Rank formulas for homological shifts of polymatroidal ideals

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Abstract. We give a partition formula for the canonical rank of every homological shift ideal of a polymatroidal ideal, over any field. Applied to powers, it determines the exact interval of Rees degrees needed to generate each homological shift module and yields $\mathrm{HS}_i(I^{s+1}) = I\,\mathrm{HS}_i(I^s)$ for $s \geq i$, with a sharp bound in every positive homological degree. Mixed powers and fixed polymatroidal twists obey the same bound because their ranks enter additively. The formula arises from the matroid independence complexes in multigraded Koszul homology; its inequalities also identify the rank of the total Betti support and show that the shift modules are integrally closed ideals of the corresponding Rees algebras.

1. Introduction

The homological shift ideals of a monomial ideal record the multidegrees occurring in its minimal resolution. For polymatroidal ideals they are again polymatroidal, as proved by Cid-Ruiz, Matherne, and Shapiro [3], so a rank function encodes each set of syzygy degrees. We determine that function explicitly from the rank of the original ideal. For powers, the formula describes exactly when new generators appear in the associated Rees module and when multiplication by the ideal supplies every subsequent piece.

In particular, it proves the stabilization bound conjectured by Ficarra and Lu [2]; the ith homological shift module of the powers is generated in Rees degrees at most i. The bound is sharp for every $i > 0$, holds over every field, and remains valid with a fixed polymatroidal factor throughout.

Let $S = k[x_j : j \in E]$, where E is finite and k is any field. For a monomial ideal L, set

$$\mathrm{HS}_i(L) = \big(x^\alpha : \mathrm{Tor}_i^S(k, L)_\alpha \neq 0\big).$$

Write $\mathbb{N} = \{0, 1, 2, \ldots\}$, $a(A) = \sum_{j \in A} a_j$, and $|a| = a(E)$. A polymatroidal ideal is generated in one degree, and its generator exponents satisfy the discrete polymatroid exchange axiom. The zero and unit ideals have no positive homological shifts.

Theorem 1.1. *If I is polymatroidal, then for every $i > 0$ and every $s \geq i$,*

$$\mathrm{HS}_i(I^{s+1}) = I\,\mathrm{HS}_i(I^s).$$

Consequently $\bigoplus_{s\geq 1} \mathrm{HS}_i(I^s)t^s$ is generated as a module over $\mathcal{R}(I)$ in Rees degrees at most i. This bound is sharp for every i. More generally, if J is any fixed polymatroidal ideal, then

$$\mathrm{HS}_i(JI^{s+1}) = I\,\mathrm{HS}_i(JI^s) \qquad (s \geq i).$$

The exact rank formula explains both the uniform bound and the earlier stabilization that can occur for an individual ideal. Recall the rank description of a discrete polymatroid [1, Propositions 1.2–1.4 and Theorem 3.4]. For a polymatroidal ideal I of degree d, its canonical upper rank is

$$g(A) = \max\{a(A) : x^a \in G(I)\}.$$

This function is nonnegative, integral, normalized, monotone, and submodular, with $g(E) = d$. The generator exponents are exactly

$$\mathcal{B}(g) = \{a \in \mathbb{N}^E : |a| = g(E),\ a(A) \leq g(A) \text{ for all } A \subseteq E\}.$$

Conversely, each such rank function defines a polymatroidal ideal, denoted I_g.

A coordinate with $g(\{j\}) = 0$ occurs in no generator. Removing it preserves the minimal resolution up to polynomial extension. For the rank formula we therefore use the active ground set, where

$$g(A) \geq 1 \qquad (\varnothing \neq A \subseteq E).$$

For a nonempty A, let $\mathcal{P}(A)$ be its set partitions into nonempty blocks, and define

$$\begin{aligned} D_g(\varnothing) &= 0, & D_g(A) &= \min_{\pi\in\mathcal{P}(A)} \sum_{B\in\pi} \big(g(B) - 1\big), \qquad (1.1)\\ u_g(A) &= |A| + D_g(A), & & \\ h_{g,i}(\varnothing) &= 0, & h_{g,i}(A) &= \min\{g(A) + i, u_g(A)\} \quad (A \neq \varnothing). \end{aligned}$$

Theorem 1.2 (Exact homological rank formula)**.** *For every $i \geq 0$, $h_{g,i}$ is an integral polymatroid rank function. Moreover,*

$$\mathrm{HS}_i(I_g) = \begin{cases} I_{h_{g,i}}, & h_{g,i}(E) = g(E) + i, \\ 0, & h_{g,i}(E) < g(E) + i. \end{cases} \qquad (1.2)$$

In the nonzero case, $h_{g,i}$ is the canonical upper rank of the shift ideal.

The formula comes from viewing a Koszul basis vector as a permitted set of coordinate deletions. In each multidegree these sets form a matroid, whose top homology is nonzero precisely when it has no coloops. This condition gives inequalities on the exponents, and uncrossing them produces the partition minimum D_g.

For I^s, the rank is sg. Each positive partition defect is therefore a positive integer multiplied by s, and truncation at i removes its effect once $s \geq i$. Tracking the defects determines the exact generation interval in Theorem 4.4. Thus the same formula gives the sharp uniform threshold and the complete stabilization behavior of each homological shift module. It also organizes all homological degrees into a single rank function by adding one coordinate, and its inequalities, viewed as linear weights on the Rees algebra, show that the shift modules are integrally closed. These consequences use the same partition data throughout.

2. **The Koszul matroid**

In a fixed multidegree, a Koszul basis vector records a set of variables whose exponents are lowered by one. We therefore ask which simultaneous deletions leave a monomial in the ideal. The lower inequalities below express the available room for deletion on every subset of variables. Their submodularity will make the permitted deletions a matroid.

We first record two basic facts in the form used below. Products of polymatroidal ideals correspond to sums of their ranks

$$I_g I_f = I_{g+f}. \tag{2.1}$$

Indeed, the integral polymatroid sum theorem states that the Minkowski sum has rank $g + f$ and that every integer point in the sum decomposes into integer points of the two summands [1, Proposition 1.4]. For a point of total degree $g(E) + f(E)$, both summands must be bases. In particular, $I_g^s = I_{sg}$.

Every I_g also has a linear resolution, since decreasing reverse lexicographic order gives linear quotients. To verify this, take an earlier generator x^a and a later generator x^b. At their last differing coordinate j, $a_j < b_j$. The exchange axiom applied to b, a, j gives $c = b - \mathbf{e}_j + \mathbf{e}_\ell$ with $b_\ell < a_\ell$. The coordinate ℓ occurs before j, so x^c is earlier than x^b, and $x^c/\gcd(x^c, x^b) = x_\ell$ divides $x^a/\gcd(x^a, x^b)$. Thus the order has linear quotients. The exact sequences for adjoining one generator, together with the Koszul resolutions of the variable-generated colon ideals, give a $g(E)$-linear resolution.

Lemma 2.1 (Lower-inequality membership)**.** *Put $d = g(E)$ and $r(U) = d - g(E \setminus U)$. For $\alpha \in \mathbb{N}^E$,*

$$x^\alpha \in I_g \quad \Longleftrightarrow \quad \alpha(U) \geq r(U) \quad \textit{for every } U \subseteq E.$$

Proof. Necessity follows from a base $b \leq \alpha$ and $b(U) = d - b(E \setminus U) \geq r(U)$. For sufficiency, choose an integer vector $b \leq \alpha$ satisfying all $b(A) \leq g(A)$ and maximal under coordinatewise increase; such a vector exists because 0 is feasible and the box is finite. If $b_j < \alpha_j$, maximality gives a set $A_j \ni j$ with $b(A_j) = g(A_j)$. The union of upper-tight sets is upper-tight, by submodularity of g. Let A be the union of these A_j, or $\varnothing$ if there are none. Then $b(A) = g(A)$ and $b(E \setminus A) = \alpha(E \setminus A)$. This b is a base whose monomial divides x^α, because

$$d \geq |b| = g(A) + \alpha(E \setminus A) \geq g(A) + d - g(A) = d. \qquad \square$$

The constraint function in our application need not be monotone; adding a variable increases its exponent sum, but can also increase the required lower bound. The following form of the matroid construction keeps precisely the hypothesis that survives, namely submodularity.

Lemma 2.2 (A submodular constraint matroid)**.** *Let $q : 2^E \to \mathbb{N}$ be submodular with $q(\varnothing) = 0$, and let $T \subseteq E$. The family*

$$\mathcal{I}_q = \{F \subseteq T : |F \cap U| \leq q(U) \textit{ for every } U \subseteq E\}$$

is the independence family of a matroid M on T. Its rank on $R \subseteq T$ is

$$r_M(R) = \min_{U \subseteq E} \big(q(U) + |R \setminus U|\big). \tag{2.2}$$

Monotonicity of q is not required.

Proof. The family is nonempty and hereditary. Choose an inclusion-maximal feasible $F \subseteq R$. For every $j \in R \setminus F$, failure to add j gives a set $U_j \ni j$ with $|F \cap U_j| = q(U_j)$. Tight sets for F are closed under unions; the left side is modular and the right side is submodular. Their union U is tight and contains $R \setminus F$. Therefore

$$|F| = q(U) + |R \setminus U|.$$

If $F = R$, use $U = \varnothing$. For every feasible $F' \subseteq R$ and every V, $|F'| \leq q(V) + |R \setminus V|$. Thus every maximal feasible subset of R has the cardinality in (2.2). The hereditary-family characterization of a matroid now applies; if independent sets F, G with $|F| < |G|$ admitted no augmentation from $G \setminus F$, then F would be a maximal independent subset of $F \cup G$ of smaller than maximum size. $\square$

The rank formula for the unit ideal is immediate, so assume $d > 0$. Fix $i \geq 0$ and $\alpha \in \mathbb{N}^E$ with $|\alpha| = d + i$. If $x^\alpha \notin I_g$, the degree-α part of the Koszul complex of I_g is zero. Otherwise put $T = \operatorname{supp} \alpha$ and

$$\Delta_\alpha = \{F \subseteq T : x^{\alpha - \mathbf{1}_F} \in I_g\}.$$

In homological degree j, the degree-α Koszul basis consists of $x^{\alpha-\mathbf{1}_F} e_F$ for the faces F of size j. Removing an exterior variable is exactly the oriented simplicial boundary of that face. The empty face corresponds to Koszul degree zero, so the degree shift is $j \mapsto j - 1$. Thus

$$\operatorname{Tor}_i^S(k, I_g)_\alpha \cong \widetilde{H}_{i-1}(\Delta_\alpha; k). \tag{2.3}$$

By Lemma 2.1, Δ_α is the family in Lemma 2.2 for

$$q(U) = \alpha(U) - r(U).$$

This is a nonnegative integer-valued submodular function, because r is supermodular. Also $q(\varnothing) = 0$ and $q(E) = i$. Consequently Δ_α is a matroid independence complex of rank at most i. Here $q(U)$ is the number of deletions allowed inside U; the total-degree condition leaves room for at most i deletions in all. This is why only the top homology of the resulting matroid can contribute to Tor_i.

3. **The homological rank**

Lemma 3.1 (Homology of a matroid independence complex)**.** *Let M be a matroid of rank r. Over every field, its independence complex has reduced homology only in degree $r - 1$. Its homology in degree $r - 1$ is nonzero if and only if M has no coloops. Rank zero uses the convention $\widetilde{H}_{-1}(\{\varnothing\}; k) = k$.*

Proof. Induct on the ground-set size. Loops can be deleted without changing the complex. A coloop makes the complex a cone, so all reduced homology vanishes. If the rank is zero, the assertion is the stated convention. In the remaining case choose a nonloop e, necessarily a noncoloop. Deletion has rank r, contraction has rank $r - 1$, and

$$\Delta(M) = \Delta(M \setminus e) \ \cup \ \big(e * \Delta(M/e)\big), \qquad \Delta(M \setminus e) \cap \big(e * \Delta(M/e)\big) = \Delta(M/e).$$

The reduced Mayer–Vietoris sequence and induction give concentration in degree $r - 1$ and a short exact sequence

$$0 \longrightarrow \widetilde{H}_{r-1}(\Delta(M \setminus e); k) \longrightarrow \widetilde{H}_{r-1}(\Delta(M); k) \longrightarrow \widetilde{H}_{r-2}(\Delta(M/e); k) \longrightarrow 0.$$

Contraction does not create coloops; for $f \neq e$,

$$r_{M/e}(T \setminus \{e\}) - r_{M/e}(T \setminus \{e, f\}) = r_M(T) - r_M(T \setminus \{f\}).$$

Thus M/e has no coloops when M has none, and the last term is nonzero by induction. The same argument includes $r = 1$ using reduced degree -1. □

For the matroid Δ_α, Lemma 3.1 says that (2.3) is nonzero exactly when its rank is i and it has no coloops. Since $T \neq \varnothing$, this is equivalent to

$$r_M(T \setminus \{j\}) = i \qquad (j \in T). \tag{3.1}$$

In (2.2), set $V = E \setminus U$. For $R = T \setminus \{j\}$,

$$q(U) + |R \setminus U| = i - \alpha(V) + g(V) + |T \cap V| - \mathbf{1}_{j \in V}.$$

Since the matroid rank is already at most i, the minimum in (2.2) equals i precisely when every term is at least i. For a set V meeting T, the strongest condition comes from choosing $j \in V \cap T$, which contributes the final -1. Hence (3.1) is equivalent to

$$\alpha(V) - |T \cap V| \leq g(V) - 1 \quad \text{whenever } V \cap T \neq \varnothing. \tag{3.2}$$

Put $\beta_j = \max(\alpha_j - 1, 0)$, so that β records the exponents remaining after removing the first copy of every variable in the support. Since $g(V) \geq 1$ for every nonempty V, the sets disjoint from T satisfy the same inequality automatically. Thus (3.2) becomes

$$\beta(V) \leq g(V) - 1 \qquad (\varnothing \neq V \subseteq E). \tag{3.3}$$

The bounds $g(V) - 1$ need not themselves form a rank function; the submodular inequality can fail for disjoint sets. But β is additive across the blocks of every partition, so all partition sums are forced bounds as well. Taking their minimum gives the equivalent system

$$\beta(V) \leq D_g(V) \qquad (V \subseteq E). \tag{3.4}$$

Indeed, (3.3) implies the bound for every partition, and the one-block partition gives $D_g(V) \leq g(V) - 1$ for the converse.

Lemma 3.2 (The partition rank)**.** *D_g is a nonnegative, integer-valued, normalized, monotone, submodular function. Moreover, for $A \subseteq B$,*

$$u_g(B) - u_g(A) \geq g(B) - g(A). \tag{3.5}$$

Proof. Let $f(C) = g(C) - 1$ for nonempty C. It is nonnegative and monotone and satisfies the submodular inequality whenever the intersection is nonempty. Restrict an optimal partition of B to A, omitting empty intersections. Its cost cannot increase, proving $D_g(A) \le D_g(B)$.

For submodularity, take optimal partitions of A and B, as a multiset of blocks. Replace intersecting incomparable blocks C, D by $C \cup D, C \cap D$. This cannot increase total cost; it preserves the multiplicity with which each element is covered; and it strictly increases the integer $\sum_C |C|^2$. The latter is bounded, so uncrossing terminates in a laminar multiset. Every element has multiplicity at most two. Consequently its outer blocks partition $A \cup B$ and its inner blocks partition $A \cap B$. If a block occurs twice, one copy is outer and one is inner. Hence the final cost is at least $D_g(A \cup B) + D_g(A \cap B)$, proving submodularity.

For (3.5), let π be an optimal partition of B and let m be the number of its blocks disjoint from A. Restricting the other blocks to A gives

$$\begin{aligned} u_g(B) - u_g(A) &\ge |B \setminus A| - m + \sum_{C \in \pi} \big(g(C) - g(C \cap A)\big) \\ &\ge \sum_{C \in \pi} \big(g(C) - g(C \cap A)\big) \ge g(B) - g(A). \end{aligned}$$

The middle inequality holds because each of the m disjoint blocks contains an element of $B \setminus A$. The last inequality follows by adding the disjoint blocks of π successively to A; submodularity bounds each marginal increase by $g(C) - g(C \cap A)$, and the increases telescope. □

Using monotonicity of D_g, (3.4) is equivalent to

$$\alpha(V) \le u_g(V) \qquad (V \subseteq E). \tag{3.6}$$

One direction follows from $\alpha = \beta + \mathbf{1}_T$. For the other, apply (3.6) to $W = V \cap T$ to obtain

$$\beta(V) = \alpha(W) - |W| \le D_g(W) \le D_g(V).$$

Finally, at total degree $d + i$, Lemma 2.1 says that membership in I_g is equivalent to

$$\alpha(V) \le g(V) + i \qquad (V \subseteq E).$$

Combining these facts gives the following field-independent description, with the inequalities required for every $V \subseteq E$

$$\operatorname{Tor}_i^S(k, I_g)_\alpha \ne 0 \quad \Longleftrightarrow \quad |\alpha| = d + i \ \text{ and } \ \alpha(V) \le h_{g,i}(V). \tag{3.7}$$

Here linearity of the resolution has excluded all other total degrees.

3.1. **A rank interpolation**

The inequality (3.5) is what makes the homological bounds into rank functions. The first interpolation below gives each shift separately, while the second places all shifts in one polymatroid by adding a coordinate for homological degree. Although the minimum of two submodular functions is not usually submodular, (3.5) orders their marginal increments and thus controls the mixed case in which the two sets select different branches.

Lemma 3.3 (Rank interpolation)**.** *Let g, u be integral polymatroid ranks on E such that*

$$u(B) - u(A) \geq g(B) - g(A) \qquad (A \subseteq B).$$

For $i \geq 0$, the function

$$h_i(\varnothing) = 0, \qquad h_i(A) = \min\{g(A) + i, u(A)\} \quad (A \neq \varnothing)$$

is a polymatroid rank. If $p = u(E) - g(E)$, then so is the function on $E \cup \{z\}$ defined by

$$F(A) = u(A), \qquad F(A \cup \{z\}) = p + g(A).$$

Proof. The hypothesis gives $0 \leq u(A) - g(A) \leq p$. Normalization, integrality, nonnegativity, and monotonicity follow, including monotonicity of F between sets of its two forms. For submodularity of h_i, if both values select the same branch, use submodularity of that branch. The function taking value $g(A) + i$ on nonempty sets and zero at the empty set is submodular as well. In a mixed case the required inequality follows from

$$g(A \cup B) - g(A) \leq g(B) - g(A \cap B) \leq u(B) - u(A \cap B).$$

The same inequality, with A and B interchanged, proves the mixed case of submodularity for F. Its two unmixed cases use the submodularity of u and g. □

Apply the lemma to $u = u_g$, using Lemma 3.2.

For completeness, an integral polymatroid rank is always attained on every subset by one of its integer bases. Choose a coordinate order placing that subset first, let A_j be its successive initial sets, and take the greedy vector $b_j = h(A_j) - h(A_{j-1})$. Monotonicity makes it nonnegative and integral, and submodularity shows $b(V) \leq h(V)$ for every V by summing diminishing marginal increments. It has $|b| = h(E)$ and attains h on each initial set. This proves the asserted canonicality without any additional compatibility hypothesis.

Now $h_{g,i}(E) \le d+i$. If the inequality is strict, (3.7) has no solutions. If equality holds, its solutions are exactly all integer bases of $h_{g,i}$; these form the generator set of $I_{h_{g,i}}$. This proves Theorem 1.2.

Lemma 3.4 (Multiplication of shifts)**.** *For any two polymatroidal ideals and every* $i \ge 0$*,*

$$I_f \operatorname{HS}_i(I_g) \subseteq \operatorname{HS}_i(I_f I_g).$$

Proof. The zero and unit cases are immediate. Otherwise take a generator multidegree α of $\operatorname{HS}_i(I_g)$ and a base $\gamma \in \mathcal{B}(f)$, and work on the union of the active ground sets. Put $\beta = (\alpha - \mathbf{1})_+$ and $\beta' = (\alpha + \gamma - \mathbf{1})_+$. If a nonempty set V meets the active ground set of g, then (3.3), applied on that ground set, gives

$$\beta'(V) \le \beta(V) + \gamma(V) \le g(V) - 1 + f(V).$$

If V misses it, then $\alpha(V) = 0$. When $\gamma(V) > 0$, $\beta'(V) \le \gamma(V) - 1 \le f(V) - 1$; otherwise $\beta'(V) = 0 \le f(V) - 1$, since f is positive on every nonempty subset of its active ground set. Thus β' satisfies (3.3) for $f + g$. The vector $\alpha + \gamma$ has the required total degree and its monomial belongs to $I_f I_g$, so it is a shift multidegree of the product. □

4. **Powers and partition defects**

For a partition π of a nonempty set A, write

$$\delta_g(\pi) = \sum_{B \in \pi} g(B) - g(A), \qquad b(\pi) = |A| - |\pi|.$$

Both are nonnegative integers; the first measures failure of rank additivity across the blocks, while the second depends only on the partition. In the homological rank they become the slope and intercept of an affine function of the power, so stabilization occurs when every positive-slope term reaches the homological cutoff.

Let $I = I_\rho$ and $J = I_\eta$, and use the union of their active coordinates. For $s \ge 1$, the rank $g_s = \eta + s\rho$ is positive on every nonempty subset. Theorem 1.2 gives

$$h_{g_s,i}(A) = g_s(A) + \min\left\{ i,\ \min_{\pi \in \mathcal{P}(A)} \big(s\delta_\rho(\pi) + \delta_\eta(\pi) + b(\pi) \big) \right\}. \tag{4.1}$$

If $s \ge i$, every term with $\delta_\rho(\pi) > 0$ is at least i. The remaining minimum is independent of s, so

$$h_{g_{s+1},i} = \rho + h_{g_s,i}.$$

The total-rank criterion in Theorem 1.2 makes the two consecutive shifts simultaneously zero or nonzero. In the latter case, the polymatroid sum identity (2.1) gives

$$\mathrm{HS}_i(JI^{s+1}) = I_{\rho+h_{g_s,i}} = I\,\mathrm{HS}_i(JI^s).$$

This proves both equalities in Theorem 1.1, the ordinary case being $J = S$.

Lemma 3.4 makes the displayed direct sums Rees modules. Iterating the equality from degree i proves their generation bound.

For ordinary powers, delete the inactive coordinates of I and put

$$c_\rho(A) = \max\{|\pi| : \pi \in \mathcal{P}(A),\ \delta_\rho(\pi) = 0\}.$$

Since the one-block partition has zero defect, this maximum is always defined. Specializing (4.1) to $\eta = 0$ gives

$$h_{s\rho,i}(A) = s\rho(A) + \min\left\{i, \min_{\pi\in\mathcal{P}(A)}\big(s\delta_\rho(\pi) + b(\pi)\big)\right\}. \tag{4.2}$$

In particular,

$$h_{s\rho,i}(A) = s\rho(A) + \min\{i, |A| - c_\rho(A)\} \qquad (s \geq i).$$

Corollary 4.1 (Several colors)**.** *Let $I_1, \ldots, I_r, J$ be polymatroidal ideals. If $s_j \geq i > 0$, then*

$$\mathrm{HS}_i\left(JI_j^{s_j+1}\prod_{\ell\neq j} I_\ell^{s_\ell}\right) = I_j\,\mathrm{HS}_i\left(J\prod_{\ell=1}^{r} I_\ell^{s_\ell}\right).$$

Thus the corresponding multigraded shift module is generated in the box $0 \leq s_j \leq i$, with the zero-degree pieces interpreted in the usual way.

Proof. Apply the mixed-power theorem with the fixed polymatroidal product $J\prod_{\ell\neq j} I_\ell^{s_\ell}$, and repeat while a coordinate exponent exceeds i. □

4.1. **Sharpness and projective dimension**

Proposition 4.2. *For each $i \geq 1$, let $n = i + 1$ and let I be the squarefree Veronese ideal of degree i in $k[x_1, \ldots, x_n]$. Then*

$$\mathrm{HS}_i(I^s) = 0 \quad (1 \leq s < i), \qquad \mathrm{HS}_i(I^i) = \big((x_1\cdots x_n)^i\big) \neq 0.$$

Therefore no universal generation bound smaller than i is possible.

Proof. The rank is $\rho(A) = \min\{|A|, i\}$. On every proper subset A, all partitions have zero defect and the singleton partition has intercept zero. Formula (4.2) therefore gives $h_{s\rho,i}(A) = s|A|$.

On E, the one-block partition has defect zero and intercept i. Every other partition has defect one, with least intercept zero at the singleton partition. Hence

$$h_{s\rho,i}(E) = si + \min\{i, s\}.$$

The total-rank criterion makes the shift zero exactly when $s < i$. At $s = i$, its singleton ranks are i and its total rank is $i(i+1)$, so its only base is $i\mathbf{1}_E$. This proves both assertions, including the case $i = 1$. □

Corollary 4.3 (Exact projective dimensions)**.** *For an active nonzero proper polymatroidal ideal I of rank ρ,*

$$\operatorname{pd}_S(I^s) = \min_{\pi \in \mathcal{P}(E)} \big(s\delta_\rho(\pi) + |E| - |\pi|\big). \tag{4.3}$$

In particular the eventual value is $|E| - c_\rho(E)$, and is reached by $s = \max\{1, |E| - c_\rho(E)\}$.

Proof. Theorem 1.2 says that $\operatorname{HS}_i(I^s) \neq 0$ exactly when $i \leq u_{s\rho}(E) - s\rho(E)$. Substituting the partition expression gives (4.3). Every positive-defect term is at least s, while the least zero-defect term is $|E| - c_\rho(E)$. This proves the stabilization assertion. □

4.2. **The exact interval of generator degrees**

The same partition formula determines the precise generation degree for each individual ideal. Continue to assume that I is nonzero and proper, and that E is its active ground set. Fix $i > 0$. For nonempty $A \subseteq E$ put

$$\kappa_i(A) = \min\{i, |A| - c_\rho(A)\}, \qquad \phi_s(A) = h_{s\rho,i}(A) - s\rho(A) \quad (s \geq 1).$$

For a given partition with positive defect, the inequality $s\delta_\rho(\pi) + b(\pi) \geq \kappa_i(A)$ gives the first power at which that partition can no longer lower the final rank. The last partition to reach this value determines the threshold. Accordingly, define

$$\tau_i(A) = \max\left(\{0\} \cup \left\{ \left\lceil \frac{\kappa_i(A) - b(\pi)}{\delta_\rho(\pi)} \right\rceil : \pi \in \mathcal{P}(A),\ \delta_\rho(\pi) > 0 \right\}\right).$$

In particular $0 \leq \tau_i(A) \leq i$. Although ϕ_s arose from powers with $s \geq 1$, its expression in (4.2) is also defined at $s = 0$, where it is identically zero.

Theorem 4.4 (Exact generation interval)**.** *If $i > |E| - c_\rho(E)$, the entire ith shift module is zero. Otherwise set*

$$r_i = \tau_i(E), \qquad s_i = \max_{\varnothing \neq A \subseteq E} \tau_i(A). \tag{4.4}$$

Then $1 \leq r_i \leq s_i \leq i$. The first nonzero homological shift is $\mathrm{HS}_i(I^{r_i})$. The Rees degrees occurring among the minimal monomial generators of $\bigoplus_{s\geq 1} \mathrm{HS}_i(I^s)t^s$ are exactly

$$r_i, r_i + 1, \ldots, s_i.$$

Thus s_i is the exact least generation bound for this ideal and homological degree.

Proof. The least zero-defect contribution to (4.2) is $|A| - c_\rho(A)$. Consequently

$$\phi_s(A) = \min(\{\kappa_i(A)\} \cup \{s\delta_\rho(\pi) + b(\pi) : \delta_\rho(\pi) > 0\}).$$

It equals its final value $\kappa_i(A)$ precisely when $s \geq \tau_i(A)$. Moreover, if $s < \tau_i(A)$, every positive-defect term increases by at least one between s and $s+1$, while all terms are integral and the constant term is strictly larger than $\phi_s(A)$. Hence

$$\phi_{s+1}(A) > \phi_s(A) \quad (s < \tau_i(A)), \qquad \phi_{s+1}(A) = \phi_s(A) \quad (s \geq \tau_i(A)). \tag{4.5}$$

The total-rank criterion gives $\mathrm{HS}_i(I^s) \neq 0$ exactly when $\phi_s(E) = i$. If the final value is less than i, all pieces vanish. Otherwise the first nonzero degree is $r_i = \tau_i(E)$, which is positive because $\phi_0(E) = 0 < i$.

For $s \geq r_i$, both consecutive shifts are nonzero, so their displayed rank functions are canonical. The product $I\,\mathrm{HS}_i(I^s)$ has canonical rank $(s+1)\rho + \phi_s$, whereas $\mathrm{HS}_i(I^{s+1})$ has canonical rank $(s+1)\rho + \phi_{s+1}$. Equations (4.4) and (4.5) show that these ideals are equal exactly when $s \geq s_i$. At every $r_i \leq s < s_i$, choose a set A with $\tau_i(A) = s_i$. Its rank increases strictly, and a base attaining the new rank on A lies outside the preceding product. Such a base requires a new module generator in degree $s+1$. It cannot come from a still earlier piece, since the module action puts every product from an earlier piece in $I\,\mathrm{HS}_i(I^s)$. All generators in the first nonzero piece are also required. This proves the exact interval assertion. □

The proof also specifies the new minimal monomial generators themselves. All bases of $\mathrm{HS}_i(I^{r_i})$ occur. For $r_i < s \leq s_i$, a base a of $h_{s\rho,i}$ gives a new module generator exactly when

$$a(A) > s\rho(A) + \phi_{s-1}(A) \quad \text{for some nonempty } A \subseteq E.$$

Indeed, the right-hand bounds are the exact rank description of $I\,\mathrm{HS}_i(I^{s-1})$. Since all candidate minimal generators in this Rees degree have ordinary degree $s\rho(E)+i$, no further divisibility among them can remove any of these new generators.

For example, take $I=(xy,xz,yz)(u,v)$ and $i=2$. Its rank is

$$\rho(A)=\min\{2,|A\cap\{x,y,z\}|\}+\min\{1,|A\cap\{u,v\}|\}.$$

The singleton partition gives $D_\rho(E)=0$, hence $h_{\rho,2}(E)=5$ and $\tau_2(E)=1$. On $A=\{x,y,z\}$, the partition formula gives $\phi_s(A)=\min\{2,s\}$, so $\tau_2(A)=2$. Since every threshold is at most $i=2$, the exact generation interval is $\{1,2\}$. Concretely, $\mathrm{HS}_2(I)=(xyzuv)$, while $x^2y^2z^2u^2\in\mathrm{HS}_2(I^2)$ lies outside $I\,\mathrm{HS}_2(I)$.

5. **The total Betti support**

The ranks of the individual shifts fit into one polymatroid when an extra coordinate records homological degree. The homogenized Betti support was shown to be a polymatroid in [3]; the interpolation in Lemma 3.3 identifies its rank explicitly and gives a direct proof.

Theorem 5.1 (Homogenized total Betti support)**.** *Let $I=I_g$ be a nonzero proper polymatroidal ideal on its active ground set E, let $d=g(E)$, and put*

$$p=u_g(E)-d=\mathrm{pd}_S(I).$$

In the polynomial ring $S[z]$, define

$$\mathcal{H}(I)=\sum_{i=0}^{p}z^{p-i}\,\mathrm{HS}_i(I). \tag{5.1}$$

Then $\mathcal{H}(I)$ is a polymatroidal ideal generated in degree $d+p$. Its canonical rank F on $E\cup\{z\}$ is

$$F(A)=u_g(A),\qquad F(A\cup\{z\})=p+g(A)\qquad(A\subseteq E).$$

Equivalently, its bases are exactly

$$\{(\alpha,p-i):\mathrm{Tor}_i^S(k,I)_\alpha\neq 0,\ 0\le i\le p\}. \tag{5.2}$$

This description is independent of the field.

Proof. The projective-dimension equality follows from Theorem 1.2. Lemma 3.3, applied to $u=u_g$, shows that F is a rank function.

An integer base (α, t) of F satisfies $|\alpha| + t = d + p$ and $0 \le t \le F(\{z\}) = p$. Set $i = p - t$. Its inequalities are exactly

$$|\alpha| = d + i, \qquad \alpha(A) \le u_g(A), \qquad \alpha(A) \le g(A) + i.$$

By (3.7), these are equivalent to $\operatorname{Tor}_i^S(k, I)_\alpha \neq 0$. Conversely every such Betti multidegree satisfies every base inequality of F. This proves (5.2), and hence (5.1) and its asserted polymatroidality. The greedy attainment argument in Section 3 proves canonicality of the displayed rank. □

When I is principal, $p = 0$ and z is a loop of this polymatroid. Variables inactive in I can likewise be reattached as loops. The unit ideal gives the rank-zero singleton base, with the same interpretation. Thus no extra positivity assumption on individual generator coordinates is involved.

6. **Integral closure inside the Rees algebra**

For fixed $i > 0$, Lemma 3.4 realizes the shift module as an ideal of the Rees algebra, since every shift multidegree belongs to the corresponding power of I. The partition inequalities describe it as an intersection of valuation ideals; each bounds a linear weight nonnegative on the Rees semigroup. This description will prove integral closure.

Theorem 6.1. *For a polymatroidal ideal I, the monomial ideal*

$$M_i = \bigoplus_{s \ge 1} \mathrm{HS}_i(I^s) t^s \;\subseteq\; \mathcal{R}(I)$$

is integrally closed in $\mathcal{R}(I)$ for every $i > 0$ and every field.

Proof. Separate the zero and unit ideals, whose positive shift ideals vanish. Otherwise let ρ be the rank on the active ground set E, and let $d = \rho(E)$. Write a Rees monomial as $x^\alpha t^s$. For a nonempty $A \subseteq E$ and a partition π of A, set

$$L_{A,\pi}(\alpha, s) = \alpha(E \setminus A) + s \left(\sum_{B \in \pi} \rho(B) - d \right). \tag{6.1}$$

We claim that membership in M_i is equivalent to membership in $\mathcal{R}(I)$ together with

$$s \ge 1, \qquad |\alpha| - sd \ge i, \tag{6.2}$$

and, for every nonempty $A \subseteq E$ and every $\pi \in \mathcal{P}(A)$,

$$L_{A,\pi}(\alpha, s) \ge i - |A| + |\pi|. \tag{6.3}$$

To see this, put $h_s = h_{s\rho,i}$. The lower-inequality membership criterion of Lemma 2.1, applied to the shift rank, says

$$\alpha(E \setminus A) \geq sd + i - h_s(A) \qquad (A \subseteq E). \tag{6.4}$$

If $h_s(E) < sd + i$, its $A = E$ inequality is impossible, exactly as required for a zero shift ideal. Otherwise it is the ordinary membership criterion for I_{h_s}. For $A \neq \varnothing$, expanding the minimum defining $h_s(A)$ yields the Rees-membership inequalities $\alpha(E \setminus A) \geq s(d - \rho(A))$ and all of (6.3). The case $A = \varnothing$ gives the second inequality of (6.2). This proves the claim.

Each weight in (6.1)–(6.3) is an integer linear function nonnegative on the Rees exponent semigroup. For $L_{A,\pi}$, nonnegativity on a polynomial variable is immediate. On a generator of Rees degree one, meaning $(\gamma, 1)$ with $\gamma \in \mathcal{B}(\rho)$, it follows from

$$L_{A,\pi}(\gamma, 1) \geq \sum_{B \in \pi} \rho(B) - \rho(A) \geq 0.$$

The other two weights are the Rees degree and the excess ordinary degree.

For any one of these linear weights L, assign to a nonzero polynomial in the Rees algebra the minimum L-weight of its monomials. This extends to a valuation on its fraction field. In fact, initial forms of nonzero polynomials multiply to a nonzero initial form in the Laurent polynomial domain, so the minimum is additive under multiplication. The ideal of elements of valuation at least a fixed integer c is integrally closed; if an element of smaller valuation satisfied an integral equation with coefficients in successive powers of that ideal, the leading power of the element would have strictly smaller valuation than every other term, an impossibility.

By the membership claim, M_i is the intersection of precisely these valuation ideals, with their displayed thresholds. An intersection of integrally closed ideals is integrally closed, proving the theorem. Inactive variables have weight zero and do not affect this argument. □

Corollary 6.2 (Twisted multi-Rees ideals)**.** *Let $I_1, \ldots, I_r, J$ be nonzero polymatroidal ideals, with the unit ideal allowed, and let*

$$R = \bigoplus_{s \in \mathbb{N}^r} I_1^{s_1} \cdots I_r^{s_r} t_1^{s_1} \cdots t_r^{s_r}.$$

For every $i > 0$, the ideal

$$M_i^J = \bigoplus_{s \in \mathbb{N}^r} \mathrm{HS}_i(J I_1^{s_1} \cdots I_r^{s_r}) t_1^{s_1} \cdots t_r^{s_r} \ \subseteq R$$

is integrally closed. This includes the multidegrees with some $s_j = 0$. A zero twist J gives the zero ideal, with the same conclusion.

Proof. Let $\eta, \rho_1, \dots, \rho_r$ be the ranks on their common ground set, and put $d_j = \rho_j(E)$ and $d_0 = \eta(E)$. Formulas (1.1)–(1.2) remain valid if inactive coordinates are retained; an inactive coordinate is optimally split off as a singleton of partition cost -1, so

$$D_g(A) = D_{g|_T}(A \cap T) - |A \setminus T|,$$

where T is the active set. Thus u_g and $h_{g,i}$ are simply extended by rank-zero loops. The same formulas include $g = 0$.

For $U = E \setminus A$, the membership inequalities for M_i^J are

$$|\alpha| - \sum_j s_j d_j \geq d_0 + i,$$

$$\alpha(U) - \sum_j s_j \big(d_j - \rho_j(A)\big) \geq d_0 - \eta(A),$$

$$\alpha(U) + \sum_j s_j \left(\sum_{B \in \pi} \rho_j(B) - d_j \right) \geq d_0 - \sum_{B \in \pi} \eta(B) + i - |A| + |\pi|.$$

The second inequality ranges over $A \subseteq E$, and the third over nonempty A and $\pi \in \mathcal{P}(A)$. They follow by expanding (6.4) for $g = \eta + \sum_j s_j \rho_j$, exactly as in the preceding theorem. Every linear weight on the left is nonnegative on the exponent semigroup of R; on a generator from the jth factor this is the base inequality or the nonnegative partition defect for ρ_j, and it is immediate on the polynomial variables. Hence the same intersection-of-valuations proof applies. The multiplication inclusion of Lemma 3.4, and the containment $J \prod_j I_j^{s_j} \subseteq \prod_j I_j^{s_j}$, justify its displayed realization as an ideal of R. □

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