Gröbner bases for symmetric rank conditions

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Abstract. The prescribed minors of a symmetric northwest rank profile generate a prime ideal and form a Gröbner basis for the folded antidiagonal order, over every field. We prove this by a common rank-reduction rule that also treats arbitrary northwest bounds together with any finite collection of symbolic powers of full symmetric determinantal ideals. Paired insertion records the rank conditions and symbolic orders in one tableau, whose first-row deletion matches Schur-complement rank reduction. The resulting Gröbner bases are monic over the integers, so their standard monomials persist under arbitrary commutative base change and give compatible degenerations for sums and intersections. For symbolic Rees algebras the same statistics yield finite SAGBI bases, together with a sharp obstruction to combining two symbolic gradings using minors alone.

1. **Introduction**

Northwest rank conditions describe a symmetric matrix through the ranks of all its initial rectangular submatrices. The corresponding minors define a closed locus; the issue is whether they generate its defining ideal and give a single irreducible scheme in every characteristic. We prove that a realizable rank profile gives a prime ideal and that its prescribed minors form a Gröbner basis for a folded antidiagonal order, in every characteristic. The same straightening rule treats arbitrary rank bounds and symbolic determinantal conditions simultaneously.

Let $U = (u_{ab})$ be a generic symmetric $n \times n$ matrix, with

$$S_n = K[u_{ab} : 1 \le a \le b \le n], \qquad u_{ba} = u_{ab}.$$

For a symmetric array $r = (r_{ij})$ of nonnegative integers, let $I(r)$ be generated by the $(r_{ij}+1)$-minors in $U_{[i]\times[j]}$. A size exceeding either side of the rectangle contributes no minors.

The *folded antidiagonal order* is graded reverse lexicographic order with variables ordered increasingly by $(\max(a,b), \min(a,b))$ in lexicographic order. At equal degree, a larger exponent at the largest differing folded index makes the monomial smaller.

For increasing lists $A = (a_1 < \cdots < a_t)$ and $B = (b_1 < \cdots < b_t)$, put

$$[A \mid B] = \det U_{A\times B}, \qquad \operatorname{anti}(A,B) = \prod_{p=1}^{t} u_{a_p b_{t+1-p}}.$$

The product retains multiplicities under symmetry; transposed entries can therefore produce a square. Let $J(r)$ be generated by the folded antidiagonals of the prescribed minors.

Theorem 1.1 (Symmetric rank loci)**.** *Suppose $r_{ij} = \operatorname{rank} w_{[i]\times[j]}$ for a symmetric matrix w over a field. Then $I(r)$ is prime over every field of coefficients, and*

$$\operatorname{in} I(r) = J(r).$$

Over an algebraically closed field, the rank locus X_w consequently has defining ideal $I_w = I(r)$ and initial ideal $J_w = J(r)$.

This proves the conjecture of Marberg and Pawlowski on symmetric rank loci [2]. Eliminating successive symmetric pivots will prove primeness, with a two-by-two pivot handling the off-diagonal case without division by two. The Gröbner argument is more general; it requires no realizability condition on the rank array and uses the same order to impose symbolic powers as well. Let $P_{n,t}$ be the ideal of all t-minors of U, which is prime by Theorem 1.1. For $s \geq 1$, set

$$\mathcal{G}_{n,t,s} = \left\{ \delta_1 \cdots \delta_k : \begin{array}{l} \delta_j \text{ is an } a_j\text{-minor}, \quad a_j \geq t, \\ \sum_j (a_j - t + 1) = s \end{array} \right\}, \qquad J_{n,t,s} = (\operatorname{LM} g : g \in \mathcal{G}_{n,t,s}).$$

The set is finite; a product has at most s factors and degree at most st. Symbolic powers of symmetric determinantal ideals are generated by products of minors [1, Theorem 2.6]. We prove that their leading terms remain compatible when symbolic orders and northwest ranks are imposed together.

Theorem 1.2 (Compatible rank and symbolic conditions)**.** *For every field K, every symmetric array r, and every finite list (t_ν, s_ν) with $1 \leq t_\nu \leq n$ and $s_\nu \geq 1$, the prescribed minors for $I(r)$ together with the sets $\mathcal{G}_{n,t_\nu,s_\nu}$ form a Gröbner basis of*

$$L = I(r) + \sum_{\nu} P_{n,t_\nu}^{(s_\nu)}.$$

In particular,

$$\operatorname{in} L = J(r) + \sum_{\nu} J_{n,t_\nu,s_\nu}.$$

After normalization of leading coefficients, this is a monic Gröbner basis over $\mathbb{Z}$. Its standard monomials form a basis of the quotient after every commutative base change, with the ideals over a general coefficient ring defined by these integral generators.

The two theorems share a rank-reduction argument. Paired insertion sends a symmetric monomial to a tableau with even row lengths, in which northwest antidiagonal ranks become tableau heights and symbolic orders become tails of the half-shape. Deleting the first tableau row gives a recurrence for the standard monomials. The Schur complement at u_{11} gives the same recurrence on the ideals, lowering positive rank bounds by one and replacing $P_{n,t}$ by $P_{n-1,t-1}$. Matching the two recurrences proves the Gröbner statements. We use the column Beissinger correspondence and its dual-equivalence rule [3, Theorems 2.12–2.13 and Lemma A.4], and establish the required antidiagonal and symbolic-order invariants here.

Compatibility of the initial ideals gives simultaneous integral bases for finite sums and intersections, while additivity of a single symbolic score yields a finite SAGBI basis for each symbolic Rees algebra. Two symbolic scores reveal a boundary of this mechanism; for the ideals of 2-minors and t-minors, $4 \le t \le n$, the natural collection of minors is a SAGBI basis of the joint symbolic Rees algebra exactly when $n = t$. A full determinant contributes additively to both scores, whereas on $t + 1$ vertices the two optimal scores can require incompatible factorizations. This distinction explains both the positive result and its sharpness.

2. **Paired insertion and antidiagonal chains**

A common tableau description must record both the largest antidiagonal in each northwest rectangle and the largest union of a given number of antidiagonals. These will become, respectively, a prefix height and a statistic of the full shape. We prove the two correspondences together by reducing to matchings, where local moves preserve both quantities and lead to explicit canonical representatives.

For a monomial m, let $\rho_m(i, j)$ be the largest size of a folded antidiagonal in $[i] \times [j]$ dividing m. Thus

$$m \notin J(r) \quad \Longleftrightarrow \quad \rho_m(i, j) \le r_{ij} \quad \text{for all } i, j. \tag{2.1}$$

Regard m as a multigraph with loops. Process its factors (a, b), $a \le b$, in increasing lexicographic order of (b, a), with their multiplicities. Row-insert a, bumping the first strictly greater entry in each row, and append b immediately to the right of the new box. Denote the resulting tableau by $T(m)$, and by $T_j(m)$ the tableau after all

factors with larger endpoint at most j have been processed. Its row lengths are even. We shall prove semistandardness along with the following two assertions.

Theorem 2.1 (Insertion statistics)**.** *For every symmetric monomial and* $1 \le i \le j \le n$*,*

$$\rho_m(i,j) = \mathrm{ht}\big(T_j(m)_{\le i}\big). \tag{2.2}$$

If $T(m)$ *has shape* $2\mu = (2\mu_1, 2\mu_2, \ldots)$*, and* $G_k(m)$ *is the largest degree of a divisor of* m *expressible as a product of* k *antidiagonals, with empty factors allowed, then*

$$G_k(m) = \sum_i \min\{k, \mu_i\} = \mu'_1 + \cdots + \mu'_k. \tag{2.3}$$

Paired insertion is a content-preserving bijection between symmetric monomials and semistandard tableaux on $[n]$ *with even row lengths.*

2.1. **Matching moves**

First let m be a perfect matching on $[N]$, and let y be its fixed-point-free involution. An admissible decreasing chain consists of points

$$(a_1, y(a_1)), \ldots, (a_t, y(a_t)), \qquad a_1 < \cdots < a_t, \quad y(a_1) > \cdots > y(a_t),$$

using each unordered matching edge at most once. These are exactly the antidiagonals dividing the matching monomial. Let $\rho_L(y)$ be the greatest size of such a chain whose rows lie in an interval L, and write $L_q(y) = \rho_{[q]}(y)$.

On matchings, paired insertion is column Beissinger insertion; ordinary insertion creates a box in an odd column, and the bottom box of the next column lies immediately to its right. Write $P(y)$ for its standard tableau. For $A_k = \{k-1, k, k+1\}$, set

$$e(t) = \begin{cases} y(t), & y(t) \notin A_k, \\ t, & y(t) \in A_k. \end{cases}$$

Let $d_k(y)$ be y, $(k-1,k)y(k-1,k)$, or $(k,k+1)y(k,k+1)$ according as the median of these three values is $e(k)$, $e(k+1)$, or $e(k-1)$. Write D_k for the elementary dual-equivalence involution on the tableau labels $k-1, k, k+1$. The cited dual-equivalence and descent rules are

$$P(d_k(y)) = D_k P(y), \qquad \mathrm{Des}\, P(y) = \{k : y(k) > y(k+1),\ y(k) \neq k+1\}. \tag{2.4}$$

The descent formula follows from [3, Lemma A.4] in both orders of two consecutive labels; paired labels give increasing modified partners. The two-label case is immediate.

Lemma 2.2 (Local invariance)**.** *A move d_k preserves ρ_L whenever $A_k \subseteq L$. It also preserves the largest number of distinct edges covered by h admissible chains, for every $h \geq 0$.*

Proof. Write $a < b < c$ for the consecutive labels in the move. Up to reversing all labels and interchanging the two matchings, its nontrivial cases are

$$\begin{array}{c|ccc} & a & b & c \\ \hline y & \alpha & \beta & \gamma \\ z & \beta & \alpha & \gamma \end{array} \qquad \alpha < \gamma < \beta, \tag{2.5}$$

with all three partners external, and

$$y : \{a, c\}, \{b, \gamma\}, \qquad z : \{b, c\}, \{a, \gamma\}, \qquad \gamma > c. \tag{2.6}$$

Indeed these are exactly the nonfixed cases of the median rule in (2.4). Reversal handles a swap of b, c and the internal case with an external partner below a.

Swapping the two labels also swaps their partners, so it preserves the underlying edge count. The only difficulty occurs when a decreasing chain uses both swapped rows or both swapped columns, and the repair can be confined to that segment. The third edge supplies a replacement; if it already lies in the segment, transposing the intervening points preserves their unordered edges. For several chains, the corresponding repair exchanges two tails instead.

In (2.5), relabeling both coordinates by (a, b) sends every y-chain to a z-chain. Failure could only involve both rows a, b or both columns a, b, and their y-points are increasing. In the reverse direction suppose first that a z-chain contains $(a, \beta), (b, \alpha)$. If the edge $\{c, \gamma\}$ is absent, replace these points by $(b, \beta), (c, \gamma)$. The points preceding the pair have row $< a$ and column $> \beta$; the following points have row $> c$ and column $< \alpha$, so the replacement fits.

If that edge is present, it occurs as (γ, c). There are two repairs, with Q^T listed in increasing row order

$$(\gamma, c), Q, (a, \beta), (b, \alpha) \longmapsto (\gamma, c), (\beta, b), Q^T, (a, \alpha),$$
$$(a, \beta), (b, \alpha), Q, (\gamma, c) \longmapsto (b, \beta), (c, \gamma), Q^T, (\alpha, a).$$

The first case forces $\alpha < \gamma < \beta < a < b < c$ and, for $(r, s) \in Q$, the bounds $\gamma < r < a$ and $\beta < s < a$. The second forces $a < b < c < \alpha < \gamma < \beta$, with $c < r < \gamma$ and $c < s < \alpha$. These inequalities prove that the new points decrease and fit the unchanged ends. In the second case, a following column is $< c$ and cannot be a or b, hence is $< a$. All new rows lie between old selected rows and labels of the triple, so

remain in L. Transposition preserves the unchanged edges, while the distinguished points use three distinct y-edges.

If instead both columns a, b are selected, their rows are $\alpha, \beta \in L$, so $\gamma \in L$ as well. Transpose, apply the appropriate repair, and transpose back. In the exceptional segments, both coordinates of Q lie between the distinguished labels; hence the repaired rows still lie in L. This proves the interval assertion for the external move.

For (2.6), relabeling gives the forward direction. The reverse repairs are simply

$$(a,\gamma),(b,c) \longmapsto (b,\gamma),(c,a), \qquad (c,b),(\gamma,a) \longmapsto (a,c),(\gamma,b).$$

The one-use condition excludes row c and columns a, b in the first case, and rows a, b and column c in the second. Consecutiveness then gives precisely the inequalities needed at both ends. The row bounds are preserved. Reversing both coordinates replaces L by an interval and preserves decreasing chains, completing the proof of ρ_L-invariance.

For the packing assertion, choose chains with disjoint unordered edge sets, deleting duplicate occurrences if necessary. The forward relabeling works for all colors. In the reverse external move, at most one chain can be bad, since it must use both affected edges. Transpose the whole family if necessary and write that chain as

$$R_-,(a,\beta),(b,\alpha),R_+.$$

If $\{c,\gamma\}$ is unused or belongs to this chain, the preceding repairs preserve the colored packing. Otherwise transpose its chain alone, if needed, and write it as $B_-,(c,\gamma),B_+$. Replace the two chains by

$$R_-,(b,\beta),(c,\gamma),B_+, \qquad B_-,(a,\alpha),R_+.$$

Disjointness excludes rows a, b from B_- and row c from R_+. Since a, b, c are consecutive, rows in B_- are $< a$, with columns $> \gamma > \alpha$; rows in R_+ are $> c$, with columns $< \alpha$. Both chains therefore decrease. Their total size and disjoint edge sets are unchanged. The internal move uses the same two-point repair as before, without affecting another color. Reversal completes the remaining cases. □

2.2. **Recovering the prefix and the shape**

Local invariance lets us compute a statistic after changing the tableau to a convenient representative of its dual-equivalence class. For prefix height, the useful representatives are column-superstandard and row-superstandard; one exhibits a long chain and the other bounds every chain. For the full shape, disjoint nested matchings realize all the optimal packing numbers at once.

We first prove $L_q(y) = \operatorname{ht} P(y)_{\le q}$. Put $\lambda = \operatorname{shape} P(y)_{\le q}$ and $h = \ell(\lambda)$. Elementary dual equivalence connects all standard tableaux of this shape [3, Theorem 2.3 and the following remark]. Moves involving only $1, \dots, q$ leave larger labels fixed, and Lemma 2.2 preserves L_q.

For the lower bound, make the prefix column-superstandard. Its first column is $1, \dots, h$, so (2.4) gives $y(1) > \cdots > y(h)$ and excludes adjacent paired positions. No two positions in $[h]$ can be paired; a decreasing involution on the interval between them would reverse it, forcing either a fixed point or a central adjacent pair. Thus these h points use distinct edges, and $L_q \ge h$.

For the upper bound, make the prefix row-superstandard. Its rows partition $[q]$ into h consecutive intervals with no tableau descents. Within one interval the only adjacent inversions of the partner sequence are adjacent paired transpositions. They are disjoint; a pair with values $p+1, p$ has preceding neighbor $< p$ and succeeding neighbor $> p+1$, since those two values are already occupied. Sorting the pairs therefore makes the sequence increasing. A decreasing chain uses at most one point from each interval, as an inversion inside it would reuse a paired edge. Hence $L_q \le h$. Empty prefixes cause no change.

For the full shape, the packing assertion of Lemma 2.2 and dual-equivalence connectivity show that every $G_k(y)$ depends only on $\operatorname{shape} P(y)$. For $\lambda = 2\mu$, put $h_j = \mu'_j$ and partition $[N]$ into successive intervals of lengths $2h_1, 2h_2, \dots$. On an interval $s+1, \dots, s+2h$, take the nested matching

$$\{s+1, s+2h\}, \{s+2, s+2h-1\}, \dots, \{s+h, s+h+1\}.$$

Paired insertion adds a two-column rectangle of height h, with rows $(s+i, s+h+i)$. At the next interval all entries are larger and the height weakly decreases, so the same bumping pattern adds the next two columns without bumping old entries. The final tableau is column-superstandard of shape 2μ. A decreasing chain cannot meet two intervals, whereas all edges of one interval form such a chain. Consequently $G_k = h_1 + \cdots + h_k$, proving (2.3) for matchings.

2.3. **Repeated endpoints and loops**

To pass from matchings to arbitrary monomials, equal endpoint labels must be separated in an order compatible with both chains and insertion. The following inflation chooses that order. The one-use condition on a matching edge is what prevents a loop, after collapse, from contributing twice to one antidiagonal.

Replace a vertex v of endpoint degree d_v by a consecutive block of d_v labels. Order its incidences by the opposite original vertex, order parallel copies consistently at both

ends, and pair loop incidences adjacently between the smaller and larger external partners. This gives the *canonical inflation* of the multigraph to a perfect matching. The partner sequence in each block is increasing except for disjoint adjacent loop pairs. An admissible chain cannot use two rows or two columns in a block, since that would reuse one such edge. Thus it collapses to an antidiagonal. Conversely, any antidiagonal lifts using distinct edge copies; distinct original rows and columns preserve all strict inequalities. This applies to a whole packing at once, so inflation preserves both prefix ranks and G_k.

Inflation also commutes with paired insertion. Assign block labels chronologically while processing the pairs; this is the incidence order just described. At every stage, (2.4) gives no descents within a block. Its labels consequently lie weakly higher and strictly farther right as they increase. They form a horizontal strip, and collapse is semistandard. A newly inserted label passes exactly the entries whose collapsed letters are at most its own. If it is bumped, all labels from its block already in the lower row are smaller, by the same no-descent property. Induction along the bumping path shows that the same boxes change before and after collapse; the appended partner occupies the same box as well.

To obtain (2.2), retain only edges with larger endpoint at most j, inflate them, and use the prefix ending at block i. The matching result gives the asserted height. The full-shape formula follows from invariance of G_k under inflation.

Finally, standardize equal entries of an even-row semistandard tableau from left to right. Column Beissinger insertion has a unique matching preimage [3, Theorem 2.12]. The standardized blocks have no descents, so their partner sequences are increasing except for adjacent paired transpositions, by the neighbor argument above. Every internal edge is one of those pairs, and the external partners are increasing. The preimage is therefore the canonical inflation of its collapsed multigraph. Insertion compatibility proves both surjectivity and uniqueness. This completes Theorem 2.1.

3. **A common rank reduction**

Put $x = u_{11}$, $b_i = u_{1i}$ for $i \geq 2$, and $C = (u_{ij})_{i,j\geq 2}$. Deleting the first tableau row lowers every positive antidiagonal rank by one. The Schur complement at x lowers the matrix ranks in exactly the same way. The comparison below matches their multigraded Hilbert series.

Lemma 3.1 (Leading terms)**.** *Every minor has leading monomial its folded antidiagonal, with coefficient* 1 *or* -1 *over* $\mathbb{Z}$.

Proof. If $a < c$ and $b < d$, the largest folded index among the four corners is that of u_{cd}, distinct from those of u_{ad} and u_{cb}. Hence $u_{ad}u_{cb} > u_{ab}u_{cd}$. Swapping any increasing pair of column assignments strictly increases a determinant term, even after folding. Repeated swaps reach the decreasing assignment, which is therefore strictly larger than every other term. □

Lemma 3.2 (Deletion of the first row)**.** *There is a weight-preserving bijection*

$$m \longmapsto (c, N)$$

from monomials in b, C to $\mathbb{N}^{n-1}$ times symmetric monomials on $2, \dots, n$, for the grading $\deg b_i = \mathbf{e}_i$ and $\deg u_{ij} = \mathbf{e}_i + \mathbf{e}_j$. At every endpoint stage, $T(N)$ is obtained from $T(m)$ by deleting its first row. In particular,

$$\rho_N(i-1, j-1) = \max\{\rho_m(i,j) - 1, 0\} \quad (2 \le i \le j \le n), \tag{3.1}$$

where the labels of N are shifted down by one on the left.

Proof. We record the first-row information in carrier exponents b_i and send every pair bumped below that row to the residual monomial. At an off-diagonal step, only the smaller of two carrier counts can be paired; at a diagonal step, the remaining demands are paired in twos. This gives the following local transformations. Their reversibility will also show that the carriers are free parameters, rather than merely an upper bound on the number of monomials.

For this proof only, write b_i, a_{ij} for the current exponents of b_i, u_{ij}. At each $j = 2, \dots, n$, first apply

$$(b_i, b_j, a_{ij}) \longmapsto (b_i - s + a_{ij},\ b_j - s + a_{ij},\ s), \quad s = \min(b_i, b_j), \tag{3.2}$$

for $i = 2, \dots, j-1$ in that order, and then apply

$$(b_j, a_{jj}) \longmapsto (b_j \bmod 2 + 2a_{jj},\ \lfloor b_j/2 \rfloor). \tag{3.3}$$

Let c_i be the final carriers and let N have the final exponents a_{ij}. Each displayed operation is an involution and preserves weight. Reversing the sequence proves bijectivity.

For example, $u_{12}u_{13}$ has tableau rows $(1,1)$ and $(2,3)$; the transformation gives $c = 0$ and $N = u_{23}$. The single factor u_{23} instead gives $c_2 = c_3 = 1$ and $N = 1$. In each case the carriers record the entries of the deleted row other than 1, which has weight zero in this grading.

The formulas describe the first row during insertion. At endpoint j, incoming pairs $(1, j)$ give b_j demands. At a letter $i < j$, exactly $s = \min(b_i, b_j)$ copies are bumped

and emit pairs (i, j) to the lower rows. The unbumped copies and the a_{ij} incoming pairs leave $b_i - s + a_{ij}$ copies of i and $b_j - s + a_{ij}$ demands. At j, each two remaining demands emit a loop; an odd demand leaves one j, and each original loop adds two. This gives (3.2)–(3.3).

The emitted pairs are in increasing endpoint order. For fixed j, the inserted lower endpoints and the letters bumped out of the first row are weakly increasing; a later bump of an appended partner emits only j. Thus the lower rows are exactly the paired-insertion tableau of N, at every stage. Deleting a nonempty top row lowers the height of each nonempty entry prefix by one. Equation (2.2) now gives (3.1). □

Lemma 3.3 (Comparison after a pivot)**.** *Let $H \subseteq K[x, b, C]$ be homogeneous for the positive grading in b, C used above, with $\deg x = 0$. Suppose a finite set $\mathcal{G} \subseteq H$ has leading monomials independent of x, generating $J \subseteq K[b, C]$. Assume these leading monomials remain the same over $F = K(x)$ in the inherited graded reverse lexicographic order. If the standard monomials of J and the quotient $F[b, C]/HF[b, C]$ have the same multigraded Hilbert series, then $\mathcal{G}$ is a Gröbner basis of H in the original order.*

Proof. The leading monomials give $JF[b, C] \subseteq \mathrm{in}_F HF[b, C]$. Equality of the Hilbert series makes this an equality, since each graded piece is finite-dimensional. The standard monomials of J are therefore linearly independent over F. Divide $f \in H$ by $\mathcal{G}$ in the original order. Its remainder has the form $\sum_{m \notin J} c_m(x)m$, with $c_m(x) \in K[x]$, and vanishes in the localized quotient. Independence forces every c_m to vanish. □

For all applications, the required coordinate change is

$$D = C - \frac{bb^T}{x}.$$

It is homogeneous in the stated grading. An invertible lower triangular congruence transforms U into $\mathrm{diag}(x, D)$. On every northwest rectangle, Cauchy–Binet in both directions lowers a positive rank bound by one. Every determinant of size at least two retains its antidiagonal leading term over F; terms containing x have smaller degree in the remaining variables, and the other comparisons are unchanged. The same holds for products of such minors.

4. **Northwest rank ideals and their prime profiles**

We first prove the rank case of Theorem 1.2; it will also supply the primeness needed for symbolic powers.

Proposition 4.1 (Northwest Gröbner bases)**.** *For every symmetric array of nonnegative bounds,* $\operatorname{in} I(r) = J(r)$ *over every field.*

Proof. Induct on n, starting with the empty matrix. Suppose first that all bounds are positive, and set $r'_{ij} = r_{i+1,j+1} - 1$. Every prescribed antidiagonal omits $x = u_{11}$. By (2.1) and Lemma 3.2, its standard monomials correspond to free carrier monomials times standard monomials of $J(r')$. Hence

$$\operatorname{Hilb} K[b, C]/J(r) = \prod_{i=2}^{n} (1 - z_i)^{-1} \operatorname{Hilb} K[D]/J(r').$$

Over $F = K(x)$, the Schur complement gives

$$F[b, C]/I(r)F[b, C] \cong \big(F[D]/I(r')F[D]\big)[b_2, \dots, b_n]. \tag{4.1}$$

Induction identifies the residual Hilbert series, so Lemma 3.3 proves the assertion.

For general r, put $r^+_{ij} = \max(r_{ij}, 1)$ and let Z be the variables in the northwest rectangles with bound zero. Their union is a northwest Ferrers region, invariant under transpose, and

$$I(r) = I(r^+) + (Z), \qquad J(r) = J(r^+) + (Z).$$

If the antidiagonal of a t-minor meets Z at its pth selected row, its first p selected rows and first $t + 1 - p$ selected columns form a zero rectangle. A permutation avoiding it would have to place p rows in the remaining $p - 1$ columns. Thus the entire minor lies in (Z). Adjoining these variables to the positive-bound Gröbner basis preserves Buchberger's criterion; a new pair either has coprime leading monomials or reduces entirely by Z. □

Any collection of northwest conditions is represented by such an array; use the least bound imposed on each rectangle and its transpose, and a bound at least n otherwise. Replacing two bounds by their minimum changes neither ideal, by Laplace expansion and divisibility of antidiagonals.

4.1. **The first rook**

Every matrix rank table can be represented by a partial permutation, because $r_{ij} - r_{i-1,j}$ is zero or one and is nondecreasing in j; a row independent on a column prefix remains independent when columns are added. Thus

$$p_{ij} = r_{ij} - r_{i-1,j} - r_{i,j-1} + r_{i-1,j-1}$$

is zero or one. Telescoping gives at most one nonzero entry in each row and column. A symmetric rank table gives a symmetric partial permutation. We may therefore take w of this form.

Lemma 4.2 (A regular pivot)**.** *If the first row of w has its rook in column k and $x = u_{1k}$, then $J_w : x = J_w$ and $I_w : x = I_w$.*

Proof. Suppose a forbidden antidiagonal m of size $t = r_w(i, j) + 1$ contains x. Transpose if necessary so that one occurrence is at $(1, k)$. The largest selected column is k, so m/x is a $(t-1)$-antidiagonal in $[i] \times [k-1]$. Since the rook in column k lies in row one,

$$r_w(i, k-1) = r_w(i, k) - 1 \leq r_w(i, j) - 1 = t - 2.$$

A sub-antidiagonal of m/x is therefore forbidden. Size one is impossible because a prefix containing x has positive rank. If $k = 1$, no larger antidiagonal contains $(1, 1)$. This proves the monomial colon equality. Proposition 4.1 transfers it to I_w; a nonzero standard remainder of f with $xf \in I_w$ would have leading monomial outside J_w whose product with x lies inside. □

Proof of Theorem 1.1. Induct on n. A zero first row forces the entire first row and column to vanish; deleting them gives the smaller rank quotient. Otherwise Lemma 4.2 embeds S_n/I_w into its localization at $x = u_{1k}$.

If $k = 1$, the Schur complement gives

$$(S_n/I_w)[x^{-1}] \cong (S_{n-1}/I_{w'})[x, x^{-1}, b_2, \ldots, b_n],$$

where w' deletes index one. This is a domain by induction.

An off-diagonal first rook instead requires a two-by-two pivot; its diagonal entry u_{11} is forced to vanish. We isolate the block on indices $1, k$ by a lower triangular congruence, so that all northwest rank conditions are preserved during elimination.

For $k > 1$, the entries $u_{11}, \ldots, u_{1,k-1}$ vanish. In the ordered blocks $1, T, k, B$, with $T = \{2, \ldots, k-1\}$ and $B = \{k+1, \ldots, n\}$, write

$$U = \begin{pmatrix} 0 & 0 & x & b^T \\ 0 & C & z & E \\ x & z^T & y & t^T \\ b & E^T & t & F \end{pmatrix}.$$

Over x^{-1}, use the lower triangular changes

$$e_i \longmapsto e_i - \frac{z_i}{x} e_1 \quad (i \in T), \qquad e_j \longmapsto e_j - \frac{b_j}{x} e_k + \left(\frac{y b_j}{x^2} - \frac{t_j}{x} \right) e_1 \quad (j \in B).$$

The congruence gives

$$\begin{pmatrix} 0 & 0 & x & 0 \\ 0 & C & 0 & E' \\ x & 0 & y & 0 \\ 0 & E'^T & 0 & F' \end{pmatrix}, \qquad E' = E - \frac{zb^T}{x}, \quad F' = F - \frac{bt^T + tb^T}{x} + \frac{ybb^T}{x^2}.$$

This is an invertible change of coordinates fixing x, y, z, b, t. Let D be the symmetric matrix on the remaining indices, with blocks C, E', F', and let w' delete indices $1, k$.

Lower triangular congruence preserves every northwest minor ideal. Put $d(i) = i - 1 - \mathbf{1}_{i \geq k}$. In prefix (i, j) the isolated pivot has rank

$$s(i,j) = \begin{cases} 0, & i, j < k, \\ 1, & \text{exactly one of } i, j \text{ is at least } k, \\ 2, & i, j \geq k. \end{cases}$$

Its nonzero block is x or $\left(\begin{smallmatrix} 0 & x \\ x & y \end{smallmatrix}\right)$, with invertible determinant. After deleting zero rows and columns, each prefix is the direct sum of that block and $D_{[d(i)] \times [d(j)]}$. Its minor ideal has size lowered by $s(i, j)$, and the rook tables satisfy

$$r_w(i,j) = s(i,j) + r_{w'}(d(i), d(j)).$$

Every smaller prefix occurs. Consequently

$$(S_n/I_w)[x^{-1}] \cong (S_{n-2}/I_{w'})[x, x^{-1}, y, z, b, t], \tag{4.2}$$

which is a domain by induction. The calculation retains y and uses no division by 2. The regular-pivot injection proves primeness. Finally $X_w = V(I_w)$ by definition, so the Nullstellensatz identifies its defining ideal with I_w over an algebraically closed field. □

5. **Symbolic orders and the mixed Gröbner basis**

While northwest conditions measure heights of tableau prefixes, symbolic order measures the area remaining below the first $t - 1$ rows of the half-shape. Deleting the first row therefore lowers both the rank threshold and the symbolic index by one, allowing the same recurrence to treat their sums.

For shape $T(m) = 2\mu$, define

$$\gamma_t(m) = \sum_{i \geq t} \mu_i.$$

The full-shape statement of Theorem 2.1 gives

$$\gamma_t(m) = \sum_j \max\{\mu'_j - t + 1, 0\} = \max_{k\geq 0}\{G_k(m) - k(t-1)\}. \tag{5.1}$$

Lemma 5.1 (Monomial symbolic order)**.** *A monomial belongs to $J_{n,t,s}$ if and only if $\gamma_t(m) \geq s$.*

Proof. A product of k antidiagonals of sizes $a_j \geq t$ and total score $\sum_j (a_j - t + 1) = s$ gives $G_k(m) - k(t-1) \geq s$. Conversely, take a packing attaining a value at least s in (5.1). Discard chains shorter than t and choose integers $0 \leq c_j \leq a_j - t + 1$ with $\sum_j c_j = s$. Discard those with $c_j = 0$ and trim each remaining chain to size $t - 1 + c_j$. Their product is a generator of $J_{n,t,s}$ dividing m. □

Lemma 5.2 (A symbolic Schur chart)**.** *Every member of $\mathcal{G}_{n,t,s}$ lies in $P_{n,t}^{(s)}$. For $t \geq 2$, the diagonal Schur change over $F = K(x)$ gives*

$$F[b, C]/P_{n,t}^{(s)}F[b, C] \cong \big(F[D]/P_{n-1,t-1}^{(s)}F[D]\big)[b_2, \ldots, b_n]. \tag{5.2}$$

The symbolic powers are homogeneous in the endpoint grading.

Proof. The case $t = 1$ is the power of the homogeneous maximal ideal. For $t \geq 2$, invert a principal $(t-1)$-minor Δ, which is outside $P_{n,t}$. The block Schur complement transforms U into $\operatorname{diag}(A, Z)$, with A invertible and Z a generic symmetric matrix over the domain of the other coordinates and Δ^{-1}. The localized ideal $P_{n,t}$ is the ideal of the entries of Z. Every a-minor has order at least $a - t + 1$ in these entries, since a determinant term uses at most $t - 1$ rows and columns from A. Products of score s lie in its sth power.

That power is primary; its associated graded ring over the coefficient domain is a polynomial domain, so multiplication by an element with nonzero degree-zero part cannot lower the least Z-degree. Its contraction is therefore $P_{n,t}$-primary and equals $P_{n,t}^{(s)}$. This proves the containment and endpoint homogeneity. The same chart shows that symbolic powers commute with localization away from the prime and with adjoining independent polynomial or rational-function variables. Applying this to $P_{n,t}F[b, C] = P_{n-1,t-1}(D)F[b, D]$ proves (5.2). □

Proof of Theorem 1.2 over a field. Induct on n, simultaneously for every array and every symbolic list; the empty matrix starts the induction. Suppose first that all rank bounds are positive and all $t_\nu \geq 2$. Every proposed leading monomial omits x. The carrier bijection transforms the northwest conditions by (3.1); the first row and

column impose no condition when their bounds are positive. Since it deletes the first tableau row, it also gives

$$\gamma_{t_\nu}(m) = \gamma_{t_\nu - 1}(N).$$

Thus the proposed standard monomials correspond to free carriers times the standard monomials for the smaller mixed ideal with bounds $r'_{ij} = r_{i+1,j+1} - 1$ and pairs $(t_\nu - 1, s_\nu)$. Equations (4.1) and (5.2) give exactly the same reduction of the localized quotient. Induction equates their Hilbert series, and Lemma 3.3 proves the Gröbner assertion.

Allowing zero bounds is the same Ferrers reduction as in Proposition 4.1. If the leading monomial of a product generator meets Z, one of its determinant factors has antidiagonal meeting Z and therefore lies entirely in (Z). Hence the product does too, and adjoining Z preserves the Gröbner basis. The extra minors introduced on replacing zero bounds by one are redundant and may be removed.

Finally, the terms with $t_\nu = 1$ sum to $\mathfrak{m}^q$, where $q = \min_{t_\nu = 1} s_\nu$. Adjoining all degree-$q$ monomials to a homogeneous Gröbner basis preserves it; every term of a new S-polynomial has degree at least q. This completes the induction. □

Corollary 5.3 (Symbolic-power bases and Hilbert series)**.** *For every field, $\mathcal{G}_{n,t,s}$ is a Gröbner basis of $P_{n,t}^{(s)}$, and*

$$m \in \operatorname{in} P_{n,t}^{(s)} \quad \Longleftrightarrow \quad \gamma_t(m) \geq s.$$

For the endpoint grading,

$$\operatorname{Hilb}(S_n/P_{n,t}^{(s)}; z_1, \dots, z_n) = \sum_{\substack{\ell(\mu) \leq n \\ \sum_{i \geq t} \mu_i < s}} s_{2\mu}(z_1, \dots, z_n).$$

In particular, the quotient by all $(q+1)$-minors has Hilbert series equal to the sum of s_λ over partitions with even row lengths and at most q rows.

Proof. The Gröbner statement is Theorem 1.2 with no northwest conditions, and the monomial criterion is Lemma 5.1. The content-preserving bijection of Theorem 2.1 turns its standard monomials into the stated semistandard tableaux, whose content generating functions are Schur polynomials. For $s = 1$ and $t = q + 1$, the condition is $\ell(\mu) \leq q$. □

6. **Integral bases and simultaneous degenerations**

Corollary 6.1. *The Gröbner bases in Theorem 1.2 are monic over $\mathbb{Z}$ after changing signs. Their standard monomials give free quotient bases over every commutative*

coefficient ring. A symmetric rank-profile quotient is a domain over every integral domain of coefficients and has geometrically integral fibers over $\operatorname{Spec}\mathbb{Z}$.

Proof. All leading coefficients are units by Lemma 3.1. Monic integral division of an S-pair gives an integral standard remainder. It vanishes over $\mathbb{Q}$ by the field theorem, and hence over $\mathbb{Z}$. Buchberger's criterion and division give the free basis, before and after any base change. Over a coefficient domain the quotient injects into its extension to the fraction field, which is a domain by Theorem 1.1. Applying that theorem after algebraic closure gives geometric integrality. □

This also completes the integral assertion of Theorem 1.2. Any other term order selecting the same folded antidiagonals gives the same initial ideals; it contains the displayed monomial ideal, and equality follows from the already equal graded Hilbert functions.

6.1. **A common triangular basis**

Compatibility of initial ideals under sums allows one triangular basis to serve all the ideals at once; intersections then correspond to intersections of basis subsets. We formulate this linear-algebra step over a general coefficient ring so that it applies equally to the deformation parameter below.

Lemma 6.2 (Simultaneous triangular bases)**.** *Let A be a commutative ring, and let $\mathcal{C}$ be a finite family of homogeneous ideals in $A[\mathbf{x}]$ with monic Gröbner bases. Suppose the union of the bases for any two members is a Gröbner basis of their sum. There is a homogeneous A-basis $\{b_m\}$ of the polynomial ring such that*

$$b_m = m + \sum_{q<m} c_{m,q}q, \qquad I = \bigoplus_{m\in \operatorname{in} I} Ab_m \quad (I \in \mathcal{C}).$$

Proof. Work in a fixed degree, in increasing monomial order. Suppose the basis has been constructed below m, and let H_I be the span of those b_q with $q \in \operatorname{in} I$. For every I with $m \in \operatorname{in} I$, choose a monic $g_I \in I$ with leading monomial m, and express $g_I - m$ in the lower basis. The difference $g_I - g_J$ lies in $I + J$ and has only smaller monomials. Division by the union basis puts it in $H_I + H_J$. Thus the lower coefficients agree at every b_q with $q \notin \operatorname{in} I \cup \operatorname{in} J$.

For each $q < m$, choose c_q to be the coefficient of $g_I - m$ for any I with $m \in \operatorname{in} I$ and $q \notin \operatorname{in} I$, if such an ideal exists, and choose zero otherwise. The preceding agreement makes this well defined. Then $b_m = m + \sum_{q<m} c_q b_q$ satisfies $b_m - g_I \in H_I$, so belongs to every required ideal. Subtracting its leading coefficient times b_m from an element of such an ideal proves the spanning assertion at this stage. For the other ideals no element can have leading monomial m. Triangularity gives a basis. □

Corollary 6.3 (Compatible ideal lattices)**.** *For fixed n, the ideals in Theorem 1.2 admit one simultaneous integral triangular basis. Over every commutative coefficient ring, the lattice they generate under finite sums and intersections is distributive, its quotients have free standard-monomial bases, and*

$$\operatorname{in}(I+J)=\operatorname{in} I+\operatorname{in} J, \qquad \operatorname{in}(I\cap J)=\operatorname{in} I\cap \operatorname{in} J.$$

All these operations commute with arbitrary base change.

Proof. A finite sum has the same form; take the entrywise minimum of the rank bounds and combine the symbolic lists. Theorem 1.2 makes the union of generators a monic Gröbner basis of that sum. Lemma 6.2 applies degree by degree. There are only finitely many distinct conditions in each degree; northwest bounds can be capped by the smaller side, and a symbolic power with $s > d$ contributes nothing in degree d. In the common basis, sums and intersections are unions and intersections of index sets. Triangularity gives their initial ideals; the complementary basis of each quotient makes the assertions persist after base change. □

6.2. **One flat family**

For a minor $\delta = [A \mid B]$, set

$$\omega(A,B)=\sum_{p=1}^{t} a_p b_{t+1-p}, \qquad \delta_\tau = \tau^{-\omega(A,B)} \det(\tau^{ab}u_{ab})_{a\in A, b\in B}.$$

The strict rearrangement inequality makes this a polynomial over $\mathbb{Z}[\tau]$, with value at zero equal to the signed antidiagonal. Replace every minor in a rank condition by δ_τ and every product generator by the product of its deformed factors.

Proposition 6.4. *These deformations, and all their finite sums and intersections, have free quotient bases over $\mathbb{Z}[\tau]$ indexed by the corresponding standard monomials. Their fibers at 1 are the integral ideals above, and their fibers at 0 are the corresponding initial ideals. Every fiber with invertible τ is a diagonal rescaling of the original one. The assertions hold after arbitrary base change, and finite sums and intersections commute with specialization.*

Proof. The deformed generators are monic up to sign in the u-variables. Over $\mathbb{Q}(\tau)$ the substitution $u_{ab} \mapsto \tau^{ab}u_{ab}$ is invertible, so the standard monomials remain independent. Monic division over $\mathbb{Z}[\tau]$ therefore has zero remainder on the defining ideal, giving the asserted free basis. Lemma 6.2 over $\mathbb{Z}[\tau]$ gives a common basis for all the ideals; its index sets describe sums, intersections, and every specialization. □

The weight ab is symmetric. The determinant family can consequently be formed before imposing $u_{ab} = u_{ba}$ and then restricted to the scheme-theoretic fixed locus of transpose. Proposition 6.4 proves flatness of this restriction, including in characteristic two. For instance, a 2×2 determinant becomes

$$\tau u_{11}u_{22} - u_{12}^2,$$

whose special fiber retains the double structure defined by u_{12}^2.

6.3. **Rank strata**

For a symmetric partial permutation w, let S_w be the stratum where every northwest rank equals its rank in w. Define $a(w), b(w)$ recursively from $a(\varnothing) = b(\varnothing) = 0$ by

first row of w	$a(w) - a(w')$	$b(w) - b(w')$
zero; delete 1	0	0
rook at $(1,1)$; delete 1	$n-1$	1
rook at $(1,k)$, $k>1$; delete $1,k$	$2n-k-1$	1.

Proposition 6.5. *Over $\mathbb{Z}$, and after every base change,*

$$S_w \cong \mathbb{A}^{a(w)} \times \mathbb{G}_m^{b(w)}.$$

It is a dense open subset of X_w over every field. In particular, $\dim X_w = a(w) + b(w)$, *and X_w is rational and generically smooth. For every finite field $\mathbb{F}_q$,*

$$|X_w(\mathbb{F}_q)| = \sum_{\substack{v \text{ symmetric partial permutation} \\ r_v \le r_w}} q^{a(v)}(q-1)^{b(v)}.$$

Proof. On S_w, the first nonzero entry of the first row is the regular pivot above and is invertible. Its diagonal chart has one invertible coordinate and $n-1$ free coordinates. The paired chart (4.2) has one invertible coordinate and $2n-k-1$ free coordinates. In both, all remaining rank equalities are exactly those for w'. The coordinate changes and their inverses are integral after inverting the pivot, proving the product description by induction. The stratum is nonempty and open in the rank locus; primeness makes it dense. Every symmetric matrix has a unique partial permutation rank profile, so the finite-field points partition into these strata, giving the count. □

7. Symbolic Rees algebras and integral closures

Retaining the power as a grading turns the symbolic score into an algebra construction. Factorizations attaining a single score give an algebra basis, while the compatibility of several scores controls integral closures and reveals the obstruction for two gradings.

7.1. A finite SAGBI basis

Write

$$\mathcal{R}_{n,t}^{\mathrm{symb}} = \bigoplus_{s\geq 0} P_{n,t}^{(s)} T^s \subseteq S_n[T], \qquad P_{n,t}^{(0)} = S_n.$$

A SAGBI basis is a set of algebra generators whose leading monomials generate the initial algebra.

Corollary 7.1. *For the order comparing T-degree first and then the folded antidiagonal order, a finite SAGBI basis is*

$$\{u_{ab} : a \leq b\} \ \cup \ \{\delta T^{a-t+1} : \delta \text{ is an } a\text{-minor}, \ t \leq a \leq n\}.$$

The initial algebra has monomial basis $\{mT^s : s \geq 0, \ \gamma_t(m) \geq s\}$.

Proof. In T-degree s, Corollary 5.3 generates the coefficient ideal by products of exact score s, and generates its initial ideal by their leading monomials. The remaining coefficient factors are monomials in the entries. These are exactly the two SAGBI assertions. □

The initial algebra need not be normal. For $n = 4$, $t = 2$, and $m = u_{12}u_{34}$, the tableau of m has shape (4), whereas that of m^2 has shape $(4, 4)$. Thus mT is absent, but

$$(mT)^2 = (u_{12}^2 T)(u_{34}^2 T)$$

belongs to the initial algebra. The element T lies in its fraction field, since $u_{12}^2 T$ and u_{12} belong to it. Hence mT is an integral element of the fraction field missing from the algebra.

7.2. Integral closures of sums of products

Let Σ be a nonempty finite set of partitions $\sigma = (\sigma_1, \ldots, \sigma_k)$ with $\sigma_1 \leq n$, and put

$$E(\Sigma) = \sum_{\sigma\in\Sigma} \prod_j P_{n,\sigma_j}, \qquad v_t(\sigma) = \sum_j \max\{\sigma_j - t + 1, 0\}.$$

Let Q_Σ be the convex hull of the vectors $v(\sigma) = (v_1(\sigma), \ldots, v_n(\sigma))$.

Corollary 7.2. *Over every field,*

$$m \in \operatorname{in} \overline{E(\Sigma)} \quad \Longleftrightarrow \quad (\gamma_1(m), \dots, \gamma_n(m)) \in Q_\Sigma + \mathbb{R}^n_{\geq 0}.$$

These integral closures and the northwest rank ideals belong to the same compatible lattice under finite sums and intersections. They have integral models with free quotient bases and the simultaneous flat degenerations of Proposition 6.4.

Proof. Henriques and Varbaro's integral-closure formula [1, Theorem 2.7], valid over every field, gives

$$\overline{E(\Sigma)} = \sum_{a \in Q_\Sigma} \bigcap_{t=1}^{n} P_{n,t}^{(\lceil a_t \rceil)},$$

where exponent zero means the whole ring. Only finitely many integer vectors $(\lceil a_t \rceil)_t$ occur. Corollary 6.3 therefore takes initial ideals through the displayed sums and intersections. The monomial criterion says that some $a \in Q_\Sigma$ satisfies $\gamma_t(m) \geq \lceil a_t \rceil$ for all t, which is the asserted condition because the left sides are integers. The same finite lattice expression over $\mathbb{Z}$ supplies the integral model; after field base change it is the stated integral closure by the same formula. □

7.3. **Two symbolic gradings**

For $1 \leq r \leq t \leq n$, set

$$\mathcal{R}_{r,t} = \bigoplus_{a,b \geq 0} \big(P_{n,r}^{(a)} \cap P_{n,t}^{(b)}\big) T^a V^b.$$

Let $\mathcal{G}_{r,t}$ consist of the entries and the elements

$$\delta T^a V^b, \qquad 0 \leq a \leq \max\{\operatorname{size}(\delta) - r + 1, 0\}, \quad 0 \leq b \leq \max\{\operatorname{size}(\delta) - t + 1, 0\},$$

with δ any minor. We use any term order restricting to the folded antidiagonal order on each (T, V)-homogeneous component. At the full matrix size, the determinant controls the second symbolic order.

Proposition 7.3 (Adjoining the determinant)**.** *For every $1 \le r \le n$ and every field, $\mathcal{G}_{r,n}$ is a SAGBI basis of $\mathcal{R}_{r,n}$. More precisely, put*

$$D = \det U, \qquad A = \prod_{i=1}^{n} u_{i,n+1-i}, \qquad h = n - r + 1.$$

Then for $a, b \ge 0$ and every monomial m,

$$P_{n,r}^{(a)} \cap P_{n,n}^{(b)} = D^b P_{n,r}^{(\max\{a-bh,0\})}, \qquad \gamma_r(A^b m) = bh + \gamma_r(m).$$

Already the following subset is a SAGBI basis

$$\{u_{ij} : i \le j\} \ \cup \ \{\delta T^{j-r+1} : \delta \text{ is a } j\text{-minor}, \ r \le j \le n\} \ \cup \ \{DT^c V : 0 \le c \le h\}.$$

Proof. The prime $P_{n,n} = (D)$ has symbolic powers (D^b). In the Schur chart of Lemma 5.2 for $P_{n,r}$, the determinant is a unit times $\det Z$, where Z is a generic symmetric $h \times h$ matrix. This determinant has exact degree h in the entries of Z; its diagonal monomial has coefficient one in every characteristic. The associated graded ring at $P_{n,r}$ is a polynomial ring over its residue field, so orders add on products. Consequently

$$D^b f \in P_{n,r}^{(a)} \quad \Longleftrightarrow \quad f \in P_{n,r}^{(\max\{a-bh,0\})}.$$

For $r = 1$ the same argument is simply total degree. This proves the ideal identity.

Taking initial ideals gives $A^b \operatorname{in} P_{n,r}^{(\max\{a-bh,0\})}$. By Corollary 6.3, this is also $\operatorname{in} P_{n,r}^{(a)} \cap (A^b)$. The monomial criterion in Corollary 5.3, for all a, gives the asserted additivity of γ_r.

In bidegree (a, b), let $q = \max\{a - bh, 0\}$. Choose integers $0 \le c_1, \dots, c_b \le h$ with $\sum_i c_i = \min\{a, bh\}$; for $b = 0$ the list is empty. The determinant factor with its gradings is

$$D^b T^{\min\{a,bh\}} V^b = \prod_{i=1}^{b} (DT^{c_i} V).$$

The remaining factor belongs to $P_{n,r}^{(q)} T^q$ and is generated by the single-grading basis of Corollary 7.1. The same factorization holds for initial monomials by the initial ideal identity. Thus the displayed subset generates the algebra and its initial algebra, proving the claim. □

The separate maxima in (2.3) need not otherwise be attained by one antidiagonal factorization with lengths $\mu'_1, \mu'_2, \dots$. For the pair $(2, t)$, one additional row and column already give an obstruction.

Proposition 7.4. *For $t \geq 4$ and $n \geq t+1$, the collection $\mathcal{G}_{2,t}$ is not a SAGBI basis of $\mathcal{R}_{2,t}$ over any field.*

Proof. Begin with the six-factor monomial on five vertices

$$m = u_{13}u_{23}u_{14}u_{24}u_{34}u_{15}.$$

Its unique size-four antidiagonal divisor is

$$d = u_{15}u_{24}u_{23}u_{14} = \operatorname{anti}\big((1,2,3,4),(1,2,4,5)\big).$$

Indeed, a size-four antidiagonal avoiding vertex 5 would have both index sets [4], requiring the square factors $u_{14}^2u_{23}^2$. Otherwise its only factor incident with 5 is u_{15}. By transposition orient this factor as $(1,5)$; the remaining rows must be $2,3,4$. Row 2 cannot use column 3; row 3 would then either reuse u_{23} or leave no smaller column for row 4. Hence the remaining entries are $(2,4),(3,2),(4,1)$, giving d. The uniqueness also excludes any larger antidiagonal. The complement $m/d = u_{13}u_{34}$ is not an antidiagonal; orienting its two edges either repeats a row or a column, or makes both coordinates increase. Thus no factorization has lengths $(4,2)$.

On the other hand,

$$m = \operatorname{anti}\big((1,2,4),(3,4,5)\big)\ \operatorname{anti}\big((1,2,3),(1,3,4)\big)$$

is a factorization of lengths $(3,3)$. Therefore $G_1(m) = 4$ and $G_2(m) = 6$.

For general t, enlarge the ordered vertex set one vertex at a time. Orient d as above and the two size-three factors as

$$(1,5),(2,4),(4,3), \qquad (1,4),(2,3),(3,1).$$

The largest antidiagonal uses every row except the extreme vertex $v = 5$, which occurs only as a column in both covering antidiagonals. Maintain these properties. If v is largest, add a new smallest vertex w and append (v,w) to the largest antidiagonal and to one covering antidiagonal. If v is smallest, add a new largest vertex w and prepend (v,w) instead. The new extreme w again occurs only as a column.

Any antidiagonal of the new maximum size must contain the new edge; deleting it leaves the previous unique largest antidiagonal monomial. Uniqueness therefore persists, as does the incompatible two-edge complement. After $t-4$ steps, relabel the vertices increasingly by $[t+1]$. The resulting monomial m_t has $G_1(m_t) = t$ and $G_2(m_t) = t+2$. Theorem 2.1 now yields

$$\mu' = (t,2), \qquad \gamma_2(m_t) = t, \qquad \gamma_t(m_t) = 1.$$

Initial-ideal compatibility for intersections gives $m_tT^tV \in \operatorname{in}\mathcal{R}_{2,t}$.

If this monomial were a product of proposed leading monomials, a factor supplying V would have size at least t, hence exactly t. It would supply at most T^{t-1}. The remaining two occurrences would have to supply at least one further T, so they would form a size-two antidiagonal. This gives the excluded $(t, 2)$ factorization. □

Together with Proposition 7.3, this shows that $n = t + 1$ is the smallest matrix size at which the minor collection fails for the pair $(2, t)$, when $t \geq 4$. The proposition concerns this collection of minors; it does not rule out a finite SAGBI basis using other generators. The compatible Gröbner bases for separate thresholds and their intersections remain as in Corollary 6.3.

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