Explicit nonpolytopal Bier spheres

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Abstract. We construct explicit nonpolytopal Bier spheres by bounding the sign rank of a matrix of face-extension data in every convex realization. The bound depends only on a fixed core of labels, and a rectangular Hadamard matrix supplies explicit extension data whose sign rank exceeds it. Finite-field formulas give an example with 290 365 vertices and dimension 145 181, defined by a complex of dimension nine. Because the obstruction uses only selected face memberships, it also gives examples with no ghost labels, with combinatorial central symmetry, and with a defining complex of dimension three.

1. Introduction

Bier's construction associates a simplicial sphere to every nonempty proper simplicial complex. Counting shows that most of these spheres are nonpolytopal [1], but does not specify an example. Holleben and Yang [3] proved that every Bier sphere with at most twelve vertices is polytopal and asked for an explicit nonpolytopal sphere. We construct one by encoding a matrix whose sign rank exceeds a bound imposed by every convex realization.

Let K be a nonempty proper simplicial complex on a labelled ground set V. Singleton labels need not be faces; an absent singleton is called a *ghost*. The Alexander dual and Bier sphere are

$$K^* = \{B \subseteq V : V \setminus B \notin K\}, \qquad \mathrm{Bier}(K) = \{A \cup B' : A \in K,\ B \in K^*,\ A \cap B = \varnothing\},$$

where B' is a copy of B on a disjoint primed ground set. The deleted join $\mathrm{Bier}(K)$ is a sphere of dimension $|V| - 2$ [1].

Theorem 1. *There is an explicitly specified complex of dimension nine whose Bier sphere is nonpolytopal, has* 290 365 *actual vertices, and has dimension* 145 181*. There are also nonpolytopal examples with all ground labels present, with a self-dual input complex, and with an input complex of dimension three.*

The obstruction separates a small core of labels from a much larger set of additional labels. For each additional label we record which selected core faces extend to a face containing it, arranging these membership signs as a column of a matrix. Convexity

bounds the sign rank of this matrix solely in terms of the number of core labels, independently of how many additional labels we use.

For a sign matrix $H \in \{-1, +1\}^{r\times s}$, define

$$\operatorname{signrank}(H) = \min\{\operatorname{rank} M : H_{it}M_{it} > 0 \text{ for all } i, t\}.$$

The precise geometric obstruction is as follows.

Theorem 2. *Let $\mathcal{K}$ be a proper complex on $E = U \sqcup \{a_1, \dots, a_s\}$, containing every singleton, where $|U| = m$. Choose faces $S_1, \dots, S_r \subseteq U$ of one fixed positive cardinality, and define*

$$H_{it} = +1 \quad \Longleftrightarrow \quad S_i \cup \{a_t\} \in \mathcal{K}.$$

Regard $\mathcal{K}$ as a complex Δ on $E \sqcup \{g\}$, with g a ghost. If $\operatorname{Bier}(\Delta)$ *is polytopal, then*

$$\operatorname{signrank}(H) \le m^2 - m + 1.$$

For selected faces of arbitrary cardinalities, the bound $m^2 - m + 2$ holds instead.

Supporting hyperplanes express a matrix with signs H through Schur complements of one matrix. The required principal inverses, padded by zeros, satisfy linear identities leaving at most $m^2 - m + 1$ dimensions. This proves the bound using only the selected memberships; the other faces of the complex are unrestricted.

For the principal example, take $m = 20$ and retain $s = 381^2 + 1$ columns of a Hadamard matrix of order $147\,456 = 384^2$. Forster's spectral bound [2] and Theorem 2 would give

$$381 < \sqrt{s} \le \operatorname{signrank}(H) \le 20^2 - 20 + 1 = 381.$$

Section 3 specifies the matrix and the face set by elementary finite-field formulas. The final constructions preserve the obstruction while changing the ghost, symmetry, or input-dimension conditions.

2. The sign-rank obstruction

Put $q = |E|$. All q unprimed labels in E and all $q + 1$ primed labels occur in $\operatorname{Bier}(\Delta)$; the complements defining the primed singletons either contain the ghost or equal the nonface E. Thus the sphere has $2q + 1$ actual vertices and dimension $q - 1$.

In a hypothetical realization, denote the labelled vertices by $x_i = i$, $p_i = i'$ for $i \in E$, and $z = g'$. The facets we need are

$$\begin{aligned} T_A &= \{x_i : i \in A\} \cup \{p_i : i \in E \setminus A\} && (A \in \mathcal{K}), \\ F_{B,a} &= \{z\} \cup \{x_i : i \in B\} \cup \{p_i : i \in E \setminus (B \cup \{a\})\} && (B \in \mathcal{K},\ B \cup \{a\} \notin \mathcal{K}). \end{aligned}$$

In each case the positive set is a face, the complement of the primed set is a nonface, and there are q vertices. These are therefore facets by the definition of the Bier sphere.

These two kinds of facets distinguish an extension from a nonextension; when $S \cup \{a\}$ is a face, replacing p_a by x_a in T_S gives another facet, whereas a nonextension supplies a facet containing z and omitting both p_a and x_a. With z as the origin, the two alternatives impose opposite signs on one coordinate of x_a in the basis supplied by T_S. Computing that coordinate gives the following Schur complement.

Lemma 3. *If* $\mathrm{Bier}(\Delta)$ *is polytopal, there are a real* $m \times m$ *matrix* G*, row vectors* u_t*, column vectors* v_t*, and scalars* d_t *such that every* $G[S_i, S_i]$ *is invertible and*

$$H_{it}\big(d_t - u_t[S_i]G[S_i, S_i]^{-1}v_t[S_i]\big) > 0.$$

Proof. The simplex facet $T_\varnothing$ and the vertex z outside its affine hull form an affine basis. Normalize the realization to

$$z = 0, \qquad p_i = e_i, \qquad x_i = -N_i,$$

where N_i is column i of a real $q \times q$ matrix N. For a selected face S, let C_S have columns x_j for $j \in S$ and e_j otherwise. These columns form the simplex facet T_S outside the origin, so C_S is invertible. Expanding in its standard basis columns shows that $N[S, S]$ is invertible as well.

For an anchor $a = a_t$, write $\alpha = C_S^{-1}x_a$. Solving first in the rows S and then in row a gives

$$\alpha_a = -N_{aa} + N[a, S]N[S, S]^{-1}N[S, a].$$

We claim that $-\alpha_a$ has the prescribed sign H_{it}.

If $S \cup \{a\}$ is a face, normalize the supporting functionals of T_S and $T_{S\cup\{a\}}$ to $h \le 1$ and $h' \le 1$. Every column of C_S has h-value one, while all except column a also have h'-value one. Since x_a lies outside the first facet and p_a outside the second,

$$\sum_j \alpha_j = h(x_a) < 1, \qquad 1 - \sum_j \alpha_j = \alpha_a\big(h'(p_a) - 1\big).$$

The left side is positive and the parenthesized factor negative, so $\alpha_a < 0$.

If $S \cup \{a\}$ is a nonface, use the facet $F_{S,a}$. Its supporting linear functional ℓ may be chosen zero on the facet and positive at every other vertex. It vanishes on all columns of C_S except column a, whence

$$0 < \ell(x_a) = \alpha_a \ell(p_a), \qquad \ell(p_a) > 0.$$

Thus $\alpha_a > 0$. The claim follows with

$$G = N[U, U], \quad u_t = N[a_t, U], \quad v_t = N[U, a_t], \quad d_t = N_{a_t a_t}. \qquad \square$$

To bound the rank, pad every principal inverse to the same matrix size. Multiplication by G, on either side, then recovers the indicator of its index set on the diagonal. The resulting commutator identities place all the padded inverses in one linear space of controlled dimension, through which the sign matrix will factor.

Lemma 4. *Let G be a real $m \times m$ matrix and let S range over subsets on which $G[S, S]$ is invertible. Pad each inverse by zeros to obtain an $m \times m$ matrix W_S, with $W_\varnothing = 0$. Then*

$$\dim \operatorname{span}\{W_S\} \le m^2 - m + 1.$$

Proof. Direct multiplication gives

$$\operatorname{diag}(GW_S) = \operatorname{diag}(W_S G) = \mathbf{1}_S, \qquad \operatorname{tr}(GW_S) = |S|.$$

Join i and j when G_{ij} or G_{ji} is nonzero, and let $m_1, \dots, m_c$ be the component sizes of this support graph. All W_S belong to the block-diagonal matrix space of dimension $\sum_j m_j^2$. For every diagonal matrix D, they satisfy

$$\operatorname{tr}([D, G] W_S) = 0.$$

The map $D \mapsto [D, G]$ has kernel the diagonals constant on each support component, and hence rank $m - c$. The trace pairing on the block-diagonal space is nondegenerate, so these give $m - c$ independent constraints. Therefore

$$\dim \operatorname{span}\{W_S\} \le \sum_j m_j^2 - m + c \le m^2 - m + 1,$$

where the last inequality follows from $m^2 - \sum_j m_j^2 = 2 \sum_{i<j} m_i m_j \ge c - 1$. $\square$

Proof of Theorem 2. Let $k > 0$ be the common cardinality of the selected faces. Lemma 3 supplies a matrix with signs H and entries

$$R_{S,t} = d_t - u_t W_S v_t.$$

Because $\operatorname{tr}(GW_S) = k$, its columns are evaluations of the linear functionals

$$W \longmapsto \frac{d_t}{k} \operatorname{tr}(GW) - u_t W v_t.$$

Lemma 4 bounds its rank by $m^2 - m + 1$. Indeed, after choosing a basis of the span of the W_S, the matrix R factors as the matrix of their coordinates times the matrix of these linear functionals. Fixed cardinality is used precisely to express the constant term d_t as a linear function of W_S. Without the common-cardinality assumption, adjoining the constant feature to this matrix space gives $m^2 - m + 2$. □

3. **An explicit Hadamard example**

To encode any sign matrix $H \in \{-1, +1\}^{r \times s}$, choose distinct k-subsets $S_1, \dots, S_r$ of $U = [m]$, with $1 \le k < m$ and $s \ge 2$. On $U \sqcup \{a_1, \dots, a_s\} \sqcup \{g\}$, define Δ_H by the following faces

$$\begin{aligned} &A \subseteq U, && |A| \le k; \\ &\{a_t\} \cup A, && A \subseteq U,\ |A| \le k-1; \\ &\{a_t\} \cup S_i, && H_{it} = +1. \end{aligned} \tag{1}$$

There are no other faces; every proper subset of a face in the third line occurs in the first two, so the prescription defines a complex. The label g is a ghost, and no face contains two anchors.

Forster's spectral bound [2, Theorem 2.2] states that

$$\operatorname{signrank}(H) \ge \frac{\sqrt{rs}}{\|H\|} \qquad \big(H \in \{-1, +1\}^{r \times s}\big),$$

where $\|H\|$ is the Euclidean operator norm. In particular, retaining s columns of a Hadamard matrix of order L gives sign rank at least $\sqrt{s}$, since its columns are orthogonal and have norm $\sqrt{L}$. We therefore need

$$(m^2 - m + 1)^2 < s \le L \le \binom{m}{k}.$$

The first inequality defeats the geometric bound; the last leaves enough core faces to encode all rows. We use $m = 20$, $k = 9$ and an order $L = 384^2$, just above the required threshold 381^2. We construct the required Hadamard matrix from a quadratic-character matrix and a binary character matrix.

Let $I = \{\infty\} \sqcup \mathbb{F}_{11}$, ordered by $\infty, 0, \dots, 10$, and let χ be the quadratic character of $\mathbb{F}_{11}$, with $\chi(0) = 0$. Define

$$K_{\infty,j} = K_{i,\infty} = 1, \qquad K_{ii} = -1 \ (i \in \mathbb{F}_{11}), \qquad K_{ij} = \chi(j - i) \ (i \ne j \text{ in } \mathbb{F}_{11}).$$

This specifies a 12×12 sign matrix. If $Q_{ij} = \chi(j - i)$, including its zero diagonal, then

$$Q\mathbf{1} = 0, \qquad Q^T = -Q, \qquad QQ^T = 11I - J.$$

The last identity is the character sum $\sum_t \chi(t)\chi(t-a) = -1$ for $a \neq 0$. One direct proof reduces to $a = 1$ and counts the solutions of $v^2 = u(u-1)$; under $X = 2u - 1$, $Y = 2v$, this becomes $(X - Y)(X + Y) = 1$, with ten solutions. Consequently

$$K = \begin{pmatrix} 1 & \mathbf{1}^T \\ \mathbf{1} & Q - I \end{pmatrix}, \qquad KK^T = 12I.$$

Let $W_{xy} = (-1)^{x \cdot y}$ for $x, y \in \mathbb{F}_2^{10}$, where the dot product is over $\mathbb{F}_2$. Character orthogonality gives $WW^T = 2^{10}I$. Thus

$$\widehat{H} = K \otimes K \otimes W, \qquad \widehat{H}\widehat{H}^T = LI_L, \qquad L = 12^2\, 2^{10} = 147\,456 = 384^2.$$

Index rows and columns of $\widehat{H}$ lexicographically by $I \times I \times \mathbb{F}_2^{10}$, ordering binary vectors by their integer values, and let H consist of its first $s = 381^2 + 1 = 145\,162$ columns. Take $m = 20$, $k = 9$, and assign its rows to the first L nine-subsets of $[20]$ in lexicographic order. This is possible because $\binom{20}{9} = 167\,960 > L$. Formula (1) now defines the principal example completely.

Principal assertion of Theorem 1. Forster's bound gives $\operatorname{signrank}(H) \geq \sqrt{s} > 381$, whereas polytopality would give $\operatorname{signrank}(H) \leq 381$ by Theorem 2. The input has a face of size ten and none larger, so its dimension is nine. There are $20 + s + 1 = 145\,183$ ground labels. Exactly the unprimed ghost is absent from the Bier sphere, giving $2(20 + s) + 1 = 290\,365$ vertices and dimension $20 + s - 1 = 145\,181$. □

4. **Variants of the construction**

4.1. **Removing the ghost**

Making the ghost a vertex changes the sphere by a single facet subdivision. Since this subdivision can be reversed in any convex realization, it preserves the nonpolytopality established above.

For the principal example, put $\Delta^+ = \Delta \cup \{\{g\}\}$. Then $\operatorname{Bier}(\Delta^+)$ is the stellar subdivision of $\operatorname{Bier}(\Delta)$ at the all-primed facet $T_\varnothing$. That facet is replaced by

$$\{g\} \cup \{i' : i \in E \setminus \{j\}\}, \qquad j \in E,$$

and all other facets remain unchanged.

A stellar subdivision of a facet can be polytopal only if the original sphere was polytopal. Indeed, in a simplicial d-polytope realizing the subdivision, the new vertex v has exactly d neighbors $u_1, \dots, u_d$. Its tangent cone has the independent edge rays $u_i - v$, so every point of the polytope has coordinates

$$v + \sum_i \lambda_i (u_i - v), \qquad \lambda_i \geq 0.$$

Every other vertex except these neighbors satisfies $\sum_i \lambda_i > 1$, since otherwise it lies in the corner simplex. Deleting v therefore reverses stacking across the hyperplane $\sum_i \lambda_i = 1$ and realizes the original sphere, a contradiction. Thus $\mathrm{Bier}(\Delta^+)$ is nonpolytopal, has 290 366 actual vertices, and has dimension 145 181. Both copies of every ground label occur.

4.2. **A self-dual input**

Duality symmetry can be imposed while retaining the original sphere as a vertex link.

Let $\mathcal{K}$ be the principal input on E before adjoining the ghost. Add two labels a, b and set

$$\mathcal{L} = 2^E \cup (\{a\} * \mathcal{K}) \cup (\{b\} * \mathcal{K}^*),$$

where the dual is relative to E and no face contains both a, b. Checking whether a subset contains neither, one, or both new labels gives $\mathcal{L}^* = \mathcal{L}$. Moreover,

$$\mathrm{lk}_a \mathrm{Bier}(\mathcal{L}) \cong \mathrm{Bier}(\Delta).$$

To verify the link identity, its positive part is a face of $\mathcal{K}$ and cannot contain b. Its primed part is either any disjoint subset of E, or $B \cup \{b\}$ with $B \in \mathcal{K}^*$. These are exactly the dual conditions for $\mathcal{K}$ with b adjoined as a ghost.

Vertex links in boundaries of convex polytopes are boundaries of vertex figures. Hence $\mathrm{Bier}(\mathcal{L})$ is nonpolytopal. It has 290 368 actual vertices and dimension 145 182. Self-duality makes the exchange of each label with its primed copy a simplicial involution taking every nonempty face to a disjoint face. Thus this example is centrally symmetric as a combinatorial sphere.

4.3. **Input dimension three**

To lower the dimension of the input, we use fewer core labels and many more anchors. Balanced column signs give the required rank obstruction through continuity of the median.

Take the first 75 triples of [9], and index the anchors by the 37-subsets $A \subseteq [75]$. Define

$$H_{i,A} = +1 \quad \Longleftrightarrow \quad i \in A.$$

Suppose a real matrix with these signs had rank at most 73. Its space of row relations would contain a two-dimensional subspace. On the unit circle in that subspace, the median of the 75 coordinates is continuous and odd, so the intermediate value theorem gives a nonzero vector λ with median zero. This vector has at most 37 positive and at most 37 negative coordinates. Choose a 37-subset containing all its positive coordinates and none of its negative ones, filling with zero coordinates as needed. Every nonzero summand in the corresponding column's dependence is positive, a contradiction. Thus $\operatorname{signrank}(H) \geq 74$. In fact equality holds, since assigning the values 38 and -37 to the positive and negative entries makes every column sum zero.

There are $\binom{9}{3} = 84$ available row sets. Apply (1) with $m = 9$, $k = 3$ and $s = \binom{75}{37}$. Theorem 2 would give $\operatorname{signrank}(H) \leq 9^2 - 9 + 1 = 73$ in a convex realization. The resulting nonpolytopal Bier sphere has $2s + 19$ actual vertices and dimension $s + 8$, with an input of dimension three.

The same argument gives input dimension k when $\binom{m}{k} \geq m^2 - m + 3$.

References

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