text{if} \quad d > |Y| - \sum_{j=n-|Y|+1}^{n} \nu'_j.$$

Here ν' is padded with zeroes to length n. These are the elementary and power relations used below. In particular, when $k = n$, g_U annihilates the Tanisaki ideal I_ν.

Proof. For a column label set C of U, let g_C be its Vandermonde. Every symmetric differential operator of positive degree in the variables indexed by C annihilates g_C.

Indeed its value would be alternating of degree smaller than $\binom{|C|}{2}$, whereas every alternating polynomial is divisible by g_C. In particular

$$\prod_{i\in C}(1+t\partial_i)g_C = g_C. \tag{2.1}$$

Fix $Y \subseteq [n]$ and set $h = n - |Y|$. At most h columns of U meet $[n] \setminus Y$. Their total size is at most the sum of the h largest column sizes, $\sum_{j=1}^{h} \nu'_j$. Consequently the columns wholly contained in Y contain at least $k - \sum_{j=1}^{h} \nu'_j = \sum_{j=h+1}^{n} \nu'_j$ labels. Applying (2.1) separately to each such column shows that

$$\prod_{i\in Y}(1+t\partial_i)G_{U,s} = \sum_{d\geq 0} t^d e_d(\partial_Y)G_{U,s}$$

has degree in t at most $|Y| - \sum_{j=h+1}^{n} \nu'_j$. This proves every claimed elementary annihilation identity. Finally, each variable of g_U has degree at most $\ell(\nu) - 1 \leq s - 1$, and every outside variable has degree $s - 1$, proving the power identities.

All these are identities between polynomials with integer coefficients. Once proved over $\mathbb{Q}$, they remain valid after base change to any coefficient field. □

We will also use the elementary inclusion $I_\mu \subseteq I_\nu$ when $\nu \trianglerighteq \mu$. Indeed, conjugation reverses dominance, so the final-column sums of ν' are at least those of μ'. Every elementary generator in the definition of I_μ is therefore also a generator of I_ν. Together with the preceding lemma, this is all the inverse-system information needed for the proof.

2.3. **Column reading and catabolism**

Number rows from zero at the bottom. For standard tableaux S, T of the same shape, index the filling $a = \mathrm{ind}(S)$ by the labels of T; a_i is the cocharge label in the box containing i.

The following choice of reading word is useful. Read the columns of S from left to right, reading each column from top to bottom. Call the resulting standard word $w_{\mathrm{col}}(S)$. It has insertion tableau S, just as the usual row-reading word does. Moreover its cocharge label on each entry of S is the same a_i. Indeed, for consecutive entries $j, j+1$ of a standard tableau, if $j+1$ is above j, its column is weakly to the left of the column of j; otherwise two strict steps along a row and a column would put an integer strictly between them. In this case $j+1$ precedes j in both reading words. If $j+1$ is not above j, its column is strictly to the right, and it follows j in both words. The inverse descents, and hence all cocharge labels, agree. Thus the new cocharge word is obtained by reading the entries of a in this column order.

Apply Blasiak's catabolism insertion algorithm to this cocharge word. Keep the original word positions fixed, scan cyclically from right to left, and cross out a position when it is accepted. A current label b is accepted in row b if adding a box there preserves the partition shape; otherwise increment its label and continue scanning. When a box is added, fill it with the label of the corresponding box of T. Denote the resulting filling by $U = U_{\mathrm{col}}(S, T)$. Blasiak's theorem gives

$$\mathrm{shape}(U) = \mathrm{ctype}(S).$$

Here we use the algorithm for a standard word and the equality of its output with the catabolizability type of its insertion tableau [3, Theorem 3.4]; no new insertion-shape assertion is required.

Let α_i be the row of i in U, and put

$$r_i = \alpha_i - a_i.$$

In each column of U, order its labels from bottom to top. With the usual choice of signs for the product of column Vandermondes,

$$g_U = \sum_{\tau \in C(U)} \mathrm{sgn}(\tau) x^{\tau\alpha}.$$

The exponent vectors $\tau\alpha$ are distinct, because the entries of α in a column of height h are $0, 1, \ldots, h-1$.

Lemma 2.2 (Two monotonicities of the column-reading witness)**.** *The vector r is nonnegative and weakly increasing from bottom to top in every column of U. Within a block of equal r in such a column, the original column indices in T weakly decrease from bottom to top, while the entries of a strictly increase.*

Proof. Number the complete right-to-left scans $0, 1, 2, \ldots$. A letter accepted during scan k has been rejected exactly k times, so it is inserted in row $a_i + k$. Hence $r_i = k \geq 0$. Every box of U is inserted after the box immediately below it, because all intermediate diagrams are partitions. Scan numbers therefore weakly increase up each column.

During a single scan the original columns of T are processed from right to left. The boxes in an equal-r block of a U-column are inserted during that same scan, in their order from bottom to top. Their original column indices consequently weakly decrease. Finally, on that block $a_i = \alpha_i - r_i$, and α_i strictly increases. □

3. **Triangular apolar pairings**

The order needed for triangularity comes from the coefficient equation itself. If the desired exponent is b and a witness has distinguished exponent α, put $r = \alpha - b$. An exponent β contributes to the pairing only when $\beta + r = \tau\alpha$ for a witness-column permutation τ. Expanding $\|\tau\alpha - r\|^2$ turns this equation into a rearrangement inequality, with its minimum at b.

When the norms tie, only permutations within the equal-r blocks remain. The earliest original columns contain the largest b-values in each block, so such a permutation can only decrease an initial-column sum, whereas the row-prefix minima on the polynomial side give the opposite inequality. The two bounds resolve the remaining freedom and lead to the two-stage order below, which will also handle the symmetric multiples.

Lemma 3.1 (A triangularity criterion)**.** *Fix a tableau T and let L_c be its first c columns. For exponent vectors put*

$$N(b) = \sum_i b_i^2, \qquad P_c(b) = \sum_{i \in L_c} b_i.$$

Let b range over distinct nonnegative fillings of one total degree, strictly increasing in the columns of T. Suppose homogeneous polynomials P_b are column-alternating, with $h_b = [x^b]P_b \neq 0$. Assume that every exponent β in P_b satisfies $N(\beta) \leq N(b)$. At equality require $P_c(\beta) \geq P_c(b)$ for every c, with simultaneous equality only when $\beta \in C(T)b$. For each b, suppose a witness has distinguished exponent α and column group $C(U)$, fixing any outside labels, such that $r = \alpha - b \geq 0$. Assume α is strictly increasing and r weakly increasing in every witness column, and original T-column indices weakly decrease within every equal-r block. Define

$$\Lambda_b(f) = \sum_{\tau \in C(U)} \operatorname{sgn}(\tau)[x^{\tau\alpha - r}]f,$$

with a coefficient of negative exponent interpreted as zero. Ordered by increasing lexicographic $(N(b), -P_1(b), -P_2(b), \ldots)$, the matrix $\Lambda_b(P_{b'})$ is upper triangular, with diagonal

$$\Lambda_b(P_b) = h_b \, |C(U) \cap \operatorname{Stab}(r) \cap C(T)|. \tag{3.1}$$

For $G = \sum_{\tau \in C(U)} \operatorname{sgn}(\tau) x^{\tau\alpha}$, the ordinary apolar functional is the positive scalar multiple

$$[x^r]f(\partial)G = \frac{\prod_i \alpha_i!}{\prod_i r_i!} \Lambda_b(f). \tag{3.2}$$

Proof. A contribution to $\Lambda_b(P_{b'})$ has $\beta + r = \tau\alpha$. Rearrangement in each witness column gives

$$N(\beta) = \|\alpha\|^2 + \|r\|^2 - 2\langle \tau\alpha, r \rangle \geq \|\alpha - r\|^2 = N(b).$$

Together with the first hypothesis, this gives $N(b) \leq N(\beta) \leq N(b')$. If the endpoint norms agree, strictness of α makes τ preserve the equal-r blocks, so $\beta = \tau b$. In each such block, L_c consists of the largest b-values, by the column-index hypothesis. Hence

$$P_c(b') \leq P_c(\beta) \leq P_c(b).$$

If the endpoint prefix vectors also agree, equality holds separately in each block. Its b-values are distinct, so the largest subset of a given size is unique. Thus τ preserves every L_c, and $\tau \in C(T)$. The second hypothesis also gives $\beta \in C(T)b'$. Since b and

b' are column-strict, equality of their column multisets implies $b = b'$. This proves triangularity, including arbitrary ties in the ordering.

On the diagonal, the contributing permutations are exactly $K = C(U) \cap \mathrm{Stab}(r) \cap C(T)$. Column alternation gives $[x^{\tau b}]P_b = \mathrm{sgn}(\tau)h_b$ for each $\tau \in K$, so the two signs cancel. Summing gives (3.1). Ordinary differentiation contributes the same factorial factor $\prod_i \alpha_i! / \prod_i r_i!$ to every summand, proving (3.2). □

Fix a shape $\lambda \vdash n$, a degree d, and one standard tableau T of shape λ. Let $\mathcal{S}$ be any set of standard tableaux of shape λ and cocharge d. For each $S \in \mathcal{S}$ construct a^S, U_S, α^S, r^S as in Section 2.3, and define the linear functional

$$\ell_S(f) = [x^{r^S}]\, f(\partial) g_{U_S}.$$

Order $\mathcal{S}$ by the norm and column-prefix key of Lemma 3.1, with $b = a^S$, breaking ties arbitrarily.

Theorem 3.2 (The higher Specht pairing)**.** *The matrix*

$$M_{S,S'} = \ell_S(F_T^{S'}) \qquad (S, S' \in \mathcal{S})$$

is upper triangular in this order. For each S, abbreviate its data by a, U, α, r and put $K = C(U) \cap C(T) \cap \mathrm{Stab}(r)$. Its diagonal entry is

$$M_{S,S} = |\,\mathrm{Stab}_{R(T)}(a)|\,|K|\,\frac{\prod_i \alpha_i!}{\prod_i r_i!} > 0.$$

Proof. Apply Lemma 3.1 with $b = a^S$ and $P_b = F_T^S$. Every monomial exponent is a permutation of b, so its norm is $N(b)$. If $\beta = \sigma\rho b$ with $\sigma \in C(T)$ and $\rho \in R(T)$, then σ preserves L_c, and the first c entries in each row of b are its smallest. Thus $P_c(\beta) \geq P_c(b)$. Equality for every prefix forces equality in each row separately; each row contributes a nonnegative excess over its smallest possible prefix sum, so none can cancel another. Successive prefix differences then give $\rho b = b$, proving $\beta \in C(T)b$.

The same argument applied to $\sigma\rho b = b$ forces $\rho b = b$ and then $\sigma = 1$, since b is column-strict. Hence $[x^b]F_T^S = |\,\mathrm{Stab}_{R(T)}(b)|$. Cocharge fillings determine their standard tableaux, so the labels b are distinct. Lemma 2.2 supplies the witness hypotheses. Triangularity and the diagonal follow from (3.1)–(3.2). □

3.1. **From one tableau to the Specht copies**

Lemma 3.3 (Multiplicity spaces)**.** *Let M be a semisimple S_n-module and let e be a primitive idempotent in the shape-λ block of its group algebra. Independent vectors $v_1, \dots, v_r \in eM$ generate a direct sum of irreducible copies of shape λ.*

Proof. Write the isotypic component as $V^\lambda \otimes E$. The image of e is a fixed one-dimensional subspace of V^λ tensored with E. Independence there is precisely independence of the corresponding multiplicity vectors, and their group orbits span the asserted sum. □

Take

$$\mathcal{S}_{\lambda,d,\mu} = \{S \in \mathrm{SYT}(\lambda) : \mathrm{cc}(S) = d,\ \mathrm{ctype}(S) \trianglerighteq \mu\}.$$

For each S in this set, ℓ_S annihilates I_μ; the witness g_{U_S} has shape $\nu = \mathrm{ctype}(S)$ and $I_\mu \subseteq I_\nu$. The triangular matrix therefore descends to R_μ, and shows that

$$\{F_T^S \bmod I_\mu : S \in \mathcal{S}_{\lambda,d,\mu}\}$$

is linearly independent for the single fixed standard tableau T.

To pass to the complete Specht copies, let M^λ be the permutation module on row tabloids. For a fixed semistandard exponent tableau $a = \mathrm{ind}(S)$, the assignment

$$\{T\} \longmapsto \sum_{\rho \in R(T)} \rho x_T^a$$

is well defined on row tabloids and equivariant under S_n. The image of the standard polytabloid $e_T = \sum_{\sigma \in C(T)} \mathrm{sgn}(\sigma)\sigma\{T\}$ is precisely F_T^S. Restrict this map to the irreducible Specht module $V^\lambda \subset M^\lambda$ and then pass to R_μ. It is nonzero by Theorem 3.2, hence injective. Its standard-polytabloid basis maps to exactly the required T-indexed family.

For the fixed T, all vectors lie in the image of the primitive idempotent ε_T/c_λ, where c_λ is the hook product and $\varepsilon_T^2 = c_\lambda \varepsilon_T$. Lemma 3.3 therefore promotes their independence to a direct sum of the complete T-indexed copies.

Distinct shapes belong to distinct isotypic components, and distinct degrees belong to distinct homogeneous components. The complete set $\mathcal{B}_\mu$ is consequently linearly independent. Its cardinality equals $\dim R_\mu$ by the established graded character formula, so it is a basis. The equivariant maps above prove the stated action on each fixed-S subspace. This completes Theorem 1.1.

3.2. **A coincident-shape example**

For $\mu = (2,2,1,1)$, take the fixed tableau and the two tableaux

$$T = \begin{array}{ccc} & 6 & \\ 4 & 5 & \\ 1 & 2 & 3 \end{array}, \qquad S_1 = \begin{array}{ccc} & 6 & \\ 3 & 4 & \\ 1 & 2 & 5 \end{array}, \qquad S_2 = \begin{array}{ccc} & 5 & \\ 4 & 6 & \\ 1 & 2 & 3 \end{array}.$$

Both have shape $(3,2,1)$ and cocharge five. Their catabolizability types are $(2,2,2)$ and $(3,1,1,1)$, respectively, and both dominate μ. Their squared norms are seven and nine, so they occur in that order. The column-reading witnesses are

$$U_1 = \begin{array}{cc} 6 & 3 \\ 5 & 4 \\ 2 & 1 \end{array}, \qquad U_2 = \begin{array}{ccc} & & 5 \\ & & 6 \\ & & 4 \\ 3 & 2 & 1 \end{array},$$

with rejection vectors $r^1 = (0,0,1,0,0,0)$ and $r^2 = (0,0,0,0,1,0)$ in the variable order $1,\dots,6$. For the unnormalized Young symmetrizers used here, the pairing matrix is

$$\begin{pmatrix} \ell_{S_1}(F_T^{S_1}) & \ell_{S_1}(F_T^{S_2}) \\ \ell_{S_2}(F_T^{S_1}) & \ell_{S_2}(F_T^{S_2}) \end{pmatrix} = \begin{pmatrix} 64 & -96 \\ 0 & 144 \end{pmatrix}.$$

The proof requires positivity only on the diagonal; off-diagonal entries may have either sign. In particular it handles two copies that cannot be distinguished by shape or degree.

4. **Higher Specht bases for Griffin rings**

The same pairing accommodates the elementary-symmetric factors in the proposed basis for Griffin rings. Fix $\lambda \vdash k$ with $1 \le k \le n$ and $s \ge \ell(\lambda)$. The partition λ defines the ring, while $\theta \vdash n$ will denote the shape of a higher Specht copy and η will index its symmetric multiplier. For a subset $Y \subseteq [n]$, put

$$p^n_{|Y|}(\lambda) = \sum_{j=n-|Y|+1}^{n} \lambda'_j,$$

where λ' is padded by zeroes through length n. The ring is

$$R_{n,\lambda,s} = \mathbb{C}[x_1,\dots,x_n]/I_{n,\lambda,s}, \qquad I_{n,\lambda,s} = \big(x_1^s,\dots,x_n^s,\ e_d(Y) : d > |Y| - p^n_{|Y|}(\lambda)\big).$$

Put $m = n - k$ and, for a partition η of length at most m, define

$$E_\eta = \prod_{j=1}^{m} e_j^{\eta_j - \eta_{j+1}}, \qquad \eta_{m+1} = 0.$$

Write $\mathrm{des}(S) = \max \mathrm{ind}(S)$ for the number of descents of S.

Theorem 4.1 (Higher Specht bases for Griffin rings)**.** *For every $\lambda \vdash k \leq n$ and $s \geq \ell(\lambda)$, the family*

$$\left\{ F_T^S E_\eta : \begin{array}{l} \mathrm{shape}(S) = \mathrm{shape}(T), \quad \mathrm{ctype}(S|_k) \trianglerighteq \lambda, \\ \mathrm{des}(S) < s, \quad \eta \subseteq (n-k) \times (s - \mathrm{des}(S) - 1) \end{array} \right\}$$

is a higher Specht basis of $R_{n,\lambda,s}$ over $\mathbb{C}$.

To apply the triangularity criterion, we need to identify the extremal exponents after multiplication by a symmetric factor and then leave enough room for them in the column witness. The rectangle bound on η will supply exactly that room.

Write $a = \mathrm{ind}(S)$ and define the boosted tableau b by

$$b(\square) = a(\square) + \eta_{n-j+1} \quad \text{if the entry of } S \text{ in } \square \text{ is } j,$$

padding η by zeroes through length n. Thus the first k entries are not boosted. The resulting tableau is semistandard; both its original cocharge and its boost are nondecreasing in the entry of S, and the original cocharge is strictly increasing in a column. Equal-cocharge boxes form a horizontal strip, and their entries in S increase in their order from leftmost to rightmost column. Consequently, among boxes with a fixed base cocharge, the boosts are allocated increasingly by original column.

4.1. **An extremal layer of an elementary-symmetric product**

Use the squared norm N and column-prefix sums P_j from Lemma 3.1, on the boxes of T.

Lemma 4.2 (Norm and prefix bounds)**.** *Every monomial x^β occurring in $F_T^S E_\eta$ satisfies $N(\beta) \leq N(b)$. If equality holds, then β has the same content as b and $P_j(\beta) \geq P_j(b)$ for every j. If all these prefix inequalities are equalities, then $\beta \in C(T) \cdot b$. Moreover*

$$[x^b](F_T^S E_\eta) = |\,\mathrm{Stab}_{R(T)}(a)| > 0.$$

Proof. An exponent δ in E_η is a sum of indicator vectors of subsets A_f, with precisely $\eta_j - \eta_{j+1}$ of the subsets having size j. For any permutation a' of a,

$$\langle a', \delta \rangle = \sum_f \sum_{i \in A_f} a'_i \leq \sum_f (\text{the sum of the } |A_f| \text{ largest entries of } a).$$

Also

$$\|\delta\|^2 = \sum_{f,g} |A_f \cap A_g| \leq \sum_{f,g} \min(|A_f|, |A_g|) = \|\eta\|^2.$$

Both bounds are attained by assigning the largest boosts to the largest entries of a, as in b. Expanding $\|a' + \delta\|^2$ proves the norm bound.

Equality in the norm bound forces equality in both displayed bounds. The subsets A_f are therefore nested, and each consists of largest a'-entries, with freedom only among equal values. Thus δ is a permutation of η aligned with a'; if $a'_i < a'_j$, then $\delta_i \leq \delta_j$. For every base value q, the occurrences of $a' = q$ receive a fixed multiset D_q of boosts, namely the corresponding consecutive portion of the sorted boosts. The resulting boosted multisets $B_q = q + D_q$ are strictly separated; every value in B_q is smaller than every value in $B_{q'}$ if $q < q'$. In particular the total content is that of b.

Since E_η is symmetric, a term can be written $\beta = \sigma\gamma$, where $\sigma \in C(T)$ and $\gamma = \rho a + \delta$ with $\rho \in R(T)$. The permutation σ preserves every L_j. Sort γ increasingly in each original row. This can only decrease every left-prefix sum. Because the ranges B_q are strictly separated, after this sorting the boxes occupied by base value q are exactly the boxes where $a = q$ in that row. Thus the first j columns select a fixed number of entries from each B_q. Their sum is at least the sum of that many smallest entries of B_q.

The tableau b realizes all these minima simultaneously; equal-q boxes have distinct original column indices, and their assigned boosted values increase in that order. This proves the prefix bounds. If equality holds for every prefix, equality in the row-sorting inequalities forces γ to be row-sorted. Indeed, each row's sorting excess and each B_q's excess over its smallest selected entries is nonnegative; equality of their total leaves no cancellation between rows or between base values. Equality for each B_q and every column threshold then fixes its entry at each equal-q box, one at a time. Hence $\gamma = b$, proving $\beta \in C(T) \cdot b$.

If $\beta = b$, column-strictness forces $\sigma = \mathrm{id}$ once $\gamma = b$. The strict separation of the B_q forces $\rho a = a$, and then $\delta = b - a$. The monomial x^{b-a} has coefficient one in E_η; its nested subsets of each prescribed size are uniquely determined. All contributing row permutations have positive sign, and there are $|\operatorname{Stab}_{R(T)}(a)|$ of them. □

4.2. **The extended apolar witness**

The witness splits at the first k entries for a specific reason; η has at most $n-k$ parts, so those entries receive no boost. We can keep their Springer column witness unchanged. On every remaining label, the exponent $s-1$ is the largest permitted by the power relations, and the rectangle bound on η leaves enough room beneath it for the boosted exponent. The difference from these witness exponents will again be a nonnegative differential order.

Apply column-reading insertion to $S|_k$, retaining its labels in T. This gives an injective filling U of shape $\nu = \mathrm{ctype}(S|_k) \trianglerighteq \lambda$ on a k-element subset $K \subseteq [n]$. Put

$$G_{U,s} = g_U \prod_{i \notin K} x_i^{s-1}, \qquad \alpha_i = \begin{cases} \mathrm{row}_U(i), & i \in K, \\ s-1, & i \notin K, \end{cases} \qquad r = \alpha - b.$$

On K there is no boost, so r is the nonnegative rejection-count vector. Outside K, the bound on η gives $b_i \le s-1$, so $r_i \ge 0$ there as well.

Lemma 2.1 gives $G_{U,s} \in I^{\perp}_{n,\nu,s}$; the same membership appears in [2, Theorem 7.8]. The elementary generators of the ideals give

$$I_{n,\lambda,s} \subseteq I_{n,\nu,s} \quad \text{when } \nu \trianglerighteq \lambda.$$

Indeed the sums of the final columns of ν' are at least those of λ', so every elementary generator of $I_{n,\lambda,s}$ is also a generator of $I_{n,\nu,s}$; the power generators are the same. We therefore obtain the well-defined functional on $R_{n,\lambda,s}$

$$\mathcal{L}_{S,\eta}(f) = [x^r] f(\partial) G_{U,s}.$$

Let $C(U)$ fix the outside labels. Lemma 2.2 gives all witness hypotheses in Lemma 3.1; outside labels are fixed singleton blocks.

Proof of Theorem 4.1. Fix a shape, total degree, and one standard tableau T. Lemma 4.2 supplies the polynomial hypotheses of Lemma 3.1, and the preceding construction supplies its witnesses. The boosted filling b determines (S, η); cocharge and boost both increase weakly with the entry of S, so a tie forces equality in both. Standardizing tied entries in their reading order recovers S, and subtracting its cocharge recovers η. Thus the fillings are distinct, as required by the criterion.

The apolar matrix is upper triangular with diagonal

$$|\operatorname{Stab}_{R(T)}(a)|\,|C(U) \cap \operatorname{Stab}(r) \cap C(T)|\,\frac{\prod_i \alpha_i!}{\prod_i r_i!} > 0.$$

Its functionals annihilate the defining ideal, proving fixed-T independence. Symmetry of E_η preserves the Specht maps and primitive-idempotent images, so Lemma 3.3

gives independence of the complete copies. The Chou–Hanada monomial basis and its indexing bijection give the matching dimension [2, Theorem B]; hence the family is a basis. □

4.3. **Other symmetric multiplier bases**

Corollary 4.3 (Dominance-triangular symmetric multipliers)**.** *For each auxiliary partition η, choose a homogeneous symmetric polynomial*

$$f_\eta = c_\eta m_\eta + \sum_{\zeta \lhd \eta} c_{\eta\zeta} m_\zeta, \qquad c_\eta \neq 0,$$

where the sum is over partitions of $|\eta|$ strictly dominated by η. Replacing E_η by f_η in Theorem 4.1 again gives a higher Specht basis of $R_{n,\lambda,s}$. In particular one may use the monomial symmetric polynomials m_η or the Schur polynomials s_η.

Proof. Strict dominance gives $\|\zeta\|^2 < \|\eta\|^2$. If $a^\downarrow$ denotes the entries of a in decreasing order, summation by parts also gives $\langle a^\downarrow, \zeta\rangle \leq \langle a^\downarrow, \eta\rangle$. Consequently every monomial of $F_T^S m_\zeta$, for $\zeta \lhd \eta$, has squared norm strictly smaller than $N(b)$. The layer of maximal squared norm in $F_T^S f_\eta$ therefore comes entirely from $c_\eta F_T^S m_\eta$.

This top layer has exactly the alignment and prefix properties in Lemma 4.2; its distinguished coefficient is $c_\eta |\operatorname{Stab}_{R(T)}(a)|$. The apolar matrix remains upper triangular, with each diagonal multiplied by the nonzero scalar c_η. This proves the corollary. The monomial case is immediate, and the Schur case uses the usual Kostka expansion $s_\eta = m_\eta + \sum_{\zeta \lhd \eta} K_{\eta\zeta} m_\zeta$. □

5. **Simultaneous harmonic bases for the Griffin ideals**

To compare different Griffin ideals, the witness for an index must be chosen independently of the quotient. We use the largest unboosted prefix whose witness fits within the exponent bound; its harmonic copy then belongs to every quotient in which that index is eligible. With this common choice, ideal sums and intersections become operations on the index sets.

For homogeneous polynomials of the same degree, use the bilinear apolar pairing $\langle f, g\rangle = [1] f(\partial) g$. It is nondegenerate, and variable permutations are orthogonal. For a homogeneous ideal I,

$$I^\perp = \{g : f(\partial) g = 0 \text{ for every } f \in I\},$$

where equality with the degreewise orthogonal complement uses that I is an ideal. These spaces are closed under differentiation, and under the group algebra when I is

S_n-stable. If ε_θ is a Young symmetrizer, write ε_θ^* for its adjoint, obtained by reversing permutations.

Fix positive integers n and s. The witnesses can be chosen simultaneously for all partitions $\lambda \vdash k \le n$ with $\ell(\lambda) \le s$. The following elementary restriction property identifies the appropriate common witness.

Lemma 5.1 (Restriction of catabolizability type)**.** *If $\kappa = \mathrm{ctype}(S)$, then $\mathrm{ctype}(S|_k)$ is obtained from κ by retaining its first k boxes in row-superstandard order. In other words, retain successive complete parts of κ until fewer than the next part remain, and append that remainder if it is positive.*

Proof. Use the greedy recursive description of catabolizability [3, Proposition 2.2 and the following paragraph]. Let m be the largest initial interval $1, \dots, m$ lying in the bottom row of S. The first part of $\mathrm{ctype}(S)$ is m, and the remaining parts are the catabolizability type of $\mathrm{Cat}_m(S)$, standardized after removing those m entries. For $k \le m$, the restriction is a single row of size k. For $k > m$, its first greedy catabolism also removes exactly m entries. Row insertion commutes with restriction to an initial alphabet; $P(w)|_{[k]} = P(w|_{[k]})$. Since catabolism is defined by deleting the initial bottom-row interval and inserting the remaining row words in the prescribed order, it follows that catabolism also commutes with this restriction. Apply induction to the standardized tableau $\mathrm{Cat}_m(S)$ and the first $k - m$ of its entries. □

Let $\mathcal{D}_{n,s}$ consist of pairs (S, η) with $S \in \mathrm{SYT}_n$, $\mathrm{des}(S) < s$, and $\eta \subseteq n \times (s - \mathrm{des}(S) - 1)$. Trailing zeroes are omitted when writing $\ell(\eta)$. For such a pair write $\kappa = \mathrm{ctype}(S)$ and put

$$k_* = \min \left\{ n - \ell(\eta),\ \sum_{j=1}^{s} \kappa_j \right\}, \qquad \nu_* = \mathrm{ctype}(S|_{k_*}).$$

The first bound ensures that the first k_* entries receive no boost, and Lemma 5.1 shows that the second bound ensures $\ell(\nu_*) \le s$.

Choose one standard tableau T_θ of each shape. Construct the column-reading witness on $S|_{k_*}$ using the labels of T_θ, and use the outside exponent $s - 1$ as before. Denote the resulting polynomial by G_*, its distinguished exponent by α_*, and put $r_* = \alpha_* - b$. If $k_* = 0$, the convention is $G_* = \prod_i x_i^{s-1}$ and the column group is trivial. For the apolar pairing, $\partial^{r_*} G_* / r_*!$ represents the functional $f \mapsto [x^{r_*}] f(\partial) G_*$. Applying the adjoint Young symmetrizer puts this representer in the primitive Specht space of shape θ. We therefore define

$$h_{S,\eta} = \frac{1}{r_*!}\, \varepsilon_\theta^* \partial^{r_*} G_*, \qquad W_{S,\eta} = \mathbb{C} S_n h_{S,\eta},$$

where ε_θ is the Young symmetrizer used earlier. These choices depend only on (n, s, S, η).

For $\lambda \vdash k \le n$ with $\ell(\lambda) \le s$, set

$$\mathcal{D}_{n,\lambda,s} = \{(S,\eta) \in \mathcal{D}_{n,s} : \ell(\eta) \le n-k,\ \mathrm{ctype}(S|_k) \trianglerighteq \lambda\}.$$

The empty partition $\lambda = \varnothing$, $k = 0$, is allowed; its ideal is $(x_1^s, \dots, x_n^s)$ and its index set is $\mathcal{D}_{n,s}$.

Theorem 5.2 (Compatible harmonic copies for fixed n, s)**.** *Each $W_{S,\eta}$ is an irreducible copy of shape* $\mathrm{shape}(S)$ *in degree* $\mathrm{cc}(S) + |\eta|$*. The same collection gives the simultaneous direct sum decompositions*

$$I^{\perp}_{n,\lambda,s} = \bigoplus_{(S,\eta)\in\mathcal{D}_{n,\lambda,s}} W_{S,\eta} \qquad (\lambda \vdash k \le n,\ \ell(\lambda) \le s).$$

Proof. Lemma 3.1 applies to the new witnesses; the retained prefix has no boosts, its output has at most s rows, and all outside boosted entries satisfy $b_i \le s-1$. It therefore proves independence of all $h_{S,\eta}$ in each primitive multiplicity space, for the entire set $\mathcal{D}_{n,s}$. Indeed, with c_θ the hook product,

$$\langle F^{S'}_{T_\theta} E_{\eta'}, h_{S,\eta}\rangle = c_\theta\, [x^{r_*}](F^{S'}_{T_\theta} E_{\eta'})(\partial) G_*.$$

The witnesses lie in the image of the primitive adjoint idempotent $\varepsilon^*_\theta/c_\theta$, so Lemma 3.3 makes their generated copies irreducible and their sum direct.

All lie in the harmonic space of $(x_1^s, \dots, x_n^s)$, which has dimension s^n. Their total dimension is s^n as well. One direct count is to standardize a word $u \in \{0, \dots, s-1\}^n$, breaking ties from left to right, to obtain a permutation w. Along the entries $1, \dots, n$ of w, subtract its cocharge labels from the corresponding letters of u. The differences are nonnegative and weakly increasing; an inverse descent forces a strict increase of the original letters, compensating for the cocharge increment. Reversing these differences gives a unique partition $\eta \subseteq n \times (s - \mathrm{des}(w^{-1}) - 1)$. This operation is reversible, and RSK replaces w by a pair (S, T) of standard tableaux of the same shape, preserving its cocharge and inverse descents on S. Thus the count is exactly $\sum_{(S,\eta)\in\mathcal{D}_{n,s}} f^{\mathrm{shape}(S)}$. This proves the decomposition for the empty partition.

Now suppose $(S,\eta) \in \mathcal{D}_{n,\lambda,s}$, and write $\nu_k = \mathrm{ctype}(S|_k)$. The length condition gives $k \le n - \ell(\eta)$. Also $\nu_k \trianglerighteq \lambda$ implies $\ell(\nu_k) \le \ell(\lambda) \le s$, so the restriction lemma gives $k \le \sum_{j=1}^{s} \kappa_j$. Hence $k \le k_*$ and $\nu_k \subseteq \nu_*$ as Young diagrams. The elementary degree thresholds consequently give

$$I_{n,\lambda,s} \subseteq I_{n,\nu_k,s} \subseteq I_{n,\nu_*,s}.$$

The first inclusion uses dominance at equal size, and the second uses diagram containment, so every tail sum of ν'_* is at least the corresponding tail sum of ν'_k. Lemma 2.1 now shows that $h_{S,\eta} \in I^{\perp}_{n,\lambda,s}$. Thus the displayed direct sum is contained in that harmonic space. Its dimension equals the dimension of the quotient by Theorem 4.1; hence it is the entire harmonic space. □

Corollary 5.3 (Sums and intersections)**.** *Fix n, s, and take any finite nonempty family of Griffin ideals $I_{n,\lambda,s}$, allowing different sizes $|\lambda|$. The quotient by their intersection has a higher Specht basis $F^S_T E_\eta$ indexed by the union of their sets $\mathcal{D}_{n,\lambda,s}$; the quotient by their sum has the analogous basis indexed by the intersection of these sets. In either case the graded Frobenius character is*

$$\sum_{(S,\eta)\in\mathcal{D}} q^{\mathrm{cc}(S)+|\eta|} s_{\mathrm{shape}(S)},$$

where $\mathcal{D}$ is the appropriate union or intersection. The same statements hold for any finite expression built from sums and intersections, and one may replace E_η by any of the multiplier families in Corollary 4.3. In particular the lattice of ideals generated by these Griffin ideals under sums and intersections is distributive; for any three ideals A, B, C in that lattice,

$$A + (B \cap C) = (A + B) \cap (A + C), \qquad A \cap (B + C) = (A \cap B) + (A \cap C).$$

Proof. Orthogonal complements exchange ideal intersections with sums of harmonic spaces and ideal sums with intersections of harmonic spaces. Theorem 5.2 identifies these operations with unions and intersections of index sets in a single direct sum. The selected witnesses annihilate the resulting ideal. Their pairing with the corresponding candidates is a principal submatrix of the same triangular matrix, with nonzero diagonal, and the harmonic decomposition supplies the matching dimension. The Specht-equivariance argument completes the claim. These statements iterate under further sums and intersections. The two ideal identities follow from the distributive laws for unions and intersections of their harmonic index sets. □

Remark 5.4 (One partition records all eligibility conditions)**.** For partitions of possibly different sizes, write $\lambda \preccurlyeq \nu$ if $\sum_{i=1}^{r} \lambda_i \le \sum_{i=1}^{r} \nu_i$ for every r. For the canonical partition ν_* above, the condition $(S, \eta) \in \mathcal{D}_{n,\lambda,s}$ is exactly $\lambda \preccurlyeq \nu_*$. Indeed the row-prefix sums of $\mathrm{ctype}(S|_k)$ are $\min\{k, \sum_{i=1}^{r} \kappa_i\}$; the height restriction $\ell(\lambda) \le s$ supplies the cutoff at the first s parts, and $|\lambda| \le k_*$ supplies the unboosted-prefix cutoff.

The same order has the equivalent tail inequalities

$$q_\lambda(t) := \sum_i (\lambda_i - t)_+ \le \sum_i (\nu_i - t)_+ = q_\nu(t) \quad (t = 0, 1, 2, \ldots).$$

This follows from the mutually inverse formulas $q_\lambda(t) = \max_r(\sum_{i=1}^r \lambda_i - rt)$ and $\sum_{i=1}^r \lambda_i = \min_{t\ge 0}(rt + q_\lambda(t))$. In particular the sum of two fixed-s Griffin ideals is again a Griffin ideal. Its partition $\lambda \vee \mu$ is determined by

$$q_{\lambda\vee\mu}(t) = \max\{q_\lambda(t), q_\mu(t)\}.$$

The right side is a nonnegative, nonincreasing, discrete convex integer sequence, eventually zero. Its successive differences are therefore the conjugate parts of a partition of size at most n and length at most s. It is the least upper bound in $\preccurlyeq$, and the defining elementary thresholds give $I_{n,\lambda,s} + I_{n,\mu,s} = I_{n,\lambda\vee\mu,s}$.

5.1. **Springer harmonic spaces**

For $\mu \vdash n$, the power relations x_i^n follow from the global elementary relations, so $I_{n,\mu,n} = I_\mu$. The common construction therefore specializes to the following decomposition, independent of μ. Write $\mathcal{H}_\mu = I_\mu^\perp$.

Choose once and for all one standard tableau T_λ of each shape $\lambda \vdash n$. For $S \in \mathrm{SYT}(\lambda)$, form the column-reading witness U_S and rejection vector r^S using T_λ. Let ε_λ be its Young symmetrizer, and let ε_λ^* be the adjoint obtained by reversing every permutation. Put

$$h_S = \frac{1}{r^S!}\,\varepsilon_\lambda^* \partial^{r^S} g_{U_S}, \qquad r^S! = \prod_i r_i^S!, \qquad W_S = \mathbb{C}S_n\, h_S. \tag{5.1}$$

These polynomials have degree $\mathrm{cc}(S)$ and are defined independently of the partition μ used for a quotient.

Corollary 5.5 (Springer harmonic decompositions)**.** *Each W_S is an irreducible S_n-module of shape* $\mathrm{sh}(S)$*. The same family of submodules gives, for every partition $\mu \vdash n$, the direct sum decomposition*

$$\mathcal{H}_\mu = \bigoplus_{\substack{S\in \mathrm{SYT}_n \\ \mathrm{ctype}(S)\trianglerighteq \mu}} W_S.$$

In particular, all these decompositions occur inside one direct sum decomposition of $\mathcal{H}_{(1^n)}$.

Proof. Take $s = n$, $k = n$, and $\eta = \varnothing$ in Theorem 5.2. Every standard tableau has fewer than n descents, and the canonical prefix is all of S. Thus its witness is precisely (5.1), and its eligible indices are $\mathrm{ctype}(S) \trianglerighteq \mu$. □

Corollary 5.6 (Finite intersections)**.** *Let $\mathcal{A}$ be a nonempty finite collection of partitions of n, and put*

$$J_{\mathcal{A}} = \bigcap_{\mu \in \mathcal{A}} I_\mu, \qquad \mathcal{S}_{\mathcal{A}} = \{S \in \mathrm{SYT}_n : \mathrm{ctype}(S) \trianglerighteq \mu \text{ for at least one } \mu \in \mathcal{A}\}.$$

Then

$$\{F_T^S \bmod J_{\mathcal{A}} : S \in \mathcal{S}_{\mathcal{A}},\ T \in \mathrm{SYT}(\mathrm{sh}(S))\}$$

is a higher Specht basis of $\mathbb{C}[x_1, \ldots, x_n]/J_{\mathcal{A}}$. Its graded Frobenius character is

$$\sum_{S \in \mathcal{S}_{\mathcal{A}}} q^{\mathrm{cc}(S)} s_{\mathrm{sh}(S)}.$$

Proof. This is the same specialization of Corollary 5.3. All auxiliary partitions are empty, and ideal intersections give unions of the eligible tableaux. □

The sum formula specializes as well

$$\mathcal{H}_\mu \cap \mathcal{H}_\nu = \mathcal{H}_{\mu \vee \nu}, \qquad I_\mu + I_\nu = I_{\mu \vee \nu},$$

where $\vee$ is dominance join. It is the equal-size case of the partition join described above.

5.2. **A simultaneously adapted higher Specht basis**

Dualizing the common harmonic decomposition gives polynomial coordinates adapted to all the ideals at once. The required change from the original higher Specht polynomials is triangular.

Corollary 5.7 (A basis adapted to every fixed-s ideal)**.** *Fix n, s. There are homogeneous polynomials $\widetilde{P}_T^{S,\eta}$, indexed by $(S,\eta) \in \mathcal{D}_{n,s}$ and $T \in \mathrm{SYT}(\mathrm{shape}(S))$, obtained from $P_T^{S,\eta} = F_T^S E_\eta$ by the same triangular change of multiplicity coordinates for every T. For every allowed λ, the eligible polynomials form a higher Specht basis of $R_{n,\lambda,s}$, while every polynomial with an ineligible index lies in $I_{n,\lambda,s}$. The change of basis is defined over $\mathbb{Z}[1/n!]$. All these statements hold over that ring and over every field of characteristic zero or greater than n, with Specht lattices in place of irreducible modules over the localized ring.*

Proof. Fix a shape θ, total degree, and the chosen tableau T_θ. Use the canonical witnesses of this section in the divided-power form of Section 8. Let M be their pairing matrix with the polynomials $P_{T_\theta}^{S,\eta}$. This integer matrix is upper triangular; its diagonal entries are the products of the two group orders appearing in the divided-power pairing. They are units in $A = \mathbb{Z}[1/n!]$. Put $C = M^{-1}$, and, for indices i, j in this shape and degree, define

$$\widetilde{P}_T^{,j} = \sum_i C_{ij} P_T^{,i}.$$

This is the asserted triangular change of multiplicity coordinates.

To check the vanishing statement first over $\mathbb{C}$, let c_θ be the hook product and project the ith divided-power witness by $\varepsilon_\theta^*/c_\theta$ after contracting its rejection monomial. Its orbit is the harmonic copy W_i already constructed, up to a nonzero scalar. The pairing with the fixed-T_θ modified vectors is the identity matrix. The invariant pairing between two Specht copies of shape θ is determined by its value on these primitive, adjoint lines. Consequently the entire copy generated by $\widetilde{P}_T^{,j}$ is orthogonal to W_i whenever $i \neq j$. Theorem 5.2 now shows that an ineligible copy is orthogonal to the full harmonic space of the quotient, so it lies in the ideal. The eligible copies pair nondegenerately with its harmonic direct sum and therefore give a basis.

For the integral statement, the master quotient $A[x]/(x_1^s, \dots, x_n^s)$ has the original family as an A-basis; the same divided-power argument and the word count in Theorem 5.2 prove this exactly as in Corollary 8.1. The change C is invertible over A. Each Griffin quotient is free over A by Lemma 6.4, so a polynomial that vanishes after extension to $\mathbb{Q}$ already vanishes over A. Thus all ineligible modified vectors vanish integrally. Their eligible complements span each quotient because the complete modified family spans the master quotient. Their number is the rank of the quotient, and independence follows over $\mathbb{Q}$, hence over A. This proves the integral basis statement and all its reductions to fields of characteristic greater than n. The Specht action is preserved because C_{ij} is independent of T. □

Remark 5.8 (Why the exponent bound is fixed)**.** The common harmonic copies above are constructed for fixed s. They cannot be chosen with the same index labels simultaneously for all s. For example, take $n = 4$, let S have bottom row $1, 3, 4$ and top row 2, and take $\eta = \varnothing$. This degree-three, shape-$(3, 1)$ index is eligible both for the squarefree quotient with $k = 0, s = 2$ and for the ordinary coinvariant quotient with $k = 4, \lambda = (1^4), s = 4$. Yet their degree-three harmonic spaces have zero intersection. Indeed a squarefree cubic has the form $f = \sum_{i=1}^{4} c_i \prod_{j \neq i} x_j$. The equation $(\partial_1 + \cdots + \partial_4)f = 0$ gives $c_i + c_j = 0$ for every pair $i \neq j$, and hence $f = 0$ over $\mathbb{C}$. The coinvariant harmonic space satisfies this equation because its ideal contains e_1. Thus a nonzero common copy for this index cannot exist across these two values of s.

6. **Untruncated ideals and higher Specht free bases**

Once the power relations are removed, the auxiliary partition has no rectangle bound, so its elementary-symmetric factors range over a polynomial parameter ring. The same higher Specht polynomials therefore form a finite free basis, and the finite-dimensional basis theorem becomes a description of a flat family.

For $\lambda \vdash k \leq n$, including $\lambda = \varnothing$, remove the power generators and define

$$J_{n,\lambda} = \big(e_d(Y) : d > |Y| - p^n_{|Y|}(\lambda)\big), \qquad A_{n,\lambda} = \mathbb{C}[x_1, \ldots, x_n]/J_{n,\lambda}.$$

Thus $J_{n,\varnothing} = 0$, while $J_{n,\lambda} = I_\lambda$ when $k = n$. These are Griffin's ideals $I_{n,\lambda}$ and rings $R_{n,\lambda}$ from [8, (1.32) and Section 6]; the notation here distinguishes them from the finite-s rings. These quotients are also the diagonal scheme intersections of Eisenbud–Saltman rank varieties identified in [8, Corollary 6.4]. Put $m = n - k$ and

$$\mathcal{S}_{n,\lambda} = \{S \in \mathrm{SYT}_n : \mathrm{ctype}(S|_k) \trianglerighteq \lambda\}.$$

Theorem 6.1 (An explicit higher Specht free basis)**.** *The images of $e_1, \dots, e_m$ are algebraically independent in $A_{n,\lambda}$, and this ring is a finite free module over $\mathbb{C}[e_1, \dots, e_m]$. A homogeneous free basis is*

$$\{F_T^S : S \in \mathcal{S}_{n,\lambda},\ T \in \mathrm{SYT}(\mathrm{shape}(S))\}.$$

Its rank is $n!/\prod_i \lambda_i!$, with the empty product equal to one. Consequently $A_{n,\lambda}$ is Cohen–Macaulay of dimension $n-k$. Its graded Frobenius series is

$$\mathrm{Frob}_q(A_{n,\lambda}) = \frac{\displaystyle\sum_{S \in \mathcal{S}_{n,\lambda}} q^{\mathrm{cc}(S)} s_{\mathrm{shape}(S)}}{\displaystyle\prod_{j=1}^{n-k}(1-q^j)}.$$

Proof. Fix a degree d and choose $s > \max\{d, n\}$. The degree-d components of $J_{n,\lambda}$ and $I_{n,\lambda,s} = J_{n,\lambda} + (x_1^s, \dots, x_n^s)$ agree. For a candidate of degree d, the inequality

$$\mathrm{des}(S) + \eta_1 \le \mathrm{cc}(S) + |\eta| = d < s$$

automatically satisfies the finite-s bound. Theorem 4.1, together with the empty-partition case in Theorem 5.2, therefore gives a vector-space basis of $A_{n,\lambda}$ consisting of

$$F_T^S E_\eta, \qquad S \in \mathcal{S}_{n,\lambda}, \quad T \in \mathrm{SYT}(\mathrm{shape}(S)), \quad \ell(\eta) \le m,$$

with no bound on the largest part of η. The map $\eta \mapsto (\eta_1 - \eta_2, \dots, \eta_m)$ is a bijection to $\mathbb{Z}_{\ge 0}^m$. Thus the E_η are exactly the monomials in $e_1, \dots, e_m$. The single-row tableau is always eligible and has $F_T^S = n!$, so the independence of its E_η multiples also proves algebraic independence of the images of the e_j. The displayed vector-space basis is precisely the asserted free-module basis expanded in these monomials.

For the rank, apply RSK to permutations of $[n]$. Restriction of the insertion tableau to $[k]$ is the insertion tableau of the subword on those values. Each permutation of $[k]$ occurs as that subword in $n!/k!$ permutations of $[n]$. The number of eligible permutations of $[k]$ is $\dim R_\lambda = k!/\prod_i \lambda_i!$. This gives the claimed rank; for $k = 0$ the same calculation uses the unique empty permutation. The invariant variables $e_1, \dots, e_m$ have degrees $1, \dots, m$, and multiplication by them preserves every Specht copy, giving the Frobenius series. Finally, finite freeness over the polynomial subalgebra gives dimension m and the regular sequence $e_1, \dots, e_m$, proving Cohen–Macaulayness. □

Remark 6.2 (Coefficients of the higher Specht free basis)**.** By the same degreewise stabilization and Corollary 8.1, the displayed higher Specht free basis works over every field of characteristic zero or characteristic $p > n$, and over $\mathbb{Z}[1/n!]$. The exact

unnormalized family cannot be a basis in characteristic $p \leq n$, since its one-row member is $n! \cdot 1 = 0$. The underlying algebra has a stronger integral freeness property, proved below using a different basis.

6.1. **A flat lifting of a Springer ring**

Put $\mu = \lambda \cup (1^m)$, the partition obtained by adjoining $m = n - k$ parts equal to one.

Proposition 6.3 (The special fiber)**.** *Over $\mathbb{Z}$, and hence over every coefficient field,*

$$I_\mu = J_{n,\lambda} + (e_1, \ldots, e_m).$$

Moreover

$$\operatorname{ctype}(S|_k) \trianglerighteq \lambda \quad \Longleftrightarrow \quad \operatorname{ctype}(S) \trianglerighteq \mu.$$

Thus the higher Specht free basis in Theorem 6.1 is an explicit flat lifting of the higher Specht basis of R_μ, and

$$\operatorname{Frob}_q(A_{n,\lambda}) = \frac{\operatorname{Frob}_q(R_\mu)}{\prod_{j=1}^{m}(1-q^j)}.$$

Proof. We have $\mu' = \lambda' + m(1, 0, 0, \ldots)$. For every proper subset $Y \subsetneq [n]$, the tail sum defining its elementary generators excludes the first entry, so the generators for I_μ and $J_{n,\lambda}$ agree. For $Y = [n]$, their listed generators are respectively $e_1, \ldots, e_n$ and $e_{m+1}, \ldots, e_n$. This proves the ideal identity over $\mathbb{Z}$.

Write $\kappa = \operatorname{ctype}(S)$. The restriction lemma shows that the left condition is equivalent to $\sum_{i=1}^{r} \kappa_i \geq \sum_{i=1}^{r} \lambda_i$ for every r. Among partitions of n, these inequalities are equivalent to $\kappa \trianglerighteq \mu$. Indeed for $r \leq \ell(\lambda)$ the required inequalities are identical. Beyond that point, either κ has fewer than r nonzero parts, so its prefix sum is n, or each additional part contributes at least one. In the latter case its prefix sum is at least $k + r - \ell(\lambda)$, which is the required prefix sum of μ until the total n is reached. The converse is immediate. The asserted lifting and Frobenius identity now follow from the free-module theorem and the displayed ideal identity. □

The parameter decomposition also holds integrally. We first record why the monomial quotients used to lift its special fiber are free. Write $I^{\mathbb{Z}}_{n,\lambda,s}$ for the ideal with the same elementary and power generators over $\mathbb{Z}$, with $s = \infty$ meaning that the power generators are omitted.

Lemma 6.4 (Integral standard monomials)**.** *The quotient by $I^{\mathbb{Z}}_{n,\lambda,s}$ is a free abelian group with Griffin's standard monomials as basis. In particular, its Hilbert function is unchanged by extension to any field.*

Proof. For the empty partition the ideal is $(x_1^s, \dots, x_n^s)$, or (0) when $s = \infty$, and the assertion is immediate. For nonempty partitions, Yu constructs a monic Gröbner basis over $\mathbb{Q}$ whose polynomials lie in the ideal generated over $\mathbb{Z}$ by the specified generators; every recursive lift is an integer combination of those elementary generators; see [6, Section 4, Lemma 4.2 and Theorem 4.3]. For finite s, append $x_1^s, \dots, x_n^s$, as in [6, Lemma 2.9]. Division by these monic polynomials works over $\mathbb{Z}$. Every original generator has remainder zero; its integral remainder is also its zero remainder over $\mathbb{Q}$. Thus this is a generating set over $\mathbb{Z}$ as well. The same division shows that the standard monomials span the integral quotient. An integral relation between them would give a relation over $\mathbb{Q}$, so they are independent. Finally, tensoring this free quotient with a field gives the quotient by the same generators. □

Corollary 6.5 (Integral freeness and all coefficient fields)**.** *Let $A^{\mathbb{Z}}_{n,\lambda} = \mathbb{Z}[x]/J^{\mathbb{Z}}_{n,\lambda}$, and choose any homogeneous integral monomial basis $\mathcal{M}$ of the Springer quotient $R^{\mathbb{Z}}_{\mu}$, where $\mu = \lambda \cup (1^{n-k})$. The same monomials, lifted to $A^{\mathbb{Z}}_{n,\lambda}$, form a free basis over $\mathbb{Z}[e_1, \dots, e_{n-k}]$. In particular this algebra is free of rank $n!/\prod_i \lambda_i!$ over that polynomial subalgebra. After base change to any field, $A_{n,\lambda}$ is Cohen–Macaulay of dimension $n-k$, in every characteristic.*

Proof. The existence of $\mathcal{M}$ follows from the integral standard monomials in Lemma 6.4. Set $B_{\mathbb{Z}} = \mathbb{Z}[t_1, \dots, t_m]$, with $\deg t_j = j$, and map t_j to e_j. The special-fiber identity gives a graded $B_{\mathbb{Z}}$-linear surjection

$$\bigoplus_{b \in \mathcal{M}} B_{\mathbb{Z}}(-\deg b) \longrightarrow A^{\mathbb{Z}}_{n,\lambda}.$$

For completeness, surjectivity follows by induction on degree; reduce a homogeneous class modulo $(e_1, \dots, e_m)$ to an integral combination of the basis monomials, and express its remainder as $\sum_j e_j a_j$, where each a_j has smaller degree.

After tensoring with $\mathbb{Q}$, source and target have the same Hilbert series by Proposition 6.3 and Theorem 6.1. The rationalized surjection is therefore an isomorphism. Its integral kernel is a submodule of a free abelian group, hence torsion-free, and its rationalization is zero; it must itself be zero. This proves integral freeness. Tensoring the isomorphism with any field gives the same free-module decomposition there, and the polynomial parameters remain a regular sequence of length m. □

6.2. **A locally finite simultaneous harmonic decomposition**

The untruncated witnesses can also be chosen simultaneously. Their outside exponents now follow the boosted tableau itself, since no power bound has to be respected;

orthogonal complements are taken degree by degree, where every space is finite dimensional.

For every standard tableau S and every partition η with $\ell(\eta) \le n$, let b be the boosted tableau and put

$$k_0 = n - \ell(\eta), \qquad \nu_0 = \mathrm{ctype}(S|_{k_0}).$$

Use the column-reading witness U_0 on this prefix, and write K_0 for its labels in the fixed tableau $T_{\mathrm{shape}(S)}$. Define

$$G_\infty = g_{U_0} \prod_{i \notin K_0} x_i^{b_i}, \qquad (\alpha_\infty)_i = \begin{cases} \mathrm{row}_{U_0}(i), & i \in K_0, \\ b_i, & i \notin K_0, \end{cases} \qquad r_\infty = \alpha_\infty - b.$$

Thus the residue is the rejection count on the prefix and zero outside it. Set

$$h^\infty_{S,\eta} = \frac{1}{r_\infty!}\, \varepsilon^*_{\mathrm{shape}(S)} \partial^{r_\infty} G_\infty, \qquad W^\infty_{S,\eta} = \mathbb{C}S_n h^\infty_{S,\eta}.$$

The empty-prefix convention is $g_{U_0} = 1$.

Theorem 6.6 (Simultaneous untruncated harmonic copies)**.** *The polynomials $h^\infty_{S,\eta}$ generate irreducible Specht copies of shape* $\mathrm{shape}(S)$ *in degree* $\mathrm{cc}(S) + |\eta|$*. They give one locally finite direct sum decomposition of the polynomial ring such that, for every $\lambda \vdash k \le n$,*

$$J^\perp_{n,\lambda} = \bigoplus_{\substack{S \in \mathcal{S}_{n,\lambda} \\ \ell(\eta) \le n-k}} W^\infty_{S,\eta}.$$

In particular the case $\lambda = \varnothing$ is the whole polynomial ring.

Proof. The elementary-annihilation part of Lemma 2.1 does not require the outside exponents to be equal; each outside variable contributes at most one to the degree in t of $\prod_{i \in Y}(1 + t\partial_i) G_\infty$. Its proof therefore shows that G_∞ annihilates J_{n,ν_0}. If $S \in \mathcal{S}_{n,\lambda}$ and $\ell(\eta) \le n - k$, then $k \le k_0$ and the restriction lemma gives

$$\mathrm{ctype}(S|_k) \subseteq \nu_0, \qquad J_{n,\lambda} \subseteq J_{n,\mathrm{ctype}(S|_k)} \subseteq J_{n,\nu_0}.$$

Hence the associated $h^\infty_{S,\eta}$ belongs to the desired harmonic space.

Lemma 3.1 applies; outside labels are fixed singleton blocks, now with zero residue, while all retained labels have zero boost and the same column-reading monotonicities. Consequently the pairing with the polynomials $F^S_T E_\eta$ is upper triangular with positive diagonal in every fixed shape and degree. The primitive-idempotent argument gives a direct sum of the irreducible copies $W^\infty_{S,\eta}$. There are only finitely many pairs (S, η)

in each degree. Theorem 6.1 supplies the matching dimension in each degree, first for $\lambda = \varnothing$ and then for every other λ. This proves the stated decompositions. □

Corollary 6.7 (Untruncated sums and intersections)**.** *For any finite expression of sums and intersections of the ideals $J_{n,\lambda}$, the quotient has a higher Specht vector-space basis $F_T^S E_\eta$ indexed by the corresponding unions and intersections of the eligible pairs (S, η); ideal intersections give unions, and ideal sums give intersections. Its graded Frobenius series is the sum of $q^{\mathrm{cc}(S)+|\eta|} s_{\mathrm{shape}(S)}$ over that index set. This lattice of ideals is distributive.*

If all partitions in the expression have the same size k, the quotient is finite free over $\mathbb{C}[e_1, \ldots, e_{n-k}]$. A free basis consists of the F_T^S indexed by the corresponding unions and intersections of the sets $\mathcal{S}_{n,\lambda}$. It is Cohen–Macaulay of dimension $n - k$.

Proof. Use the common harmonic decomposition degree by degree, as in Corollary 5.3. All the relevant vector spaces within a fixed degree are finite dimensional, so orthogonal complements exchange sums and intersections. The same principal triangular submatrices prove the quotient-basis assertion. For partitions of a common size k, every eligible set has the same unrestricted multiplier condition $\ell(\eta) \leq n - k$. Unions and intersections preserve this product structure, and the free-module argument of Theorem 6.1 applies directly. The single-row tableau remains eligible for every expression, so these quotients are nonzero. □

7. **Fibers and primary components**

The free parameters determine a characteristic polynomial whose distinct roots separate the coordinates into groups. The rank conditions remain on the zero group, while translation by a nonzero root leaves an ordinary coinvariant ring on its group. Working directly in the Artin local factors includes repeated roots and positive characteristic in the same calculation, and gives both the fiber decomposition and the primary components.

Let K be any field, and let a point of the parameter space have parameter polynomial

$$Q(t) = t^m - a_1 t^{m-1} + \cdots + (-1)^m a_m = t^{r_0} \prod_{j=1}^{a} (t - c_j)^{r_j},$$

where $c_1, \ldots, c_a \in K$ are pairwise distinct and nonzero, $r_j > 0$ for $j > 0$, and $r_0 + \cdots + r_a = m$. Put $k_0 = k + r_0$ and $\mu_0 = \lambda \cup (1^{r_0})$.

Theorem 7.1 (Every fiber)**.** *The fiber algebra is*

$$A^K_{n,\lambda}/(e_j - a_j : 1 \le j \le m) \;\cong \prod_{n!/(k_0!\prod_{j=1}^{a} r_j!)} \left(R^K_{\mu_0} \otimes_K \bigotimes_{j=1}^{a} R^K_{(1^{r_j})} \right).$$

We use $R^K_\varnothing = K$. The factors are indexed by assigning k_0 coordinate positions to zero and r_j positions to each c_j. In particular, distinct nonzero roots give $\prod_{n!/k!} R^K_\lambda$, whereas the fiber at the origin is $R^K_{\lambda\cup(1^m)}$. Over a field in which Q does not split, the displayed formula holds after extension to a splitting field.

Proof. The fiber algebra D is finite-dimensional by Corollary 6.5. Its global characteristic-polynomial identity is

$$\prod_{i=1}^{n}(t - x_i) = t^k Q(t).$$

Consequently every closed point assigns the specified roots to coordinate positions with the stated multiplicities. These points are K-rational because Q splits over K. Fix one closed point, with zero group A of size k_0 and nonzero groups B_j of sizes r_j, and work in its Artin local factor.

The monic group polynomials satisfy

$$\prod_{i\in A}(t - x_i) = t^{k_0}, \qquad \prod_{i\in B_j}(t - x_i) = (t - c_j)^{r_j}. \tag{7.1}$$

Here is a direct justification over an arbitrary characteristic. Modulo the nilpotent maximal ideal these identities hold, and the proposed factors on the right are pairwise coprime; a factor of degree zero may simply be omitted. Suppose the identities hold modulo its vth power. Write the errors modulo its $(v+1)$st power as $\delta_j(t)$, each of degree smaller than its monic factor $p_j(t)$. The product identity gives $\sum_j \delta_j \prod_{h\neq j} p_h = 0$. Reducing modulo p_j and using coprimality shows $\delta_j = 0$; its degree is smaller than $\deg p_j$. Induction through the nilpotent powers proves (7.1). Thus the zero group has its global elementary polynomials equal to zero, and each shifted group $(x_i - c_j)_{i\in B_j}$ has the same property.

We check the remaining ideal on the zero group exactly. Write $B = \bigcup_{j=1}^{a} B_j$, $b_0 = |B|$, and $q_\lambda(h) = \sum_{j>h} \lambda'_j$. An original generator on $Y = D_0 \sqcup B'$, where $D_0 \subseteq A$, $B' \subseteq B$, $d_0 = |D_0|$ and $b = |B'|$, has degree

$$d > d_0 + b - q_\lambda(h), \qquad h = n - d_0 - b \ge k_0 - d_0.$$

Its expansion is

$$e_d(D_0 \sqcup B') = \sum_{j=0}^{b} e_j(B') e_{d-j}(D_0).$$

If $D_0 \subsetneq A$, then $k_0 - d_0 \geq 1$, so $q_{\mu_0}(k_0 - d_0) = q_\lambda(k_0 - d_0)$ and

$$d - j > d_0 - q_\lambda(h) \geq d_0 - q_{\mu_0}(k_0 - d_0).$$

Each factor $e_{d-j}(D_0)$ therefore lies in I_{μ_0}. If $D_0 = A$, then $d-j > k_0 - q_\lambda(h) \geq r_0 \geq 0$, so it belongs to the global elementary ideal on A. This proves that every original generator vanishes in

$$C = R^K_{\mu_0} \otimes_K \bigotimes_{j=1}^{a} R^K_{(1^{r_j})},$$

where the variables of the latter factors are shifted by c_j. The global fiber equations also hold in C by (7.1).

Conversely, for any proper $D_0 \subsetneq A$, a Tanisaki generator $e_r(D_0)$ satisfies $r > d_0 - q_\lambda(k_0 - d_0)$. Hence the original ideal contains $e_{b_0+r}(D_0 \sqcup B)$. Its expansion has the form

$$e_{b_0}(B) e_r(D_0) + \sum_{j<b_0} e_j(B) e_{b_0+r-j}(D_0).$$

The coefficient $e_{b_0}(B) = \prod_{i \in B} x_i$ is a unit in the local ring, since each of these coordinates has nonzero residue. Descending induction on r, starting at $r = d_0$, therefore recovers every such Tanisaki generator. The global generators on A and the elementary generators in each shifted nonzero group already follow from (7.1). Thus the local ideal is exactly the ideal defining C.

The algebra C is supported at the chosen assignment, so the two ideal inclusions give the asserted local isomorphism and also show that every assignment occurs. There are $n!/(k_0! \prod_j r_j!)$ assignments. The decomposition of the finite algebra into its Artin local factors proves the formula. □

Proposition 7.2 (Trivialization over ordered distinct roots)**.** *Let*

$$B = \mathbb{Z}[t_1, \dots, t_m], \qquad B' = \mathbb{Z}[c_1, \dots, c_m, (\prod_j c_j \prod_{i<j} (c_i - c_j))^{-1}],$$

and map t_j to $e_j(c_1, \dots, c_m)$. Then there is an S_n-equivariant B'-algebra isomorphism

$$A^{\mathbb{Z}}_{n,\lambda} \otimes_B B' \;\cong \prod_{\iota:[m]\hookrightarrow[n]} \big(R^{\mathbb{Z}}_\lambda \otimes_{\mathbb{Z}} B'\big).$$

In the factor indexed by ι, the coordinate $x_{\iota(j)}$ equals c_j, and the remaining k coordinates carry their Springer algebra. Thus the underlying S_n-module over B' is

$$\mathrm{Ind}_{S_k}^{S_n}(R^{\mathbb{Z}}_\lambda \otimes B'),$$

where S_k permutes the remaining coordinates. The same formulas hold after base change to every coefficient field. For $m = 0$, the inverted product is one and the formula is simply the identity on $R^{\mathbb{Z}}_\lambda$.

Proof. For each injection, impose $x_{\iota(j)} = c_j$ and the Tanisaki ideal on the complementary k coordinates. The elementary expansion in the proof of Theorem 7.1 shows integrally that every generator of $J_{n,\lambda}$ vanishes in this quotient. Its global elementary functions are exactly the $e_j(c)$, so these assignments define a map from the displayed base change to the product.

The assignment ideals in $B'[x]$ are pairwise comaximal. Two distinct assignments either put different constants in one coordinate, producing the unit $c_i - c_j$, or put a constant c_j in a coordinate which is nilpotent in the other assignment. In the latter case the other ideal contains x_i^k, so their sum contains the unit c_j^k. When $k = 0$, only the first case is needed. The Chinese remainder theorem therefore makes the map surjective.

Both source and target are finite free B'-modules of rank $n!/\prod_i \lambda_i!$. A surjection between free modules of the same finite rank is an isomorphism. Variable permutations permute the injections transitively, and the stabilizer of an injection is the symmetric group on its complementary positions. This proves the equivariance and the induced-module description. The empty partition has $k = 0$ and $n!$ scalar factors, so is included in the argument. □

Proposition 7.3 (Primary components and their multiplicities)**.** *Over every field K, the untruncated ideal has the primary decomposition*

$$J^K_{n,\lambda} = \bigcap_{\substack{A\subseteq[n]\\|A|=k}} I_\lambda(x_i : i \in A)\, K[x_1,\dots,x_n].$$

The same identity holds over $\mathbb{Z}$, with the displayed terms again primary. Over a field, its reduced support is the union of the coordinate $(n-k)$-planes. Each such plane has generic multiplicity $k!/\prod_i \lambda_i!$, and the standard graded multiplicity of the whole quotient is

$$\frac{n!}{(n-k)!\prod_i \lambda_i!}.$$

For $\lambda = \varnothing$, the intersection has its single term (0).

Proof. Work over a coefficient field or over $\mathbb{Z}$, writing R for the coefficient ring and $S = R[x_1,\dots,x_n]$. For $|A| = k$, put $Q_A = I_\lambda(x_i : i \in A)S$ and $q_\lambda(h) = \sum_{j>h} \lambda'_j$. Expanding an elementary generator on $Y = D \sqcup B'$, with $D \subseteq A$ and $B' \subseteq [n]\backslash A$, shows that $J^R_{n,\lambda} \subseteq Q_A$; each elementary factor on D has degree greater than $|D| - q_\lambda(k-|D|)$. For $m = 0$, there is only one component and the ideal identity is immediate. We prove the identity for $m > 0$ next.

The quotient S/Q_A is $R^R_\lambda \otimes_R R[x_i : i \notin A]$. Its global elementary parameters equal the elementary functions of the m outside variables, because all positive global elementary functions on A vanish. The polynomial ring on those m variables is free of rank $m!$ over its elementary-symmetric subalgebra, by the empty-partition case of integral freeness and base change. Therefore S/Q_A is free over $B = R[t_1,\dots,t_m]$, with t_j acting by e_j, of rank $m!k!/\prod_i \lambda_i!$.

In S/Q_A, the parameter e_m equals $\prod_{i\notin A} x_i$. After inverting e_m, every outside coordinate is a unit and every coordinate on A is nilpotent. The ideals Q_A consequently become pairwise comaximal; for two distinct k-subsets, one coordinate is nilpotent in one quotient and a unit in the other. The Chinese remainder theorem gives a surjection

$$(S/J^R_{n,\lambda})[e_m^{-1}] \longrightarrow \prod_{|A|=k} (S/Q_A)[e_m^{-1}].$$

Both modules are free over $B[t_m^{-1}]$ of rank $n!/\prod_i \lambda_i!$, so this map is an isomorphism. Since $S/J^R_{n,\lambda}$ is B-free, multiplication by e_m is injective. Thus the kernel before localization is already zero, proving $J^R_{n,\lambda} = \bigcap_{|A|=k} Q_A$. This also includes $k = 0$, when there is only the component (0).

For the primary assertion, valid also when $m = 0$, put $P_A = (x_i : i \in A)$. The quotient by Q_A is graded by degree in A, with each graded piece free over the domain

$R[x_i : i \notin A]$. An element outside P_A has a nonzero degree-zero residue, which acts injectively on the lowest nonzero graded piece of any element. Every element of P_A is nilpotent in the quotient. Thus Q_A is P_A-primary. The primes P_A are distinct and incomparable, so the primary decomposition is irredundant and these are exactly its minimal primes.

Over a field, the localization at P_A has length $k!/\prod_i \lambda_i!$. There are $\binom{n}{k}$ coordinate components, each of degree one, giving standard graded multiplicity $n!/((n-k)!\prod_i \lambda_i!)$. □

Corollary 7.4 (Reducedness and the étale locus)**.** *Over every field K, the algebra $A^K_{n,\lambda}$ is reduced if and only if λ has at most one part, allowing the empty partition.*

For a parameter fiber, factor Q over an algebraic closure as in Theorem 7.1. That fiber is geometrically reduced *if and only if $\lambda \cup (1^{r_0})$ has at most one part and every nonzero-root multiplicity r_j equals one. Consequently the finite flat map to $\mathbb{A}^m_K$ is étale over the squarefree parameter polynomials Q when $\lambda = \varnothing$, and over the squarefree Q with $Q(0) \neq 0$ when $\lambda = (k)$ with $k > 0$. Its étale locus is empty when $\ell(\lambda) > 1$. For $m = 0$, the constant polynomial $Q = 1$ satisfies the squarefree and nonzero-constant conditions by convention. Ordinary reducedness of a nonsplit fiber over an imperfect field is not asserted by this criterion.*

Proof. If λ has one part, the primary components in Proposition 7.3 are the coordinate prime ideals themselves, since $R^K_{(k)} = K$. Their intersection is radical; the empty-partition case is the polynomial ring. If λ has more than one part, the localization along each minimal component has length $k!/\prod_i \lambda_i! > 1$ and residue field $K(x_i : i \notin A)$. It is a nonreduced Artin local algebra, so $A^K_{n,\lambda}$ cannot be reduced.

After algebraic closure, each local factor in the all-fiber formula is a tensor product of Artin local algebras with the same residue field. It is reduced exactly when its dimension is one. The Springer factor has dimension one exactly when its partition has at most one part; a nonzero-root factor has dimension $r_j!$, which equals one exactly when $r_j = 1$. These are dimensions as integers, and the conclusion is unchanged in small characteristic. This proves the geometric-reducedness criterion.

A finite flat morphism is étale at a point exactly when its local geometric fiber is reduced. Substituting $\mu_0 = \lambda \cup (1^{r_0})$ in the criterion gives the three listed cases. If $m = 0$, the map is the single Springer algebra R^K_λ over K; it is étale exactly when λ has at most one part, consistently with the stated convention. □

7.1. **Regularity and the canonical module**

The regular sequence of elementary parameters transfers graded invariants from the Springer fiber to the untruncated ring. Its top degree determines regularity, and

concentration of its socle in that degree determines the canonical module.

Corollary 7.5 (Regularity)**.** *For $m = n - k$ and $n(\lambda) = \sum_i (i-1)\lambda_i$, over every field one has*

$$\operatorname{reg}(A_{n,\lambda}) = n(\lambda) + m\ell(\lambda).$$

Proof. Put $\mu = \lambda \cup (1^m)$. The top degree of R_μ is $n(\mu)$. This also follows directly from our Springer basis; for an eligible S, its column witness has $r = \alpha - a \geq 0$, so

$$\operatorname{cc}(S) = \sum_i a_i \leq \sum_i \alpha_i = n(\operatorname{ctype}(S)) \leq n(\mu).$$

The last inequality is the usual reversal of $n(\cdot)$ under dominance, obtained by summing the row-prefix inequalities. The row-superstandard tableau of shape μ has catabolizability type μ and cocharge $n(\mu)$, so equality is attained. Integral freeness makes the Hilbert series unchanged in every characteristic.

Write the standard graded Hilbert series of $A_{n,\lambda}$ as $h(q)/(1-q)^m$. The flat-lifting identity gives

$$h(q) = \frac{\operatorname{Hilb}(R_\mu; q)}{\prod_{j=1}^{m}(1 + q + \cdots + q^{j-1})}.$$

The quotient is Cohen–Macaulay over every field, so its regularity is the degree of this Hilbert numerator. Therefore

$$\operatorname{reg}(A_{n,\lambda}) = n(\mu) - \binom{m}{2} = n(\lambda) + m\ell(\lambda). \qquad \square$$

Corollary 7.6 (Levelness, type and the Gorenstein cases)**.** *Let $\mu = \lambda \cup (1^{n-k})$, and let f^μ be the number of standard tableaux of shape μ. Over every coefficient field, $A_{n,\lambda}$ is level of Cohen–Macaulay type f^μ; its canonical module is generated in the single degree*

$$\binom{n-k+1}{2} - n(\mu).$$

It is Gorenstein exactly when μ is empty, a single row, or a single column. Equivalently, λ is a column, including the empty partition, or $k = n$ and $\lambda = (n)$. These are also exactly the complete-intersection cases.

Proof. We first verify the necessary Springer socle statement in every characteristic. The empty partition gives the field itself. For $\nu \vdash N > 0$, put $D = n(\nu)$. Use divided-power symbols $x^{[a]}$, with monomial contraction $x^b \lrcorner x^{[a]} = x^{[a-b]}$ when $a \geq b$, and zero otherwise. For a filling U of shape ν, define

$$\widetilde{g}_U = \sum_{\tau \in C(U)} \operatorname{sgn}(\tau) x^{[\tau\alpha]}, \qquad \alpha_i = \operatorname{row}_U(i).$$

These degree-D alternants annihilate I_ν integrally. Indeed over $\mathbb{Q}$ the divided-power identification sends $\widetilde{g}_U$ to $g_U/\prod_i \alpha_i!$; the direct annihilation lemma therefore proves the contraction identities over $\mathbb{Q}$, and their integer coefficients prove them over $\mathbb{Z}$ and every field.

Let $\mathcal{C}_\nu$ be the nonnegative vectors bounded coordinatewise by a shuffle of the column staircases

$$(\nu'_1 - 1, \dots, 0),\ (\nu'_2 - 1, \dots, 0),\ \dots.$$

At exponent bound $s = \ell(\nu)$, the Griffin quotient is R_ν integrally. Indeed, modulo the global elementary generators, $e_r([N] \setminus \{i\}) = (-x_i)^r$, and the proper-subset threshold makes $e_{\ell(\nu)}([N] \setminus \{i\})$ zero. When $\ell(\nu) = N$, that polynomial is identically zero. Thus $x_i^{\ell(\nu)} \in I_\nu$ over $\mathbb{Z}$. Griffin's monomial-basis theorem gives $|\mathcal{C}_\nu| = \dim R_\nu$; see [7, Section 3.2 and Theorem 4]. For $c \in \mathcal{C}_\nu$, choose such a shuffle $a \geq c$ and place its indices in the corresponding columns of a filling U. The column alternant then has lexicographic leading exponent a and coefficient ± 1.

Contract $\widetilde{g}_U$ by x^{a-c}. Its leading term is $x^{[c]}$ with coefficient ± 1; every other surviving exponent remains lexicographically smaller after the same subtraction. The $\mathcal{C}_\nu$-indexed contractions are consequently linearly independent over every field. The integral standard monomials give the same dimension in all characteristics, so these contractions form the entire divided-power inverse system of I_ν. It is therefore generated under contraction in the single degree D. Dualizing the multiplication maps shows that the socle of R_ν is concentrated in degree D.

Its dimension is f^ν. Over characteristic zero, the top space of alternants is the usual Specht space; the map from row tabloids to monomials with their row exponents is injective and sends polytabloids to the column alternants. Its dimension is f^ν. The characteristic-independent Hilbert function then gives the same top dimension over every field. Thus every R_ν is level of type f^ν.

Now use the regular sequence $e_1, \dots, e_m$ in $A = A_{n,\lambda}$, where $m = n - k$, whose quotient is R_μ. For canonical modules, regular reduction gives

$$\omega_{R_\mu} \cong \big(\omega_A/(e_1, \dots, e_m)\omega_A\big) \left(\sum_{j=1}^{m} j\right).$$

Graded Nakayama therefore transfers generation in one degree and the number of generators from ω_{R_μ} to ω_A. The former is generated in degree $-n(\mu)$ by f^μ elements. This proves the asserted degree and type, with no restriction on the characteristic. Finally $f^\mu = 1$ exactly for the empty partition, a row, or a column, which gives the Gorenstein criterion. The empty-partition case has $\mu = (1^n)$ and recovers the polynomial ring; when $m = 0$, the statement is exactly the Springer socle statement.

For $\lambda = (1^k)$, the proper-subset generators vanish identically, so $J_{n,\lambda} = (e_{m+1}, \dots, e_n)$. This is a regular sequence; the polynomial ring is free over its full elementary-symmetric polynomial subalgebra. For $m = 0$ and $\lambda = (n)$, the ideal is $(x_1, \dots, x_n)$. Thus every listed case is a complete intersection. Conversely, every complete intersection is Gorenstein, so there are no others. □

8. **The exact characteristic range**

Since the integral standard-monomial basis in Lemma 6.4 fixes the dimension in every characteristic, the coefficient problem is entirely one of independence for the specified higher Specht polynomials. The coefficient pairing removes the factorials introduced by ordinary differentiation, including those depending on the exponent bound; only orders of subgroups of S_n remain.

Corollary 8.1 (The exact characteristic range)**.** *Over a field, the specified unnormalized Springer and Δ-Springer higher Specht families, with the prescribed E_η factors, are bases if and only if the characteristic is zero or is a prime $p > n$. In these characteristics each fixed-S Springer family, and each fixed-(S, η) Δ-Springer family, is an irreducible Specht module of its stated shape. There is no restriction involving s.*

Moreover, the displayed polynomials form bases of the corresponding quotients over $\mathbb{Z}[1/n!]$ in both cases.

Proof. Use the coefficient functionals Λ_b of Lemma 3.1. They are defined over $\mathbb{Z}$. Equivalently, in the divided-power module with contraction $x^\beta \lrcorner x^{[\gamma]} = x^{[\gamma-\beta]}$ when $\gamma \geq \beta$ and zero otherwise, they pair against

$$\widetilde{G} = \sum_{\tau \in C(U)} \operatorname{sgn}(\tau) x^{[\tau\alpha]}$$

and extract the coefficient of $x^{[r]}$. Over $\mathbb{Q}$, the identification $x^{[\gamma]} \mapsto x^\gamma/\gamma!$ takes $\widetilde{G}$ to $G/\prod_i \alpha_i!$. The annihilation identities therefore hold over $\mathbb{Q}$, hence over $\mathbb{Z}$ because their coefficients are integral. The functionals descend after any field base change.

Their triangular diagonals are already given by (3.1)

$$|\operatorname{Stab}_{R(T)}(a)|\,|C(U) \cap \operatorname{Stab}(r) \cap C(T)|.$$

Both group orders divide $n!$. Thus every diagonal is a unit after inverting $n!$, with no condition on the outside exponent $s - 1$.

For $p > n$, the symmetric-group algebra is split semisimple, and the usual standard-polytabloid Specht modules are irreducible. The equivariant argument that promotes

fixed-T independence to independence of all the Specht copies therefore applies unchanged. One may equivalently extend first to an algebraic closure and then descend linear independence. Lemma 6.4 gives the required dimension in every characteristic, so the independent families are bases. Conversely, the one-row tableaux $S = T$ and the empty auxiliary partition always occur among the candidates. Their polynomial is $n! \cdot 1$, which is zero in every characteristic $p \leq n$. The specified family therefore cannot be a basis in any such characteristic. The quotients here are nonzero; $1 \leq |\lambda| \leq n$ implies $p_m^n(\lambda) \leq m$, so every defining elementary generator has positive degree, as do the power generators.

For the last assertion, express the proposed integral polynomials in the integral standard-monomial basis of Lemma 6.4. Their square coefficient matrix has a nonzero integer determinant. The field assertion shows that no prime larger than n divides this determinant. It is therefore a unit in $\mathbb{Z}[1/n!]$, proving the claimed integral bases. □

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