# Sublattice resolutions and Betti preservation in all powers

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**Abstract.** For ideals whose monomial initial ideals come from sublattices of products of chains, equality of Betti numbers in the first power forces equality in every power. We prove this through an integral decomposition of the sublattice Koszul complexes; their relative homology is generated in Rees degree one, so lifting first-power classes lifts all their products. The decomposition commutes with arbitrary base change and retains integral torsion. For closed binomial edge ideals it gives Betti preservation in every power over every field, together with a common integral family of minimal resolutions; the same principle applies to unit interval determinantal facet ideals.

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## 1. **Introduction**

We prove that equality of Betti numbers in a first-power Gröbner degeneration forces equality in every power when the initial ideal is defined by a finite sublattice of a product of chains. The propagation has a multiplicative explanation; every relative Koszul homology class of the Rees algebra is a sum of products of classes from its first power. At the level of complexes, the same structure gives an integral decomposition compatible with every coefficient base change.

For closed binomial edge ideals, this proves the conjecture of Ene, Rinaldo, and Terai [2]; Peeva's equality of the first-power Betti tables [3] propagates to every power over every field.

A vertex-labeled graph $G$ is *closed* if, at each vertex, its smaller neighbors form a clique and its larger neighbors form a clique. In $S = \Bbbk[x_1, \dots, x_n, y_1, \dots, y_n]$, put

$$J_G = (x_i y_j - x_j y_i : \{i,j\} \in E(G),\ i < j), \qquad I_G = (x_i y_j : \{i,j\} \in E(G),\ i < j).$$

**Theorem 1.1** (Closed binomial edge ideals)**.** *For every closed graph $G$, every field $\Bbbk$, every $m \geq 1$, and all $p, q$,*

$$\beta^S_{p,q}(J_G^m) = \beta^S_{p,q}(I_G^m).$$

The equality is part of a common integral resolution; for each power, a single complex over $\mathbb{Z}[z]$ specializes to resolutions of the binomial and monomial ideals and is minimal after specialization to either endpoint over any field (Theorem 9.1). A recursion for the coefficient ideals computes these common Betti tables independently of the field. The same recursion covers unit interval determinantal facet ideals; for at least three rows, degree congruences give the first-power comparison directly.

Let $E$ be a finite nonempty sublattice of $[n_1] \times \cdots \times [n_d]$, and set

$$S = \Bbbk[x_{r,j} : 1 \leq r \leq d,\ 1 \leq j \leq n_r], \qquad I_E = (u_a : a \in E), \qquad u_a = \prod_{r=1}^{d} x_{r,a_r}.$$

The closed-graph initial ideals above are of this form after an order-preserving relabeling of each component. For $a \in E$ and a coordinatewise multichain $w = (a_1 \leq \cdots \leq a_m)$, define

$$C_a = (u_b : b <_{\text{lex}} a) : u_a, \qquad C_w = \sum_{s=1}^{m} C_{a_s},$$

where the lexicographic order is increasing. We use the fine grading with one degree for each variable. Write $\mathcal{R}(I) = \bigoplus_{m \geq 0} I^m t^m$ and $\mathcal{H}_S(A) = \operatorname{Tor}^S(A, \Bbbk)$ for the Rees algebra and relative Koszul homology algebra.

**Theorem 1.2** (All powers of a sublattice ideal)**.** *For every finite sublattice $E$, every field, and every $m \geq 1$,*

$$\beta_{p,\alpha}(I_E^m) = \sum_{a_1 \leq \cdots \leq a_m} \beta_{p,\alpha-\sum_s \deg u_{a_s}}(S/C_w). \tag{1.1}$$

*Moreover, $\mathcal{H}_S(\mathcal{R}(I_E))$ is generated in Rees degree one.*

The predecessor quotients $S/C_w$ occur in a monomial degeneration of the Rees algebra. Their minimal generators separate into disjoint variable blocks, each already present at one member of $w$. Tensor products of the block Koszul complexes therefore express every homology class as a sum of products of first-power classes. The first-power filtration is minimal, so these classes lift through the degeneration; multiplication lifts their products.

Consequently, if a homogeneous ideal $J$ satisfies $\operatorname{in} J = I_E$ and $\beta_{p,q}(J) = \beta_{p,q}(I_E)$ for all $p, q$, then Theorem 4.3 gives, for every $m \geq 1$,

$$J^{(m)} = J^m, \qquad \operatorname{in}(J^m) = I_E^m, \qquad \beta_{p,q}(J^m) = \beta_{p,q}(I_E^m).$$

The symbolic and initial identities follow from normal torsion-freeness of $I_E$, which we prove by realizing its generators as ideal cuts in a finite network. The homological equality then follows from multiplicative lifting. Applied to closed graphs, this propagates Peeva's comparison and proves Theorem 1.1. The construction also accommodates mixed products of compatible sublattice ideals.

Theorem 7.1 identifies the whole Koszul complex over the integers, up to chain homotopy, with a direct sum of the coefficient complexes in (1.1). Consequently the comparison preserves torsion as well as Betti numbers after passage to a field. The same description locates the possible characteristic dependence; Section 10 realizes finite-complex homology inside a single coefficient quotient, where it persists through all powers. We develop the chain normal forms and multiplicative lifting first, then use their integral form to distinguish the torsion-free families from general sublattices.

## 2. **Chain normal forms**

Sorting a product of generators produces a multichain, but it does not yet determine which polynomial coefficients remain independent. The predecessor ideals below record exactly those coefficient relations. Once the chain is fixed, we will separate these relations into disjoint sets of variables.

The meet and join in $E$ are coordinatewise minimum and maximum. For $a, b \in E$, put

$$q_{b,a} = \frac{u_b}{\gcd(u_b, u_a)}.$$

**Lemma 2.1** (Predecessor ratios)**.** *The ideal $C_a$ is generated by $q_{b,a}$ for $b < a$ in the coordinatewise partial order. Every variable in these generators has index strictly less than $a_r$ in its own row $r$.*

*Proof.* For a lexicographic predecessor $b$, put $c = b \wedge a$. Then $c \in E$, $c < a$ coordinatewise, and $q_{c,a} \mid q_{b,a}$. The ratio $q_{c,a}$ uses exactly the rows where $b_r < a_r$; the original ratio may additionally use rows where $b_r > a_r$. This proves both claims. □

Put $P = S[T_a : a \in E]$, with $\deg T_a = (d, 1)$ and $\deg x_{r,j} = (1, 0)$, and define

$$Q = (T_a T_b : a, b \text{ incomparable}) + \sum_{a \in E} C_a T_a, \qquad B = P/Q.$$

For a multichain $w = (a_1 \le \cdots \le a_m)$ write $T^w = \prod_s T_{a_s}$ and $C_w = \sum_s C_{a_s}$. The presentation gives

$$B_{*,m} = \bigoplus_{a_1 \le \cdots \le a_m} (S/C_w) T^w. \tag{2.1}$$

Each summand has ordinary internal shift $dm$ and fine shift $\sum_s \deg u_{a_s}$. Multiplication is the natural quotient multiplication if the combined support is a chain, and zero otherwise.

**Proposition 2.2** (A universal Rees presentation)**.** *The Rees algebra $\mathcal{R}(I_E)$ has a homogeneous Gröbner degeneration to $B$. A Gröbner basis of its presentation kernel consists of*

$$\begin{aligned} T_a T_b - T_{a \wedge b} T_{a \vee b} \quad & (a, b \textit{ incomparable}), \\ q_{b,a} T_a - q_{a,b} T_b \quad & (b < a \textit{ coordinatewise}). \end{aligned}$$

*The leading monomials are the first displayed monomials.*

*Proof.* Place the $T$-variables first, with $T_a > T_b$ when $a <_{\text{lex}} b$, followed by the row variables in row order and with smaller column index first. Use degree reverse lexicographic order. For a sorting relation, $T_{a \vee b}$ is the last variable at which the two terms differ, so the unsorted term leads. For a predecessor relation, the two coefficient monomials have equal degree. In the last differing row, the second term contains the later variable $x_{r,a_r}$; hence the first term leads. Lemma 2.1 shows that these leading monomials generate exactly $Q$.

Let $\varphi : P \to S[t]$ fix $S$ and send $T_a$ to $u_a t$. All displayed relations belong to its kernel. We show that distinct $Q$-standard monomials have distinct images. Write one as $vT_{a_1} \cdots T_{a_m}$, where $a_1 \leq \cdots \leq a_m$. Its image is $Mt^m$, with $M = v \prod_s u_{a_s}$. The element $a_1$ is the earliest facet $b$ in lexicographic order such that $u_b \mid M$. Indeed, if an earlier such $b$ existed, set $c = b \wedge a_1$. A variable of $q_{c,a_1}$ has row index smaller than the corresponding coordinate of every $a_s$, so it must divide $v$. Thus $q_{c,a_1} \mid v$, contrary to standardness because $q_{c,a_1} \in C_{a_1}$.

The pair $(M, m)$ therefore determines $a_1$. Remove this factor and repeat; the remaining standard monomial is a divisor of the original one. Induction recovers every $a_s$ and finally $v$. This proves the claimed injectivity on standard monomials. Division of an element of $\ker \varphi$ by the displayed relations has a standard remainder. Its image is zero, so the injectivity forces the remainder to vanish. The displayed relations are consequently a Gröbner basis, as claimed. □

**Lemma 2.3** (Minimal first-power filtration)**.** *For every $p$ and every fine degree $\alpha$,*

$$\beta_{p,\alpha}(I_E) = \sum_{a \in E} \beta_{p,\alpha - \deg u_a}(S/C_a) = \beta_{p,\alpha}(B_{*,1}).$$

*Proof.* Adjoin the generators in lexicographic order. The corresponding exact sequence of ideals is

$$0 \longrightarrow I_{<a} \longrightarrow I_{\leq a} \longrightarrow (S/C_a)(-\deg u_a) \longrightarrow 0.$$

By Lemma 2.1, every fine Tor degree of the final module has rowwise maximum-index vector exactly $a$; the shift supplies $x_{r,a_r}$ in every row, and the colon uses only earlier variables. This follows, for example, by using the Taylor resolution of the colon; unused polynomial variables add no Tor degrees. Inductively, the fine Tor degrees already present have the maximum-index vector of an earlier facet. Therefore no connecting map in the long Tor sequence can be nonzero in any fine degree. The Betti tables add at each step. The last equality is the degree-one instance of (2.1). □

### 2.1. **Crossing blocks**

The following elementary observation is the structural point of the argument. It applies to every finite sublattice, without any assumption on linear quotients or on the degrees of the colon generators. As a chain increases, a row variable becomes available for a predecessor ratio when the chain first passes its column index. If a minimal relation used variables first passed at different steps, taking a meet at the earliest step would give a proper relation dividing it. Thus a minimal relation can involve only one first-crossing block.

**Lemma 2.4** (Crossing blocks)**.** *Fix a multichain $w = (a_1 \leq \cdots \leq a_m)$. For each variable define*

$$p(r, j) = \min\{s : j < (a_s)_r\},$$

*with value $\infty$ if the set is empty. Let $X_p$ be the variables with first crossing $p$. There are monomial ideals $R_p \subset \Bbbk[X_p]$ such that*

$$C_w = R_1 + \cdots + R_m, \qquad R_p \subseteq C_{a_p}.$$

*In particular, the $R_p$ use pairwise disjoint sets of variables.*

*Proof.* Let $g$ be a minimal monomial generator of $C_w$. By Lemma 2.1, $g = q_{b,a_s}$ for some $s$ and some $b < a_s$ coordinatewise. Every variable of $g$ crosses by time $s$. Let $p$ be the least crossing time among these variables and set $c = b \wedge a_p \in E$. We claim that

$$h = q_{c,a_p}$$

is precisely the product of the variables of $g$ with crossing time $p$. If a row contributes no variable to $g$, then $b_r = (a_s)_r \geq (a_p)_r$ and it contributes none to $h$. Otherwise it contributes $x_{r,b_r}$ to $h$ exactly when $b_r < (a_p)_r$. This says that its first crossing is at most $p$, hence exactly $p$ by the choice of $p$.

Thus $h$ is a nonempty divisor of $g$, and $h \in C_{a_p} \subseteq C_w$. Minimality of $g$ forces $h = g$. Every minimal generator consequently lies in one block and belongs to the corresponding $C_{a_p}$. Let $R_p$ be generated by the minimal generators assigned to block $p$. These ideals have all the stated properties. Variables in $X_\infty$ do not occur in $C_w$. □

**Proposition 2.5** (Generation of relative Koszul homology)**.** *The algebra $\mathcal{H}_S(B)$ is generated by its Rees-degree-one part and its unit, over every field.*

*Proof.* In the summand belonging to $w$, Lemma 2.4 gives

$$S/C_w \cong \Bbbk[X_\infty] \otimes_\Bbbk \bigotimes_{p=1}^{m} \Bbbk[X_p]/R_p.$$

The relative Koszul complex is the tensor product of the block Koszul complexes and the complex on the unused polynomial variables. Over a field, its homology is the tensor product of the block homologies; the last polynomial factor contributes only $\Bbbk$ in degree zero.

Choose cycle representatives $c_p$ of block homology classes, with coefficients in $\Bbbk[X_p]$. The same polynomial and exterior expression is a cycle over $S/C_{a_p}$; its differential

has coefficients in $R_p \subseteq C_{a_p}$. Hence $T_{a_p} c_p$ is a cycle in the Rees-degree-one part of $\mathcal{K}_S(B)$. Their product is, up to the usual exterior sign, exactly

$$T^w(c_1 \wedge \cdots \wedge c_m),$$

which represents the chosen tensor class in the $w$-summand. A factor with trivial block homology contributes $T_{a_p}$ in homological degree zero. Tensor classes span the homology, so this proves generation on each summand of (2.1), while at Rees degree zero the homology consists only of the unit. □

## 3. **A multiplicative degeneration principle**

The block decomposition expresses special-fiber homology classes as sums of products of classes from the first power. A product lifts to a cycle as soon as its factors do. The following proposition turns this observation into a criterion for preserving every Betti table.

Let $R = \Bbbk[z_1, \ldots, z_N]$. For an $R$-algebra $A$, write

$$\mathcal{K}_R(A) = A \otimes_{\Bbbk} \bigwedge_{\Bbbk}(e_1, \ldots, e_N), \qquad \partial(e_i) = z_i,$$

and write $\mathcal{H}_R(A) = H(\mathcal{K}_R(A)) = \operatorname{Tor}^R(A, \Bbbk)$ for its homology algebra. Its product is induced by multiplication of coefficients and exterior multiplication.

Let $A$ be a bigraded $R$-algebra with a nonnegative auxiliary grading $m$, with $A_{*,0} = R$ and finite-dimensional components in each fixed internal and auxiliary degree. Give $A$ a homogeneous weight filtration, finite, exhaustive, and separated on each such component, restricting to a variable-weight filtration on $R$. Let $B$ be its associated graded algebra, viewed over $\operatorname{gr} R = R$.

**Proposition 3.1** (Propagation from Rees degree one)**.** *Suppose that*

*(i) the algebra $\mathcal{H}_R(B)$ is generated by its auxiliary-degree-one part, together with the unit;*

*(ii) the graded Betti numbers of the auxiliary-degree-one $R$-modules $A_{*,1}$ and $B_{*,1}$ are equal.*

*Then, for every $m, p, q$,*

$$\beta^R_{p,q}(A_{*,m}) = \beta^R_{p,q}(B_{*,m}).$$

*Moreover, $\mathcal{H}_R(A)$ is itself generated in auxiliary degree one.*

*Proof.* Let $\mathcal{O} = \Bbbk[t]_{(t)}$ and $K = \Bbbk(t)$. Localize the weight-homogenized family over $\mathcal{O}$, and form its relative Koszul differential graded algebra $C$, assigning each exterior generator the weight of its corresponding variable. Its special fiber is $\mathcal{K}_R(B)$, and its generic fiber is $\mathcal{K}_R(A) \otimes_{\Bbbk} K$ after rescaling variables. In every fixed internal and auxiliary degree, $C$ is a finite complex of finite free $\mathcal{O}$-modules.

In one such degree write

$$H_p(C) \cong \mathcal{O}^{r_p} \oplus \bigoplus_{j=1}^{t_p} \mathcal{O}/(t^{a_{p,j}}).$$

The exact sequence of multiplication by $t$ gives

$$0 \longrightarrow H_p(C)/tH_p(C) \longrightarrow H_p(C/tC) \longrightarrow H_{p-1}(C)[t] \longrightarrow 0. \tag{3.1}$$

Thus the special and generic homology dimensions differ by $t_p + t_{p-1}$. Assumption (ii) forces every $t_p$ in auxiliary degree one to vanish. Every special class in that degree consequently lifts to a cycle in $C$.

By (i), every special homology class is a sum of products of these classes. Multiplying their cycle lifts shows that $H_p(C) \to H_p(C/tC)$ is surjective in all degrees. Equation (3.1) now gives $H_{p-1}(C)[t] = 0$ everywhere, including the top Koszul degree by taking the next, zero, homological degree. All the modules $H_p(C)$ are therefore free over $\mathcal{O}$, and their special and generic dimensions agree.

Let $L$ be the subalgebra of $H(C)$ generated by its auxiliary-degree-one part. The quotient $H(C)/L$ vanishes modulo $t$ in every finite homogeneous component. Nakayama's lemma gives $L = H(C)$. Passing to $K$ and descending along the faithful field extension $\Bbbk \subset K$ proves both assertions. □

**Remark 3.2.** A term-order degeneration appearing below can be replaced by one integer weight vector realizing the finitely many leading terms of a finite homogeneous Gröbner basis. A positive internal-degree weight can be added if necessary. Thus the finite-filtration hypothesis in this proposition does not require a limiting argument over infinitely many term-order comparisons.

*Proof of Theorem 1.2.* Apply Proposition 3.1 to the degeneration in Proposition 2.2, using Lemma 2.3 and Proposition 2.5. All constructions preserve the fine grading, so the conclusion holds in that grading. Formula (2.1) then gives (1.1). □

### 3.1. **Rees degenerations of homogeneous ideals**

**Lemma 3.3** (A finite weight for an all-powers initial identity)**.** *Let $J = (f_a : a \in E)$ be an ideal generated by homogeneous forms of degree $d$, with $\operatorname{in}_{\prec} f_a = u_a$. If*

$\mathrm{in}_\prec(J^m) = I_E^m$ *for every* $m$*, there is a homogeneous weight degeneration* $\mathcal{R}(J) \rightsquigarrow \mathcal{R}(I_E)$ *preserving internal and Rees degree.*

*Proof.* The all-powers identity says that the finite set consisting of the variables of $S$ and the elements $f_a t$ is a SAGBI basis; every leading monomial in $\mathcal{R}(J)$ is a product of their leading monomials. A weight realizing finitely many subduction relations will give the required degeneration.

Let $K_I$ be the toric presentation kernel of $\mathcal{R}(I_E)$ in $P$ and choose finite binomial generators $U - V$ for it. The images of $U$ and $V$ in $\mathcal{R}(J)$ have the same leading monomial, so it cancels in their difference. Subduct that difference using the finite SAGBI basis. The process terminates because the leading monomial strictly decreases in a term order at each step. It produces a relation $F = U - V - \sum c_H H$ in the presentation kernel $K_J$, where the monomial image of each $H$ under the monomial presentation is strictly smaller than the common image of $U, V$. All terms have the same internal and Rees degree.

There are finitely many such relations and comparisons, together with the finitely many comparisons selecting $u_a$ in $f_a$. One positive integer weight on the variables of $S$ realizes them all; this is the usual finite weight realization of a monomial order. Give $T_a$ the weight of $u_a$. The weight-initial form of $F$ is then $U - V$, so $K_I \subseteq \mathrm{in}\, K_J$. The two quotient rings have the same Hilbert function in every internal and Rees degree, by the assumed initial identities. A weight degeneration preserves these Hilbert functions. The inclusion is consequently equality. Weight homogenization gives the asserted degeneration, with finite filtrations in each bidegree. □

## 4. **Symbolic powers and first-power transfer**

The finite Rees degeneration of Lemma 3.3 requires initial identities in every power. For sublattice ideals these follow from the first power; ordinary powers are already detected by the minimal vertex covers. For a radical ideal $H$, write $H^{(m)} = \bigcap_{P \in \mathrm{Min}(H)} (P^m S_P \cap S)$. A vertex cover of a squarefree monomial ideal is a set of variables meeting the support of every minimal generator; its inclusion-minimal covers index the minimal primes.

**Theorem 4.1** (Sublattice ideals are normally torsion-free)**.** *For every finite nonempty sublattice* $E$ *and every* $m \geq 1$*,*

$$I_E^m = I_E^{(m)} = \bigcap_{C \text{ a minimal cover of } I_E} P_C^m, \qquad P_C = (x : x \in C).$$

*Proof.* We seek a factorization of a given monomial into as many generators as its vertex-cover bounds permit. In the network below, generators become order-ideal cuts and covers become paths. Integer shortest-path distances then supply the required cuts, with each variable used no more often than its prescribed exponent.

*Coordinate chains.* Let $P$ be the poset of join-irreducibles of the finite distributive lattice $E$. The map $a \mapsto \{p \in P : p \le a\}$ identifies $E$ with the lattice $\mathcal{J}(P)$ of order ideals; join-irreducibles are join-prime by distributivity, and every element is their join. Compress the distinct attained values of coordinate $r$ to $0, \dots, \ell_r$. Each nonconstant threshold $\chi_j(D) = \mathbf{1}_{\{a_r(D) \ge j\}}$ preserves meets and joins. Its 1-fiber is a prime filter with a least element $D_j$. Primality makes $D_j$ a join-irreducible order ideal, hence $D_j = {\downarrow}p_{r,j}$ for a unique $p_{r,j} \in P$. Consequently $\chi_j(D) = 1$ exactly when $p_{r,j} \in D$. The nested, distinct thresholds give a chain

$$p_{r,1} < \cdots < p_{r,\ell_r}, \qquad a_r(D) = |D \cap \{p_{r,1}, \dots, p_{r,\ell_r}\}|.$$

Every $p \in P$ occurs in some coordinate chain; the distinct ideals ${\downarrow}p$ and $({\downarrow}p) \setminus \{p\}$ must be distinguished by a coordinate. Unattained row variables are unused polynomial variables and may be suppressed.

*The network and its covers.* Adjoin a bottom vertex $s$ and a top vertex $t$ to $P$. For each row, use the directed path

$$s = p_{r,0} \longrightarrow p_{r,1} \longrightarrow \cdots \longrightarrow p_{r,\ell_r} \longrightarrow p_{r,\ell_r+1} = t,$$

labeling its $j$-th arc $p_{r,j} \to p_{r,j+1}$ by the row variable $x_{r,j}$, for $0 \le j \le \ell_r$. Parallel arcs retain distinct labels. Also include an unlabeled reverse arc $v \to u$ for every strict order relation $u < v$ in the extended poset. The outgoing labeled arcs of the cut $\{s\} \cup D$, for $D \in \mathcal{J}(P)$, are exactly the variables of $u_D$; each row path crosses once, and no reverse arc leaves the cut.

A set $C$ of labeled arcs is a cover precisely when $C$, together with all reverse arcs, contains an $s$–$t$ path. Such a path must cross each ideal cut. Conversely, if no such path exists, the vertices reachable from $s$ form an order ideal containing $s$ and excluding $t$, whose outgoing labeled arcs avoid $C$.

*Integer cut packing.* Give each labeled arc $e$ a nonnegative integer cost $b_e$ and each reverse arc cost zero. The shortest $s$–$t$ distance $h$ equals

$$h = \min_{C \text{ a minimal cover}} \sum_{e \in C} b_e.$$

Indeed every cover contains a path, and every path's labeled arcs contain a minimal cover; paths can be chosen simple because costs are nonnegative. Write $d(v)$ for the shortest distance from $s$ to $v$. Every vertex is reachable. Reverse arcs imply

$0 = d(s) \le d(v) \le d(t) = h$ and $d(u) \le d(v)$ whenever $u < v$. For $j = 1, \dots, h$, therefore,

$$D_j = \{p \in P : d(p) < j\}$$

is an order ideal. A labeled arc $e : u \to v$ occurs in exactly $d(v) - d(u)$ of these cuts, at most $b_e$ by the shortest-path inequality. Hence $\prod_{j=1}^{h} u_{D_j}$ divides $\prod_e x_e^{b_e}$. If every minimal cover has exponent sum at least $m$, then $h \ge m$, and any $m$ of these cuts give a product of $m$ generators dividing the monomial. Conversely, every such product uses at least one variable from each cover in each factor. Thus a monomial belongs to $I_E^m$ exactly when every minimal cover has exponent sum at least $m$. This is precisely membership in $\bigcap_C P_C^m$, which is the symbolic power since $I_E$ is squarefree. □

**Lemma 4.2** (Initial ideals of symbolic powers over any field)**.** *If $J \subseteq \Bbbk[\mathbf{x}]$ has squarefree initial ideal $I$ for a term order $\prec$, then $J$ is radical and, over every field,*

$$\mathrm{in}_\prec(J^{(m)}) \subseteq I^{(m)} \qquad (m \ge 1).$$

*Proof.* If a nonzero standard remainder $g$ had $g^N \in J$, then $\mathrm{in}(g)^N \in I$ would imply $\mathrm{in}(g) \in I$, a contradiction. Thus $J$ is radical. Let $D^{\boldsymbol{\nu}}$ be the Hasse derivative, defined on monomials by

$$D^{\boldsymbol{\nu}}\mathbf{x}^{\boldsymbol{\gamma}} = \prod_i \binom{\gamma_i}{\nu_i} \mathbf{x}^{\boldsymbol{\gamma}-\boldsymbol{\nu}}.$$

Its product rule extends to every localization $S_P$ by expanding $f(\mathbf{x}+\mathbf{T})/g(\mathbf{x}+\mathbf{T})$, since the constant term of the denominator is a unit there. That rule gives $D^{\boldsymbol{\nu}}(P^m S_P) \subseteq P^{m-|\boldsymbol{\nu}|} S_P$ when $|\boldsymbol{\nu}| < m$; in each product of $m$ factors from $PS_P$, at most $|\boldsymbol{\nu}|$ factors receive a positive derivative order.

Suppose $f \in J^{(m)}$ has leading monomial $\mathbf{x}^{\boldsymbol{\gamma}} \notin I^{(m)}$. There is a minimal cover $C$ of $I$ with $r = \sum_{x_i \in C} \gamma_i < m$. Take $\nu_i = \gamma_i$ on $C$ and $\nu_i = 0$ elsewhere. The preceding localization argument gives $D^{\boldsymbol{\nu}} f \in \bigcap_{P \in \mathrm{Min}(J)} P = J$. Its leading monomial is $\mathbf{x}^{\boldsymbol{\gamma}-\boldsymbol{\nu}}$; the leading coefficient is unchanged, since every relevant binomial coefficient is 1, and dividing surviving monomials by the same monomial preserves their order. This leading monomial avoids $C$, contradicting $\mathrm{in}(J) = I \subseteq P_C$. No division by a factorial or perfect-field assumption is involved. □

**Theorem 4.3** (A first-power criterion)**.** *Let $E$ be a finite nonempty sublattice and $J$ a homogeneous ideal with $\mathrm{in}_\prec J = I_E$. Then, for all $m \ge 1$,*

$$J^{(m)} = J^m, \qquad \mathrm{in}_\prec(J^{(m)}) = \mathrm{in}_\prec(J^m) = I_E^m.$$

*If also* $\beta_{p,q}(J) = \beta_{p,q}(I_E)$ *for all* $p, q$*, then*

$$\beta_{p,q}(J^{(m)}) = \beta_{p,q}(J^m) = \beta_{p,q}(I_E^m) \quad \textit{for every } m, p, q,$$

*and* $\mathcal{H}_S(\mathcal{R}(J))$ *is generated in Rees degree one. The same Betti equality holds in every finer grading respected by the degeneration and refining ordinary internal degree.*

*Proof.* Theorem 4.1 and Lemma 4.2 give the inclusions

$$I_E^m \subseteq \operatorname{in}(J^m) \subseteq \operatorname{in}(J^{(m)}) \subseteq I_E^{(m)} = I_E^m.$$

Thus all are equal. The contained ideals $J^m \subseteq J^{(m)}$ with the same initial ideal are equal, by division. The degree-one identity supplies homogeneous generators of $J$ with leading monomials $u_a$, and Lemma 3.3 gives the Rees degeneration $\mathcal{R}(J) \rightsquigarrow \mathcal{R}(I_E)$. Theorem 1.2 gives generation of the special homology in Rees degree one. Under the first-power Betti hypothesis, Proposition 3.1 therefore gives equality in every power and generation of $\mathcal{H}_S(\mathcal{R}(J))$. For a finer grading, generic Betti numbers are at most special Betti numbers in each degree. Ordinary equality makes the sum of these nonnegative differences zero, so each difference is zero. □

## 5. **Betti formulas and mixed powers**

The coefficient ideal of a multichain depends only on which lattice elements occur, so varying their multiplicities introduces no new homological data. Grouping by these support chains turns the formula of Theorem 1.2 into a finite expression for all powers. Write $\mathsf{B}_M(s, v) = \sum_{p,q} \beta_{p,q}(M) s^p v^q$ and, for a nonempty chain $\sigma$ of distinct elements of $E$, put $C_\sigma = \sum_{a \in \sigma} C_a$.

**Corollary 5.1** (A finite formula for every power)**.** *For every finite sublattice* $E$ *and every* $m \geq 1$*,*

$$\mathsf{B}_{I_E^m}(s, v) = v^{dm} \sum_{\substack{\varnothing \neq \sigma \subseteq E \\ \sigma \text{ a chain}}} \binom{m-1}{|\sigma| - 1} \mathsf{B}_{S/C_\sigma}(s, v). \tag{5.1}$$

*The same formula holds for* $J^m$ *under the Betti comparison hypothesis of Theorem 4.3. The binomial coefficient is zero when* $|\sigma| > m$*.*

*Proof.* Group the multichains in (1.1) by their support. A fixed support chain of size $r$ occurs $\binom{m-1}{r-1}$ times, one for each assignment of positive multiplicities with sum $m$, and every occurrence has the same coefficient ideal. □

In particular,

$$\sum_{m\geq 1} z^m v^{-dm} \mathsf{B}_{I_E^m}(s,v) = \sum_{\substack{\varnothing\neq\sigma\subseteq E\\ \sigma \text{ a chain}}} \left(\frac{z}{1-z}\right)^{|\sigma|} \mathsf{B}_{S/C_\sigma}(s,v).$$

Thus finitely many coefficient quotients determine the Betti tables of all powers.

**Corollary 5.2** (Polynomial behavior from the first power)**.** *For each $p, q$, the normalized entry $\beta_{p,dm+q}(I_E^m)$ is, for all positive integers $m$, a polynomial*

$$\sum_{r\geq 1} c_{p,q,r}\binom{m-1}{r-1}, \qquad c_{p,q,r} = \sum_{|\sigma|=r} \beta_{p,q}(S/C_\sigma) \geq 0.$$

*All its forward differences are nonnegative on the positive integers. Consequently the projective dimensions of the powers are nondecreasing and the depths of their quotient rings are nonincreasing. These conclusions also hold for every $J$ satisfying the Betti hypothesis in Theorem 4.3.*

*Proof.* Use (5.1) and $\binom{m}{r-1} - \binom{m-1}{r-1} = \binom{m-1}{r-2}$, with the usual zero conventions. A nonzero normalized Betti entry therefore remains nonzero at the next power. This gives projective-dimension monotonicity; the depth statement follows from $\operatorname{pd}(S/I_E^m) = \operatorname{pd}(I_E^m) + 1$ and the Auslander–Buchsbaum formula. □

**Corollary 5.3** (First occurrence of a strand)**.** *If $p \geq 1$ and $\beta_{p,dm+q}(I_E^m)$ is nonzero for some $m$, it is already nonzero for some $m \leq p$. If $\mu$ is the minimum element of $E$, multiplication by $u_\mu$ induces an injection*

$$\operatorname{Tor}_p^S(\Bbbk, I_E^m) \longrightarrow \operatorname{Tor}_p^S(\Bbbk, I_E^{m+1})$$

*with the appropriate degree shift. Under the Betti hypothesis of Theorem 4.3, multiplication by the corresponding generator $f_\mu$ has the same property for $J$.*

*Proof.* Express a nonzero class as a sum of products of first-Rees-degree classes and select a nonzero product term. At most $p$ factors have positive homological degree. The product of those factors is nonzero, since otherwise the full product would be zero. Removing the homological-degree-zero factors removes exactly $d$ from the internal degree per factor, preserving the normalized strand $q$. This proves the first assertion.

For the second assertion, $C_\mu = 0$ and $\mu$ is comparable with every element of $E$. Multiplication by $T_\mu$ on $B$ sends the summand $(S/C_w)T^w$ isomorphically onto the summand indexed by $w+\mu$; different multichains have different images. It is therefore injective on relative Koszul homology. The proof of Proposition 3.1 makes the family homology free over the local parameter ring and identifies its reduction with the

special homology. A map between finite free modules that is injective modulo the parameter is injective over its fraction field. Apply this first to multiplication by a lift of $T_\mu$, and then to multiplication by $f_\mu$ in the degeneration from $J$. □

**Remark 5.4** (Several auxiliary degrees)**.** Proposition 3.1 extends verbatim to an auxiliary grading by $\mathbb{N}^s$. More generally, let $\Gamma$ be any set of auxiliary degrees which generates the special homology algebra. Equality of Betti tables in the components indexed by $\Gamma$ makes their special classes lift to cycles. Products then lift every special class, so (3.1) gives equality in all components. We shall apply this form to multi-Rees algebras whose generating degrees are the coordinate units.

### 5.1. **Mixed powers**

Adding a color coordinate to the lattice lets the same decomposition distinguish factors from different ideals. We realize a mixed product as the component of an ordinary power with the smallest possible total color degree; that degree leaves no room for exterior color variables in the Koszul complex. The first-power classes on this diagonal are therefore exactly those of the individual ideals.

Let $E_1, \dots, E_r \subseteq [n_1] \times \cdots \times [n_d]$ be nonempty sets such that

$$\widetilde{E} = \{(c, a) : 1 \le c \le r,\ a \in E_c\} \subseteq [r] \times [n_1] \times \cdots \times [n_d]$$

is a sublattice, and put $I_c = (u_a : a \in E_c) \subset S$.

**Theorem 5.5** (Mixed powers from a colored sublattice)**.** *For this colored sublattice, the algebra*

$$\operatorname{Tor}^S(\Bbbk, S[I_1 t_1, \dots, I_r t_r])$$

*is generated in the auxiliary degrees* $\mathbf{e}_1, \dots, \mathbf{e}_r$*. Suppose homogeneous ideals* $J_c \subset S$ *satisfy* $\beta(J_c) = \beta(I_c)$ *for every* $c$*, and one monomial order gives*

$$\operatorname{in}\left(\prod_{c=1}^{r} J_c^{m_c}\right) = \prod_{c=1}^{r} I_c^{m_c} \quad \textit{for every } \mathbf{m} \in \mathbb{N}^r.$$

*Then*

$$\beta_{p,q}\left(\prod_{c} J_c^{m_c}\right) = \beta_{p,q}\left(\prod_{c} I_c^{m_c}\right) \quad \textit{for every } \mathbf{m}, p, q.$$

*Proof.* Introduce new variables $z_1, \dots, z_r$, put $R = S[z_1, \dots, z_r]$, and let $K = \sum_c z_c I_c$. This is exactly the sublattice ideal $I_{\widetilde{E}}$, with the $z$-variables forming an additional row. Theorem 1.2 says that $H(\mathcal{K}_R(\mathcal{R}(K)))$ is generated in Rees degree one.

Give $z_c$ and its Koszul exterior generator the fine $z$-degree $\mathbf{e}_c$. In Rees degree $m$, every coefficient in $K^m$ has total $z$-degree at least $m$. Fix $\mathbf{a} \in \mathbb{N}^r$ with $|\mathbf{a}| = m$. A Koszul term of fine $z$-degree $\mathbf{a}$ containing any exterior $z$-generator would require a coefficient of total $z$-degree less than $m$, so it is zero. The whole subcomplex in these degrees is therefore literally

$$\mathcal{K}_S\left(\prod_c I_c^{a_c}\right) z^{\mathbf{a}} t^m.$$

These subcomplexes, over all $\mathbf{a}$, form a differential graded subalgebra naturally isomorphic to the Koszul complex of the multi-Rees algebra. Fine $z$-degree is preserved by the differential, so taking this diagonal commutes with homology.

Choose homogeneous Rees-degree-one generators of the full homology algebra. Each has total $z$-degree at least one. A product of $m$ such generators contributing to the diagonal of total $z$-degree $m$ must use only generators of total $z$-degree one. Their fine $z$-degrees are the coordinate units $\mathbf{e}_c$, and the identified complexes show that these are precisely the components $\operatorname{Tor}^S(\Bbbk, I_c)t_c$. The Rees-degree-zero homology is just $\Bbbk$. This proves the asserted generation for the multi-Rees algebra.

The mixed initial identities provide a finite SAGBI degeneration

$$S[J_1 t_1, \dots, J_r t_r] \rightsquigarrow S[I_1 t_1, \dots, I_r t_r].$$

The finite weight construction of Lemma 3.3 applies unchanged, retaining the multi-colored Rees degree. The multigraded form of Proposition 3.1 now gives Betti equality and homology generation in every multidegree. □

For example, take any finite sublattice $E$ and any chain $b_1 \le \cdots \le b_r$ in it, and set $E_c = \{a \in E : a \le b_c\}$. These sets satisfy the colored-sublattice hypothesis. Thus the theorem applies to all mixed products of principal lattice prefixes, without assuming that their individual resolutions are linear.

The same diagonal retains the coefficient decomposition, giving an explicit formula for mixed Betti numbers. Set $C_a^{(c)} = (u_b : b \in E_c,\ b <_{\mathrm{lex}} a) : u_a$. For a multichain $w = ((c_1, a_1) \le \cdots \le (c_m, a_m))$ in $\widetilde{E}$, let $\mathrm{col}(w)$ count its colors and put $C_w^S = \sum_s C_{a_s}^{(c_s)}$. Then

$$\beta_{p,\alpha}\left(\prod_c I_c^{m_c}\right) = \sum_{\substack{w \text{ a multichain in } \widetilde{E} \\ \mathrm{col}(w)=\mathbf{m}}} \beta_{p,\alpha-\sum_s \deg u_{a_s}}(S/C_w^S). \tag{5.2}$$

For $\mathbf{m} = 0$, use the unique empty chain and $C_\varnothing^S = 0$; the formula then states the identity for $S$. For positive total degree, apply the fine formula for $K^m$ and take the same minimal total $z$-degree diagonal. Only chains with the indicated colors can

contribute; the fine $z$-degree-zero Koszul homology of a coefficient quotient is the homology of its restriction to $S$. Cross-color predecessor ratios contain a $z$-variable, so that restriction is exactly $C_w^S$. For $m_c \geq 1$ for every $c$, grouping by strict colored chains gives a polynomial in $\mathbf{m}$ for each normalized ordinary Betti entry, with nonnegative coefficients in the basis $\prod_c \binom{m_c-1}{s_c-1}$, where $s_c$ counts color $c$ in the strict chain. The same ordinary graded specialization holds for the products of the $J_c$ in Theorem 5.5.

The hypothesis on all mixed initial ideals can itself be checked by a single ordinary initial identity. The colored sublattice is what allows the first-power criterion to enforce every product identity.

**Corollary 5.6** (A finite condition for mixed transfer)**.** *Under the colored-sublattice hypothesis of Theorem 5.5, let $J_c \subset S$ be homogeneous ideals and put $L = \sum_c z_c J_c \subset S[\mathbf{z}]$. Suppose one term order $\prec'$ on $S[\mathbf{z}]$ satisfies*

$$\operatorname{in}_{\prec'} L = K := \sum_c z_c I_c = I_{\widetilde{E}}.$$

*Then its restriction $\prec$ to $S$ satisfies*

$$\operatorname{in}_{\prec}\left(\prod_c J_c^{m_c}\right) = \prod_c I_c^{m_c} \qquad (\mathbf{m} \in \mathbb{N}^r).$$

*If $\beta(J_c) = \beta(I_c)$ for every $c$, all the mixed products have equal graded Betti tables, and the multi-Rees Koszul homology for the $J_c$ is generated in the coordinate unit degrees.*

*Proof.* Theorem 4.3, applied to $L$ without a Betti hypothesis, gives $\operatorname{in}_{\prec'}(L^M) = K^M$. Both ideals are fine $z$-graded, and taking initial ideals commutes with their fine $z$-components. For $|\mathbf{m}| = M$, their components are respectively $z^{\mathbf{m}} \prod_c J_c^{m_c}$ and $z^{\mathbf{m}} \prod_c I_c^{m_c}$. Theorem 5.5 now applies; lifting the associated-graded algebra generators gives its generation conclusion as in Proposition 3.1. □

The network also gives a finite primary description of every monomial mixed product

$$\prod_c I_c^{m_c} = \bigcap_{C \text{ a minimal cover of } K} P_{C\cap S}^{\sum_{c:z_c\notin C} m_c}, \qquad P_{C\cap S} = (x_{r,j} : x_{r,j} \in C).$$

Here an exponent zero means $S$; the intersection may be redundant. Indeed take the fine $z$-degree $\mathbf{m}$ component of $K^{|\mathbf{m}|} = \bigcap_C P_C^{|\mathbf{m}|}$ and subtract the exponent contribution $\sum_{c:z_c\in C} m_c$ of the $z$-variables.

For nested families, this single initial identity follows from the individual ones; in each fine color degree, the coefficient ideal is the largest participating member.

The network description then also expresses a mixed product as an intersection of ordinary powers.

**Corollary 5.7** (Nested families and initial prefixes)**.** *Let $J_1 \subseteq \cdots \subseteq J_r \subset S$ be homogeneous ideals with $\mathrm{in}_\prec(J_c) = I_c$ for one term order, and suppose the $I_c$ satisfy the colored-sublattice hypothesis of Theorem 5.5. For $\mathbf{m} \in \mathbb{N}^r$, put $M_c = \sum_{b \le c} m_b$. Then*

$$\prod_c J_c^{m_c} = \bigcap_{c=1}^{r} J_c^{M_c}, \qquad \prod_c I_c^{m_c} = \bigcap_{c=1}^{r} I_c^{M_c}, \qquad \mathrm{in}_\prec\left(\prod_c J_c^{m_c}\right) = \prod_c I_c^{m_c}. \tag{5.3}$$

*An exponent zero means the whole ring. If in addition $\beta(J_c) = \beta(I_c)$ for every $c$, every mixed product has the same graded Betti table as its monomial initial product, and the multi-Rees Koszul homology is generated in coordinate unit degrees.*

*Proof.* Put $L = \sum_c z_c J_c$ and $K = \sum_c z_c I_c$. For a nonzero fine $z$-degree $\mathbf{a}$, the coefficient ideal of $L$ is $J_{\max \operatorname{supp} \mathbf{a}}$, because the family is nested. The analogous assertion holds for $K$. Therefore any extension of $\prec$ to $S[\mathbf{z}]$ has $\mathrm{in}(L) = K$. Corollary 5.6 supplies the mixed initial identities and the assertions under the Betti hypothesis.

For the monomial intersection formula, one may use all vertex covers in the primary formula for $K$. Every such cover contains one of the form $\{z_{c+1}, \dots, z_r\} \cup D$, where $D$ covers $I_c$; take $c$ to be the largest color whose $z_c$ is absent. A cover containing every $z_c$ gives only an exponent-zero condition. Conversely every displayed set is a cover, since $I_b \subseteq I_c$ for $b \le c$. The primary formula thus reduces to

$$\prod_c I_c^{m_c} = \bigcap_c \bigcap_{D \text{ covers } I_c} P_D^{M_c} = \bigcap_c I_c^{M_c},$$

using Theorem 4.1 for each color. Nestedness gives $Q := \prod_c J_c^{m_c} \subseteq \bigcap_c J_c^{M_c}$. The reverse equality follows from the initial-ideal sandwich

$$\mathrm{in}(Q) \subseteq \mathrm{in}\left(\bigcap_c J_c^{M_c}\right) \subseteq \bigcap_c \mathrm{in}(J_c^{M_c}) = \bigcap_c I_c^{M_c} = \mathrm{in}(Q).$$

Contained ideals with equal initial ideals are equal. □

### 5.2. **Regular-sequence substitutions**

The construction also survives replacing the row variables by a homogeneous regular sequence. Flat base change carries both the resolutions and their Koszul products, while positive degrees preserve minimality.

**Corollary 5.8** (Homogeneous regular-sequence substitutions)**.** *Replace the variables of the sublattice ideal $I_E$ by any homogeneous regular sequence $f_{r,j}$ of positive degrees in a polynomial ring $R$ over $\Bbbk$, and put*

$$K_E = \left( \prod_{r=1}^{d} f_{r,a_r} : a \in E \right) \subset R.$$

*The minimal multigraded resolutions of all powers of $I_E$ base change to minimal free resolutions of the corresponding powers of $K_E$. Their fine shifts are regraded by $\deg x_{r,j} = \deg f_{r,j}$. Moreover, $\operatorname{Tor}^R(\Bbbk, \mathcal{R}(K_E))$ is generated in Rees degree one. Thus the same first-power propagation theorem applies to any homogeneous Rees degeneration with this substituted target. Disjoint monomial substitutions are a particular case.*

*Proof.* The homomorphism $S \to R$, $x_{r,j} \mapsto f_{r,j}$, is graded free; lifts of a homogeneous basis of $R/(f_{r,j})$ generate over $S$, and the regular-sequence Hilbert series makes the resulting free-module surjection an isomorphism. Base change therefore preserves resolutions and gives $K_E^m = I_E^m R$. Positive degrees keep every differential entry in the homogeneous maximal ideal, preserving minimality. Derived base change identifies the Tor algebras

$$\operatorname{Tor}^R(\Bbbk, \mathcal{R}(I_E) \otimes_S R) \cong \operatorname{Tor}^S(\Bbbk, \mathcal{R}(I_E)),$$

with the stated regrading, and preserves their products. Apply Theorem 1.2 and Proposition 3.1. □

## 6. **Unit interval determinantal ideals**

For unit interval complexes, the coefficient ideals have an additional recursive form; each is a sum of linear variables and a smaller restriction of the original ideal. This recursion computes the Betti numbers over every field. For at least three rows it also separates the possible homological degrees, so the determinantal degeneration cannot cancel any Betti entries.

A pure simplicial complex with $d$-element facets is *unit interval* if, whenever $a_1 < \cdots < a_d$ is a facet, every $d$-subset of $[a_1, a_d]$ is a facet. Equivalently, its facets

are the union of the complete $d$-uniform families on its maximal interval cliques. Throughout this section $d \geq 2$. Let $E$ be its facet set and $I = (u_a : a \in E)$ its diagonal monomial ideal. Coordinatewise meet and join preserve $E$, so the preceding results apply.

List the maximal interval cliques with increasing left and right endpoints, and let $c(j)$ be the left endpoint of the first clique whose right endpoint is at least $j$. For a facet $a$, that clique contains $a$, and $c(a_d) \leq a_1$. Set

$$L_a = (x_{1,h} : c(a_d) \leq h < a_1) + \sum_{r=2}^{d} (x_{r,h} : a_{r-1} < h < a_r),$$
$$D_a = (u_b : b \in E,\ b_1 < c(a_d),\ b_r \leq a_{r-1}\ (2 \leq r \leq d)).$$

For a set $U$ of retained variables, a superscript $U$ denotes intersection with $S_U = \Bbbk[U]$. For an ideal $H$ in its indicated polynomial ring, write $\mathsf{Q}_H(s,v) = \mathsf{B}_{S/H}(s,v)$ for the quotient Betti polynomial.

**Lemma 6.1** (Coefficient ideals and their recursion)**.** *The predecessor ideal of $u_a$ is*

$$C_a = L_a + D_a.$$

*The variables generating $L_a$ are disjoint from those of $D_a$, and $D_a \subseteq D_b$ when $a \leq b$. For every variable restriction $I_U = I \cap S_U$, its quotient Betti polynomial satisfies*

$$\mathsf{Q}_{I_U}(s,v) = 1 + sv^d \sum_{a:\, u_a \in S_U} (1+sv)^{\ell_a^U} \mathsf{Q}_{D_a^U}(s,v), \qquad \ell_a^U = \#G(L_a^U), \quad \mathsf{Q}_0 = 1.$$

*Each residual $D_a^U$ is a variable restriction of $I$ with fewer generators. In particular, all the fine-graded Betti numbers are independent of the field, and*

$$\beta_{p,q}(S_U/I_U) \neq 0 \quad \Longrightarrow \quad q - p \equiv 0 \pmod{d-1}.$$

*Proof.* By Lemma 2.1, it suffices to consider ratios from $b < a$ coordinatewise. Each generator of $L_a$ comes from decreasing one coordinate of $a$ inside a clique containing it. Suppose a ratio avoids these variables. If $b_1 \geq c(a_d)$, avoidance forces $b_1 = a_1$, then successively $b_r = a_r$ for every $r$, a contradiction. Thus $b_1 < c(a_d)$. For $r \geq 2$, avoidance gives either $b_r \leq a_{r-1}$ or $b_r = a_r$. If $b_r = a_r$ and $r < d$, strict increase of the facet gives $b_{r+1} > a_r$; avoiding the next interval of linear variables therefore forces $b_{r+1} = a_{r+1}$. Repeating gives $b_d = a_d$. But no facet can start below $c(a_d)$ and end at $a_d$. Consequently $b_r \leq a_{r-1}$ for every $r \geq 2$, and its ratio is $u_b \in D_a$. The converse inclusions are immediate. The displayed bounds also prove disjointness and nesting.

For a retained generator, taking its predecessor colon commutes with intersection with $S_U$; a variable outside $U$ in a preceding generator cannot cancel against $u_a$. The

resulting colon is $L_a^U + D_a^U$. Lemma 2.3 makes this generator filtration minimal in every fine degree. Splitting off the linear variables gives the factor $(1+sv)^{\ell_a^U}$; the generator shift and passage from an ideal to its quotient give $sv^d$. This proves the recursion.

The residual retains only columns below $c(a_d)$ in row 1 and columns at most $a_{r-1}$ in row $r \geq 2$, in addition to the restriction $U$. Every residual generator precedes $a$, so induction terminates at the empty ideal. Its coefficients are nonnegative integers. The same recursion, with monomial degree weights in place of powers of $v$, proves fine-graded characteristic independence. A linear Koszul factor changes $p$ and $q$ equally; the factor $sv^d$ changes $q-p$ by $d-1$. This proves the congruence. □

For a nonempty support chain $\sigma$, let $V_\sigma$ be the variables generating $L_\sigma = \sum_{a\in\sigma} L_a$. Nesting gives $C_\sigma = L_\sigma + D_{\max\sigma}$. In the ring $S_\sigma$ on the remaining variables, put $D_\sigma = D_{\max\sigma} \cap S_\sigma$. Then

$$\mathsf{B}_{S/C_\sigma}(s,v) = (1+sv)^{|V_\sigma|}\mathsf{B}^{S_\sigma}_{S_\sigma/D_\sigma}(s,v). \tag{6.1}$$

The chain formula therefore proves characteristic independence for every power of $I$, together with

$$\beta_{p,q}(I^m) \neq 0 \quad\Longrightarrow\quad q-p \equiv dm \equiv m \pmod{d-1}. \tag{6.2}$$

### 6.1. **Closed binomial edge ideals**

We first arrange that each connected component occupies a consecutive block of labels and every edge spans a clique. In a closed labeling, a shortest path has no local maximum or minimum of its labels, since either would give a shortcut; every shortest path is therefore label-monotone. Consecutive vertices in the induced order of a component must be adjacent, and these consecutive edges, together with closedness, force all vertices between the endpoints of any edge to lie in one clique.

Concatenating the components while preserving their internal orders gives the required interval labeling. The corresponding permutation of the pairs $(x_i, y_i)$ carries both $J$ and its specified monomial initial ideal to the new ideals, preserving their Betti numbers. Isolated vertices contribute only polynomial variables, so they can be omitted and reinserted without affecting the comparison. If there are no edges, both ideals and all their positive powers are zero.

Henceforth assume this interval labeling. In particular, for $i<j<\ell$,

$$\{i,\ell\} \in E(G) \quad\Longrightarrow\quad \{i,j\},\{j,\ell\} \in E(G). \tag{6.3}$$

For a vertex $j$ with a smaller neighbor, put

$$a(j) = \min\{h < j : \{h, j\} \in E(G)\};$$

put $a(j) = j$ if there is no smaller neighbor. The function $a(j)$ is nondecreasing. Indeed, if $j < \ell$ and $a(\ell) < j$, equation (6.3) gives $\{a(\ell), j\} \in E(G)$ and hence $a(j) \le a(\ell)$; if $a(\ell) \ge j$, the same inequality is immediate.

Let $E = \{(i, j) : i < j,\ \{i, j\} \in E(G)\}$. It is a sublattice of $[n] \times [n]$. For incomparable edges $(i, j), (k, \ell)$, say $i < k$ and $j > \ell$, we have $i < k < \ell < j$. The outer edge $(i, j)$ spans a clique, so $(i, \ell)$ and $(k, j)$ are edges. These are the coordinatewise meet and join. Comparable pairs need no argument. The ideal $I_G$ is therefore exactly the sublattice ideal of Section 2, with $x_{1,i} = x_i$ and $x_{2,j} = y_j$.

*Proof of Theorem 1.1.* For a graph with an edge, use the edge sublattice just described. The ordinary initial identity and Peeva's first-power theorem verify the hypotheses of Theorem 4.3. It yields the displayed equality for every $m, p, q$ over the arbitrary field $\Bbbk$. The labeling reduction and the zero-ideal case above include disconnected graphs, isolated vertices, and the empty edge set. □

In row size two, $c(j) = a(j)$ for every edge ending at $j$, and the coefficient ideals in Lemma 6.1 become

$$L_{ij} = (x_h : a(j) \le h < i) + (y_\ell : i < \ell < j),$$
$$D_{ij} = (x_h y_\ell : (h, \ell) \in E,\ h < a(j),\ \ell \le i).$$

Thus the recursion and (6.1) give finite formulas for all graph powers.

The degeneration respects the vertex and row gradings, while the coefficient recursion is independent of the field. These two features refine the ordinary Betti equality as follows.

**Corollary 6.2** (Natural multigraded preservation)**.** *Give $S$ the grading*

$$\deg(x_i) = (\mathbf{e}_i, 1), \qquad \deg(y_i) = (\mathbf{e}_i, 0) \qquad \text{in } \mathbb{Z}^n \oplus \mathbb{Z}.$$

*Then for every $m, p$ and every natural multidegree $(\alpha, r)$,*

$$\beta_{p,\alpha,r}(J_G^m) = \beta_{p,\alpha,r}(I_G^m).$$

*For a disconnected graph, the number of $x$-variables can be recorded separately in each connected component, with the same conclusion.*

*Proof.* The homogeneous degeneration $J_G^m \rightsquigarrow I_G^m$ respects these gradings. In each such degree, the finite filtered Koszul complex gives the usual inequality from generic to special Betti numbers. Theorem 1.1 says that the sums of those nonnegative differences at every fixed $p, q$ vanish. Each difference must therefore vanish. The relations are also homogeneous for separate row counts in each component. □

**Corollary 6.3** (Independence of the field)**.** *For every closed graph, all powers of $I_G$ have field-independent Betti numbers in the finest monomial grading. All powers of $J_G$ have field-independent Betti numbers in the natural grading of Corollary 6.2, and hence in the ordinary grading.*

*Proof.* For a support chain $\sigma$, killing the variables in $V_\sigma$ leaves $D_\sigma$, an induced variable restriction of $I$. Its Betti table is field-independent by Lemma 6.1. The exterior factor and the chain formula in Corollary 5.1 therefore give the same assertion for every $I^m$. The proof of that formula respects the finest monomial grading. Corollary 6.2 transfers the assertion to $J^m$. □

For $K_4$-free closed graphs, the linear-strand comparison was proved by Dohadwala et al. [4]. The coefficient formula gives the following expression for every closed graph.

**Corollary 6.4** (Linear strands for arbitrary closed graphs)**.** *For every $m \geq 1$ and $p \geq 0$,*

$$\beta_{p,2m+p}(J^m) = \beta_{p,2m+p}(I^m) = \sum_{\substack{\varnothing \neq \sigma \subseteq E \\ \sigma \text{ a chain}}} \binom{m-1}{|\sigma|-1} \binom{|V_\sigma|}{p}.$$

*Proof.* The ideal $D_\sigma$ is generated by quadrics. Its quotient has no positive-homological-degree Tor in internal degree equal to homological degree. Hence the entries on this diagonal in (6.1) come entirely from the exterior factor $(1+sv)^{|V_\sigma|}$. □

### 6.2. **Higher minors**

Let $J_\Delta$ be the ideal of maximal minors associated with the facets of a unit interval complex in a generic $d \times n$ matrix. Its diagonal initial ideal is $I_\Delta$ [1, Theorem 2.7].

**Theorem 6.5.** *If $d \geq 3$, then, over every field and for every $m \geq 1$,*

$$\beta_{p,q}(J_\Delta^m) = \beta_{p,q}(I_\Delta^m).$$

*These numbers are independent of the field, and a nonzero entry satisfies $q - p \equiv m \pmod{d-1}$.*

*Proof.* The assertion is immediate for the zero ideal. Otherwise, Theorem 4.3 gives $\mathrm{in}(J_\Delta^m) = I_\Delta^m$. By (6.2), the latter has no two nonzero Betti entries in adjacent homological degrees and the same internal degree, since $d-1 \geq 2$. In the corresponding weight degeneration, a torsion summand of the family homology would contribute to both such entries by (3.1). There is therefore no torsion, and the Betti numbers agree. Lemma 6.1 and the chain formula supply characteristic independence and the congruence. □

For $d = 2$ the congruence is vacuous. Peeva's first-power theorem and multiplicative transfer give precisely the comparison needed in that case.

### 6.3. **Mixed prefixes**

Passing to an initial-column prefix does not change the predecessor colon of any retained facet. Consequently the mixed chain formula uses the same coefficient quotients as ordinary powers, and the same recursion computes its Betti tables.

Closed graphs. Use the interval labeling of Section 6.1, choose $n_1 \leq \cdots \leq n_r$, and let $J_c$ be the nonzero binomial edge ideal of the induced graph on $[n_c]$, extended to the common ring $S$. Its edge set is $E_c = \{(i,j) \in E(G) : j \leq n_c\}$. These are nested sublattices, and their colored set is a sublattice; the meet belongs to the smaller prefix and the join to the larger. Each induced graph is closed, so Peeva's theorem supplies every first-power Betti equality. Thus (5.3), all mixed Betti equalities, and multi-Rees homology generation hold over every field.

Moreover these mixed Betti tables are independent of the field. For $a \in E_c$, every coordinatewise predecessor $b \leq a$ belongs to the same prefix. Lemma 2.1 therefore gives $C_a^{(c)} = C_a$, the colon in the full edge lattice. The coefficient ideals in (5.2) are consequently the same sums $\sum_s C_{a_s}$ as for ordinary graph powers. The positive recursion of Lemma 6.1 makes their fine Betti tables characteristic independent. The mixed chain formula gives fine independence for the monomial products, and the mixed comparison gives ordinary independence for the binomial ones.

Higher-row prefixes. The same conclusions hold for nonzero initial-column prefixes of a unit interval determinantal facet ideal. Retaining facets with last column at most $n_c$ again gives nested colored sublattices, and their prefix colons equal those in the full facet lattice. Theorem 6.5 supplies the individual first-power comparison; its positive recursion for coefficient quotients supplies characteristic independence. Thus every such mixed product satisfies (5.3) and the same Betti-preservation and field-independence conclusions.

## 7. **Integral chain decomposition for sublattice powers**

Over a field, the block decomposition expresses homology classes as products of first-power classes. Over the integers, tensor products can also contribute torsion, so lifting homology classes alone no longer suffices. We retain that information by lifting each entire block complex to Rees degree one and then multiplying the chain maps. The separation of first-power strands by their rowwise maximum indices makes these stronger lifts possible.

All monomial ideals in this section are defined by the same generators over the indicated coefficient ring. Write $S_{\mathbb{Z}} = \mathbb{Z}[\mathbf{x}]$, and retain the notation $C_a, C_w, B$ of Sections 2 and 2.1.

**Theorem 7.1** (Integral chain decomposition)**.** *For every finite sublattice $E$, every $m \geq 1$, and every fine degree $\alpha$, there is a chain homotopy equivalence over $\mathbb{Z}$*

$$\mathcal{K}_{S_{\mathbb{Z}}}(I_E^m)_\alpha \simeq \bigoplus_{a_1 \leq \cdots \leq a_m} \mathcal{K}_{S_{\mathbb{Z}}}(S_{\mathbb{Z}}/C_w)_{\alpha - \sum_s \deg u_{a_s}}. \tag{7.1}$$

*More precisely, for a monic integer-weight Rees degeneration to $B$, each fixed fine/auxiliary component of its relative Koszul complex is chain homotopy equivalent over $\mathbb{Z}[\tau]$ to its constant special fiber complex.*

*Consequently, for every commutative coefficient ring $\Lambda$, with $S_\Lambda = \Lambda[\mathbf{x}]$,*

$$\operatorname{Tor}_p^{S_\Lambda}(I_E^m, \Lambda)_\alpha \cong \bigoplus_{a_1 \leq \cdots \leq a_m} \operatorname{Tor}_p^{S_\Lambda}(S_\Lambda/C_w, \Lambda)_{\alpha - \sum_s \deg u_{a_s}}. \tag{7.2}$$

*In particular, over $\mathbb{Z}$ this preserves every torsion summand and its exponent. The equivalences are noncanonical; no $S$-linearity or multiplicative coherence between different powers is asserted. Without an additional torsion-freeness hypothesis, homology-algebra generation in Rees degree one is asserted only over fields.*

*Proof. The integral family.* The relations in Proposition 2.2 are monic with coefficients $\pm 1$, and its normal-form injectivity proof works over $\mathbb{Z}$. Choose a positive integer weight $\omega$ realizing their leading terms, and homogenize them with a parameter $\tau$ of weight one. Their standard monomials form a $\mathbb{Z}[\tau]$-basis of the resulting algebra $\mathcal{A}$; division gives spanning, and after inverting $\tau$ their images are distinct monomials of the original Rees algebra, proving independence. Thus $\mathcal{A}/(\tau) = B_{\mathbb{Z}}$ and $\mathcal{A}/(\tau - 1) = \mathcal{R}(I_{E,\mathbb{Z}})$. Give each exterior generator the weight of its base variable. For fixed fine degree $\alpha$ and Rees degree $m$, let $\mathcal{C}_{m,\alpha}$ be the resulting relative Koszul complex over $\mathbb{Z}[\tau]$, and let $\mathcal{B}_{m,\alpha}$ be its special fiber. These are bounded complexes; the former has finite free $\mathbb{Z}[\tau]$-terms. Their weight gradings are bounded below.

*Chain sections in the first power.* At fine degree $\beta$, the summand $(S_{\mathbb{Z}}/C_a)T_a$ of $B_{*,1}$ has the single total Koszul weight

$$\lambda_a(\beta) = \omega(T_a) + \omega_{\mathbf{x}}(\beta - \deg u_a),$$

independent of homological degree. Its homology can be nonzero only when $a$ is the rowwise maximum-index vector of $\beta$. Indeed a different nonzero summand has a higher row variable in its residual fine degree, unused by $C_a$; the corresponding polynomial Koszul strand is contractible over $\mathbb{Z}$. Concretely, that variable contributes the two-term complex $0 \to \mathbb{Z} \xrightarrow{1} \mathbb{Z} \to 0$ in its positive fine degree. The contraction therefore uses a unit and survives every coefficient change. Thus, when this maximum vector is $a \in E$, all first-power special homology is concentrated at the one weight $\lambda = \lambda_a(\beta)$.

The weight-$v$ exact sequence is

$$0 \longrightarrow (\mathcal{C}_{1,\beta})_{v-1} \xrightarrow{\tau} (\mathcal{C}_{1,\beta})_v \longrightarrow (\mathcal{B}_{1,\beta})_v \longrightarrow 0.$$

Induction from the lower weight bound shows that $(\mathcal{C}_{1,\beta})_v$ is acyclic for $v < \lambda$. At $v = \lambda$, the kernel is therefore a contractible bounded complex of finite free abelian groups. The surjection onto $(\mathcal{B}_{1,\beta})_\lambda$ admits a chain section; choose a degreewise section and correct its differential defect with a contracting homotopy of the kernel. Explicitly, if $s$ is the chosen section and $h$ contracts the kernel, put $e = \partial s - s\partial$. Then $e$ takes values in that kernel and $\partial e + e\partial = 0$, so

$$s' = s - he, \qquad \partial s' - s'\partial = 0.$$

The correction does not change the projection of $s$. Hence every chain map from a finite free abelian complex into this special component lifts to the family.

*Lifting block complexes.* Fix a multichain $w = (a_1 \le \cdots \le a_m)$ and put $\delta = \alpha - \sum_s \deg u_{a_s}$. A negative component gives zero. Use the crossing blocks $X_s, R_s$ of Lemma 2.4. Their disjoint supports and $R_s \subseteq C_{a_s} \subseteq C_w$ give

$$R_s = C_w \cap \mathbb{Z}[X_s] = C_{a_s} \cap \mathbb{Z}[X_s].$$

If $\delta$ has positive degree in $X_\infty$, the corresponding special strand is contractible and can be omitted. Otherwise, writing $\delta_s = \delta|_{X_s}$, that strand is exactly the tensor complex

$$D_w = \bigotimes_{s=1}^{m} \mathcal{K}_{\mathbb{Z}[X_s]}(\mathbb{Z}[X_s]/R_s)_{\delta_s}.$$

The $s$-th block complex maps to the first-power special complex by the quotient map and multiplication by $T_{a_s}$, since $R_s \subseteq C_{a_s}$. Its fine degree is $\beta_s = \delta_s + \deg u_{a_s}$. Every variable of $X_s$ lies below $a_s$ in its row, so the maximum vector of $\beta_s$ is exactly $a_s$. The chain section just constructed therefore lifts this entire block map, not merely its homology classes. Extend the lifts $\mathbb{Z}[\tau]$-linearly, tensor them, and multiply in the family Koszul differential graded algebra. This gives a homogeneous chain map

$$\Phi : \left( \bigoplus_w D_w \right) \otimes_{\mathbb{Z}} \mathbb{Z}[\tau] \longrightarrow \mathcal{C}_{m,\alpha}.$$

Its reduction modulo $\tau$ is the usual block decomposition on every retained summand. The omitted summands are integrally contractible, so this reduction is a quasi-isomorphism. Tensoring whole complexes retains the integral Künneth torsion classes.

*The cone and arbitrary base change.* Assign $D_w$ the weight $\sum_s \lambda_{a_s}(\beta_s)$. The cone of $\Phi$ is weight graded and bounded below, has finite free $\mathbb{Z}[\tau]$-terms, and is

acyclic modulo $\tau$. Its weight-component exact sequences, as above, prove its acyclicity by induction on the weight. A bounded acyclic complex of projective modules is contractible, splitting successively from its lowest degree. Thus $\Phi$ is a chain homotopy equivalence over $\mathbb{Z}[\tau]$. Reinstating the contractible omitted summands identifies its source up to homotopy with the constant complex $\mathcal{B}_{m,\alpha} \otimes \mathbb{Z}[\tau]$.

Specializing $\tau = 1$ proves (7.1). Tensoring the homotopies with any commutative ring $\Lambda$ preserves them. Monomial ideals, their powers, and monomial quotients base change literally. The polynomial variables are a regular sequence over $\Lambda[\mathbf{x}]$, so these relative Koszul complexes compute Tor also when $\Lambda$ is not a field. This proves (7.2). □

**Corollary 7.2** (Integral mixed formula)**.** *Under the colored-sublattice hypothesis of Theorem 5.5, for every commutative coefficient ring $\Lambda$ the mixed formula holds as an isomorphism of fine-graded $\Lambda$-modules*

$$\operatorname{Tor}_p^{S_\Lambda}\left(\prod_c I_c^{m_c}, \Lambda\right)_\alpha \cong \bigoplus_{\substack{w=((c_1,a_1)\le\cdots\le(c_m,a_m))\\ \operatorname{col}(w)=\mathbf{m}}} \operatorname{Tor}_p^{S_\Lambda}(S_\Lambda/C_w^S, \Lambda)_{\alpha-\sum_s \deg u_{a_s}},$$

*where $C_w^S = \sum_s C_{a_s}^{(c_s)}$ as in* (5.2).

*Proof.* If $|\mathbf{m}| = 0$, the formula is the identity for $S_\Lambda$. Otherwise apply Theorem 7.1 to $K = \sum_c z_c I_c$ over $S_\Lambda[\mathbf{z}]$ and take fine $z$-degree $\mathbf{m}$ in Rees degree $|\mathbf{m}|$. At the chain level this diagonal has no exterior $z$-terms. Only chains with the prescribed colors contribute, and their residual $z$-degree is zero. Cross-color predecessor ratios contain a $z$-variable; restricting the coefficient ideals to $S_\Lambda$ therefore gives precisely $C_w^S$. This proves the assertion without any fieldwise Künneth argument. □

## 8. **Alcove resolutions and stabilization**

The chain decomposition determines the homology in each power, but its possible homological degrees are more easily bounded geometrically. Products of $m$ generators correspond to the integer points of a dilated order polytope, whose dimension bounds the length of a cellular resolution. A maximal lattice chain supplies a complete intersection attaining that bound; together with first-power generation, this also determines when the normalized Betti support stabilizes.

Let $P$ be the poset of join-irreducibles of $E$, let $h = |P|$, and write $N$ for the number of polynomial variables, including unused ones. Retain the coordinate chains obtained in the proof of Theorem 4.1. Compress their attained values to $0, \dots, \ell_r$, so row $r$ corresponds to $p_{r,1} < \cdots < p_{r,\ell_r}$.

**Theorem 8.1** (An integral cellular resolution)**.** *For every $m \geq 1$, the ideal $I_E^m$ has an explicit, generally nonminimal, cellular free resolution over $\mathbb{Z}[\mathbf{x}]$ of length at most $h$. Its base change to any commutative coefficient ring is still a free resolution. Over every field, the analytic spread of $I_E$ is $h+1$ and*

$$\operatorname{pd}(I_E^m) \leq h \quad (m \geq 1), \qquad \operatorname{pd}(I_E^m) = h \quad (m \geq M := \max\{1, h\}).$$

*For each fixed field, the ordinary normalized Betti support*

$$\{(p, q - dm) : \beta_{p,q}(I_E^m) \neq 0\}$$

*is nondecreasing with $m$ and is constant for $m \geq M$. Put $c_{E,\Bbbk} = \operatorname{reg}(I_E^M) - dM$. Consequently, for $m \geq M$,*

$$\operatorname{depth}(S/I_E^m) = N - h - 1 \quad (d \geq 1), \qquad \operatorname{reg}(I_E^m) = dm + c_{E,\Bbbk}.$$

*Proof. Vertices and labels.* The minimal generators of $I_E^m$ correspond bijectively to the integer order-reversing maps $f : P \to \{0, \ldots, m\}$. For a product of $m$ generators, let $f(p)$ count the order ideals containing $p$; conversely, its level sets form a multichain of order ideals. The row exponents of its monomial label $u_f$ are

$$m - f(p_{r,1}), \quad f(p_{r,1}) - f(p_{r,2}), \quad \ldots, \quad f(p_{r,\ell_r}),$$

with exponent $m$ for a constant row. These exponents determine $f$ because every $p \in P$ belongs to some coordinate chain. All labels have degree $dm$, so distinct labels are minimal generators.

Let $\mathcal{O}(P) = \{f \in [0,1]^P : f(p) \geq f(q) \text{ if } p < q\}$. Triangulate $\mathbb{R}^P$ by the affine braid hyperplanes

$$f(p) = k, \qquad f(p) - f(q) = k \qquad (k \in \mathbb{Z}).$$

Equivalently, subdivide each integer unit cube by ordering the fractional coordinates. The resulting simplices have integer vertices. The polytope $m\mathcal{O}(P)$ is a union of these simplices; denote its finite triangulation by $\mathcal{T}_m$. Label its vertex $f$ by $u_f$ and each face $\sigma$ by the least common multiple $u_\sigma$ of its vertex labels.

*Exactness in each fine degree.* Use one free generator of degree $u_\sigma$ in homological degree $\dim \sigma$, with differential

$$\partial[\sigma] = \sum_{\tau \text{ a facet of } \sigma} \epsilon(\sigma, \tau) \frac{u_\sigma}{u_\tau} [\tau], \qquad [f] \longmapsto u_f$$

for the augmentation to $I_E^m$. For a monomial $\mathbf{x}^{\mathbf{b}}$, the fine strand is the augmented simplicial chain complex of the subcomplex induced by vertices whose labels divide $\mathbf{x}^{\mathbf{b}}$. The divisor conditions are

$$m - f(p_{r,1}) \le b_{r,0}, \quad f(p_{r,j}) - f(p_{r,j+1}) \le b_{r,j}, \quad f(p_{r,\ell_r}) \le b_{r,\ell_r}.$$

Together with $m\mathcal{O}(P)$ these define a convex polytope $Q_{\mathbf{b}}$, possibly empty. A constant row either imposes no additional condition or makes it empty. Every bounding hyperplane belongs to the displayed affine arrangement. Thus $Q_{\mathbf{b}}$ is a union of its simplices, and the induced divisor subcomplex is exactly this polytope's triangulation. It is contractible when nonempty. Its augmented integral chain complex is therefore exact; if empty, the whole augmented fine strand is zero. This proves exactness over $\mathbb{Z}$, and the same argument works over every coefficient ring. Since $\dim \mathcal{T}_m = h$, the length bound follows. The resolution need not be minimal; a face and a facet may have the same monomial label. Thus its existence does not preclude torsion in the Koszul homology discussed in Section 10.

*The lower bound.* Suppose $h > 0$, choose a linear extension $p_1, \dots, p_h$ of $P$, and let $a_j \in E$ correspond to $D_j = \{p_1, \dots, p_j\}$. In the chain $a_0 < a_1 < \cdots < a_h$, the newly crossed variables at step $j$ are the old state variables in rows whose coordinate chain contains $p_j$. Their product is

$$g_j = q_{a_{j-1},a_j} \in C_{a_j}.$$

The crossing block created at step $j$ is precisely $(g_j)$. Indeed, a block generator can be taken as $q_{b,a_j}$ by Lemma 2.4. If the order ideal of $b$ omitted an earlier $p_i$, choose the first such $i$ and a row containing $p_i$. Its state variable has first crossing $i < j$, a contradiction. Thus $b$ contains $D_{j-1}$ and differs from $D_j$ only by $p_j$. Each $g_j$ is nonconstant, and their supports are disjoint. Therefore

$$C_{(a_0,\dots,a_h)} = (g_1, \dots, g_h)$$

is a complete intersection of length $h$. Since $C_{a_0} = 0$, omitting $a_0$ leaves the same coefficient ideal. Its top Koszul homology is nonzero, so (1.1) gives $\operatorname{pd}(I_E^h) \ge h$. Repeating chain factors gives the same lower bound for every $m \ge h$. When $h = 0$, the ideal is principal and all assertions are immediate.

*Support, depth, and spread.* Corollary 5.3 says that any nonzero normalized strand in homological degree $p \ge 1$ has appeared by power $p$. Since $p \le h$, every such strand has appeared by $M$. Corollary 5.2 makes it persist thereafter; in degree zero the only normalized internal degree is zero, present from the first power. This proves support stabilization. The regularity formula follows by taking the maximum of $(q - dm) - p$

on that finite support. The depth formula follows from $\operatorname{pd}(S/I_E^m) = \operatorname{pd}(I_E^m) + 1$ and Auslander–Buchsbaum.

Finally the fiber algebra is the monomial algebra on the labels $u_a$. Its exponent semigroup is the image of the pairs $(m, f)$ under the injective linear map given by the row-difference formula. The vectors of the bottom and a maximal ideal chain span a group of rank $h + 1$; successive differences recover the $h$ coordinate directions of $f$, in addition to $m$. The monomial algebra therefore has dimension $h + 1$, which is the analytic spread. □

## 9. **An integral family for closed graphs**

For closed graphs, the positive recursion removes integral torsion from the coefficient homology. We can therefore lift bases through the degeneration and assemble them into one free complex whose fibers resolve both the monomial and binomial powers. In this section $z$ is a deformation parameter of ordinary internal degree zero.

Let $G$ be a closed graph with the interval labeling used above, and set

$$\mathcal{J} = (x_i y_j - z^{j-i} x_j y_i : \{i, j\} \in E(G),\ i < j) \subset \mathbb{Z}[z, x_1, \dots, x_n, y_1, \dots, y_n].$$

**Theorem 9.1** (A universal minimal complex)**.** *For each $m \geq 1$ there is a finite graded free resolution of $\mathcal{J}^m$ whose matrices have entries in $(x_1, \dots, x_n, y_1, \dots, y_n)$. Every base change from $\mathbb{Z}[z]$ preserves its exactness. In particular, specialization to $z = 0$ and $z = 1$ over every field gives minimal free resolutions of $I_G^m$ and $J_G^m$, with the same ranks and shifts.*

To construct this complex, we first prove that the relative Tor modules are free over the parameter ring. A positive parameter weight allows integral cycle lifts to be chosen as a basis, after which the resolution is built by lifting generators of successive syzygies.

**Lemma 9.2** (Integral multiplicative transfer)**.** *Put $R = \mathbb{Z}[z]$ and $S_R = R[x_1, \dots, x_N]$. Let $\mathcal{A}$ be an $R$-flat graded $S_R$-algebra, with auxiliary degree $m$ and ordinary internal degree $q$, whose fixed $(m, q)$ components are finite free over $R$. Suppose it has an additional bounded-below integer weight grading with $\mathrm{wt}(z) = 1$, and every fixed $(m, q, \mathrm{wt})$ component is finite free over $\mathbb{Z}$. Let $B = \mathcal{A}/z\mathcal{A}$. Assume*

*(i) $H(K_x(B))$ is free as a graded abelian group and is generated as a $\mathbb{Z}$-algebra by auxiliary degree one and its unit;*

*(ii) over every field the first-degree special and generic Betti tables agree, and after inverting $z$ the family and its Koszul complex are obtained from a constant fiber by diagonal rescaling.*

*Then every fixed $(m, q)$ part of $H(K_x(\mathcal{A}))$ is a finite free graded $R$-module, its reduction modulo $z$ is $H(K_x(B))$, and it is generated over $R$ in auxiliary degree one. Relative Koszul homology commutes with arbitrary base change from $R$.*

*Proof.* Write $C = K_x(\mathcal{A})$, assigning each exterior variable the weight of its coefficient variable. First work in auxiliary degree one over a field $k$. Over the DVR $k[z]_{(z)}$, the universal-coefficient formula is

$$\dim_k H_p(C/zC) = \operatorname{rank} H_p(C) + t_p + t_{p-1},$$

where $t_i$ counts the cyclic $z$-primary torsion summands. Equality of special and generic Betti numbers forces all $t_i$ to vanish. The diagonal rescaling away from zero rules out torsion supported at any other irreducible polynomial. Thus first-degree homology over $k[z]$ is free. A homogeneous basis has the weights of its basis modulo $z$, whose dimensions are independent of $k$; by assumption (i), the special integral homology is free, so universal coefficients commute with passage to every field in each retained weight. Thus the first-degree family homology has field-independent dimensions in every fixed weight.

Now apply universal coefficients over $\mathbb{Z}$ to each finite free complex $C_{1,q,\mathrm{wt}}$. Comparison of $\mathbb{Q}$ with each $\mathbb{F}_\ell$ shows that no integral homology group in auxiliary degree one has integer torsion; the difference of the dimensions is the sum of two nonnegative adjacent torsion counts. There is also no $z$-torsion in that degree. A class killed by $z$ vanishes after tensoring with $\mathbb{Q}$, where $z$ is regular, and is therefore integer torsion.

The long exact sequence of $0 \to C \xrightarrow{z} C \to C/zC \to 0$ consequently lifts every integral special class of auxiliary degree one to a family cycle. Products of these cycles lift all special classes by assumption (i). Hence $H(C) \to H(C/zC)$ is surjective in every degree, and the same long exact sequence gives

$$H(C)[z] = 0, \qquad H(C)/zH(C) = H(C/zC).$$

The next, zero, term of the sequence gives the assertion in top homological degree as well.

Fix $p, m, q$, choose a homogeneous integral basis of the special homology, and lift it to $H = H_p(C)_{m,q}$. Let $F \to H$ be the map from the graded free $R$-module on these lifts. Its cokernel is zero modulo $z$. Because the grading is bounded below and $z$ has positive weight, induction on weight makes the cokernel zero. If $K$ is the kernel, regularity of $z$ on $H$ gives an injection $K/zK \to F/zF$; its image is zero because the chosen bases identify $F/zF$ with $H/zH$. Thus $K/zK = 0$, and the same bounded-below weight induction gives $K = 0$. This proves freeness. Applying that induction to the quotient by the algebra generated in degree one proves the generation claim.

Finally, a bounded complex of flat $R$-modules with flat homology commutes with arbitrary base change. In the present finite free complexes this also follows by induction from the bottom; the exact sequences defining boundaries and cycles split whenever their quotient is projective, so every homology summand survives tensoring. This proves the last assertion. □

*Proof of Theorem 9.1.* Every fine Koszul homology group of $\mathbb{Z}[x, y]/C_w$ is free; the recursion of Lemma 6.1 makes its dimensions independent of the field, and universal coefficients exclude integral torsion. Theorem 7.1 therefore makes $H(K_x(\mathcal{R}(I_{G,\mathbb{Z}})))$ free over $\mathbb{Z}$. Its proof also gives integral generation in Rees degree one. Indeed each crossing-block complex is a fine strand of a coefficient quotient with all other block degrees zero, so its homology is free. Integral Künneth expresses every block-tensor class as a product. The chain equivalence in that proof is defined by multiplying lifts of the whole block complexes into Rees degree one. After specializing $\tau = 1$, every homology class is therefore a sum of products of first-power classes. This generation assertion requires no compatibility between the choices for different multichains.

For the displayed binomial family assign

$$\operatorname{wt}(z) = 1, \qquad \operatorname{wt}(x_i) = n - i + 1, \qquad \operatorname{wt}(y_i) = 1.$$

Every generator is homogeneous. After inverting $z$, substituting $x_i \mapsto z^{-i}x_i$ transforms its generator into $z^{-i}(x_i y_j - x_j y_i)$, so the family and its Koszul complex are diagonal rescalings of the constant binomial fiber.

For each $m$, the products of $m$ displayed generators form a monic Gröbner basis over $\mathbb{Z}[z]$. Over $\mathbb{Q}(z)$ this follows from the all-powers initial identity and the diagonal rescaling. Division is monic and therefore stays in $\mathbb{Z}[z]$; its remainder vanishes over $\mathbb{Q}(z)$, hence already over $\mathbb{Z}[z]$. Standard monomials give a free $\mathbb{Z}[z]$-basis of the quotient. The ideal itself has a free basis indexed by the monomials of $I^m$; for each

such monomial choose a monic multiple of a basis element with that leading monomial. Triangular division proves spanning and independence. These choices are weight homogeneous. Fixed ordinary degree components are finite free over $\mathbb{Z}[z]$, and fixing the weight as well gives finite free abelian groups, bounded below in weight.

The special fiber of $\mathcal{R}(\mathcal{J})$ is therefore exactly $\mathcal{R}(I_{G,\mathbb{Z}})$, and arbitrary field base changes give the intended powers. The first paragraph supplies integral freeness and generation of its homology. The fine first-power recursion supplies weightwise characteristic independence, and Peeva's theorem supplies first-power equality in every field. Lemma 9.2 now applies. In particular, for $M = \mathcal{J}^m$, all modules $\operatorname{Tor}_i^{S_R}(R, M)$ are finite free over $R = \mathbb{Z}[z]$.

Lift a homogeneous $R$-basis of $M/(x, y)M$ to a map $F_0 \to M$; graded induction in ordinary internal degree makes it surjective. Its kernel is $R$-flat because $M$ and $F_0$ are $R$-flat, and its quotient modulo $(x, y)$ is the free relative $\operatorname{Tor}_1$ module. Lifting a basis of this quotient repeats the construction. At every step, the syzygy remains $R$-flat, every matrix entry lies in $(x, y)$, and the next quotient modulo $(x, y)$ is the corresponding free relative Tor module. The construction terminates after at most $2n$ steps, the length of the Koszul resolution of $R$ over $S_R$; when the next relative Tor is zero, graded induction makes the next syzygy zero.

The syzygies are $R$-flat, so base change preserves exactness. Over a field the entries remain in the homogeneous maximal ideal, giving minimality. □

## 10. **Characteristic dependence for general sublattices**

The integral decomposition locates all characteristic dependence in the coefficient quotients. For general sublattices these can carry finite-complex homology, including torsion; we first encode a complex by a bipartite incidence graph and a squarefree ideal, then realize that ideal as a single predecessor colon. The fine grading isolates its homology, and repeating the top lattice element carries the same group into every power.

**Lemma 10.1.** *Let $G$ be a bipartite graph with parts $L, R$, no isolated vertices, and $\ell = |L| \geq 1$. In $A = \mathbb{Z}[z_a : a \in L,\ x_{ar} : ar \in E(G)]$, put*

$$X_a = \prod_{r \in N(a)} x_{ar}, \qquad W_r = \prod_{a : r \in N(a)} x_{ar}, \qquad C = \prod_{a \in L} (z_a, X_a) + (W_r : r \in R).$$

*Let $K = \{Q \subseteq R : N(a) \nsubseteq Q \text{ for all } a \in L\}$, assume $K$ has a vertex, and let $\sigma$ be the product of all variables of $A$. Then*

$$\operatorname{Tor}_i^A(A/C, \mathbb{Z})_\sigma \cong \widetilde{H}_{i-\ell-2}(K; \mathbb{Z}). \tag{10.1}$$

*The same assertion holds over every field.*

*Proof.* Let $\Delta$ be the Stanley–Reisner complex of $C$, on the full variable set $V$, and $\Gamma = \Delta^*$. Write $Z = \{z_a : a \in L\}$ and $E_a = \{x_{ar} : r \in N(a)\}$. Directly from the generators,

$$\Gamma = U_0 \cup B, \qquad U_0 = \bigcup_{r \in R} 2^{V \setminus \operatorname{supp} W_r}, \qquad B = \underset{a \in L}{*} \big(2^{\{z_a\}} \sqcup 2^{E_a}\big).$$

Here each factor of $B$ consists of an isolated vertex and a disjoint simplex. Thus $B \simeq S^{\ell-1}$, while $U_0$ is a cone on every $z_a$. For nonempty $Q \subseteq R$, set $B_Q = B \cap 2^{V \setminus \bigcup_{r \in Q} \operatorname{supp} W_r}$. Its $a$-th join factor has leaves indexed by $N(a) \setminus Q$. Consequently $B_Q$ is a sphere up to homotopy if $Q \in K$, and is contractible otherwise. Every $B_Q$ contains the simplex $2^Z$.

The relative Mayer–Vietoris spectral sequence for $U = U_0 \cap B = \bigcup_r B_{\{r\}}$, relative to $2^Z$, has just one nonzero row; $H_{\ell-1}(B_Q, 2^Z) = \mathbb{Z}$ for $Q \in K$. Orient its generator as the ordered join of the zero-cycles $z_a - x_{ar_a}$, choosing $r_a \in N(a) \setminus Q$. Different leaf choices are homologous inside their leaf simplex, so all nonzero inclusion maps are $+1$. The surviving row is exactly the ordinary simplicial chain complex of $K$. Hence

$$H_n(U, 2^Z) = H_{n-\ell+1}(K), \qquad \widetilde{H}_i(\Gamma) = \widetilde{H}_{i-\ell}(K).$$

For the second identity, use excision $\widetilde{H}_i(\Gamma) = H_i(B, U)$ and the triple $2^Z \subseteq U \subseteq B$; the map $H_0(K) \to H_{\ell-1}(B, 2^Z) = \mathbb{Z}$ is augmentation. All statements hold integrally, since the spectral sequence has only one nonzero row. Finally, with $N = |V|$, a basis vector in the squarefree Koszul strand is $x^{V \setminus F} e_F$, and it survives precisely when $V \setminus F$ is a face of $\Delta$. Thus homological degree $i = |F|$ corresponds to cochain degree $N - i - 1$. The strand is the augmented *cochain* complex of $\Delta$, with this reversed indexing. Integral combinatorial Alexander duality gives

$$\operatorname{Tor}_i^A(A/C, \mathbb{Z})_\sigma = \widetilde{H}^{N-i-1}(\Delta; \mathbb{Z}) = \widetilde{H}_{i-2}(\Gamma; \mathbb{Z}),$$

which proves the formula. The cohomology convention is essential for placing integral torsion in the correct degree. □

**Proposition 10.2.** *There is a sublattice $E \subseteq \{0,1,2\}^{30}$ with 1087 elements such that, writing $I_E \subseteq S = \Bbbk[x_{e,j} : 1 \le e \le 30,\ 0 \le j \le 2]$,*

$$\beta_{13,90}(I_E) = \beta_{14,90}(I_E) = \begin{cases} 1, & \operatorname{char} \Bbbk = 2, \\ 0, & \operatorname{char} \Bbbk \neq 2. \end{cases}$$

*Over $\mathbb{Z}$, its fine Koszul homology in the product $\Omega$ of all 90 variables is $\mathbb{Z}/2$ in degree 13 and zero otherwise.*

*Proof.* Take the complex $K$ on $R = [6]$ with facets and minimal nonfaces

$$\begin{aligned} \text{facets} \quad & 123, 124, 135, 146, 156, 236, 245, 256, 345, 346, \\ \text{nonfaces} \quad & 125, 126, 134, 136, 145, 234, 235, 246, 356, 456. \end{aligned}$$

Every four-element set contains a listed nonface. Its one-skeleton is complete. Collapse the spanning tree with edges $1j$, $2 \le j \le 6$. The first five facets kill the cycle-basis elements $23, 24, 35, 46, 56$. For the remaining elements $a = 25, b = 26, c = 34, d = 36, e = 45$, the last five boundaries impose

$$d - b = e - a = a - b = e + c = c - d = 0.$$

Thus $\widetilde{H}_1(K; \mathbb{Z}) = \mathbb{Z}/2$ and all other reduced integral homology vanishes; these ten boundaries have full rational rank, with determinant $-2$ on the cycle basis. Over characteristic 2, reduced homology has dimension one in degrees $1, 2$; over every other field it vanishes.

Let $L$ index the ten nonfaces, and set $N(a)$ equal to the indexed triple. Then $G$ has 30 edges, with degrees 3 on $L$ and 5 on $R$. Put every element of $L$ below every element of $R$ in a height-two poset $P$; the edges of $G$ select only the coordinate chains. For each edge $e = ar$, record the intersection of an order ideal with the chain $a < r$. Its size is a coordinate in $\{0, 1, 2\}$. The image is the required sublattice; it consists of the vectors

$$(\mathbf{1}_{a \in D})_{ar} \quad (D \subseteq L), \qquad (1 + \mathbf{1}_{r \in F})_{ar} \quad (F \subseteq R).$$

The stages meet only at the all-one vector, so $|E| = 2^{10} + 2^6 - 1$. The coordinate map is injective and preserves meet and join, since every poset element occurs in a selected chain.

At the all-two top element, the exact colon is obtained from $C$ in Lemma 10.1 by

$$z_a \longmapsto \prod_{r \in N(a)} x_{ar,0}, \qquad x_{ar} \longmapsto x_{ar,1},$$

and adjoining the unused variables $x_{ar,2}$. Indeed the lower stage gives the product of the two-generated row-block ideals, and the upper stage gives the $W_r$. This disjoint monomial substitution is free; in each new block, divide a monomial by the largest power of the complete block product; the remaining monomials form a basis over the old polynomial ring. It therefore preserves resolutions and their minimality over fields, and identifies integral Tor after augmentation. The full old degree becomes $\sigma' = \prod_e x_{e,0} x_{e,1}$. Equation (10.1) gives integral Tor $\mathbb{Z}/2$ in degree 13 at $\sigma'$.

To isolate this group in $I_E$, use the exact prefix sequences $0 \to F_{<a} \to F_{\le a} \to S/C_a(-\deg u_a) \to 0$ over $\mathbb{Z}$. If $a \neq \text{top}$, then $\Omega - \deg u_a$ contains some $x_{e,2}$ with $a_e < 2$,

while $C_a$ uses only variables below $a_e$ in that row by Lemma 2.1. A polynomial extension has no Koszul homology of positive fine degree in an unused variable. The long exact sequences therefore identify $\operatorname{Tor}_i^S(I_E,\mathbb{Z})_\Omega$ with $\operatorname{Tor}_i^S(S/C_{\mathrm{top}},\mathbb{Z})_{\sigma'}$. Universal coefficients give the asserted field dimensions. Since $I_E$ is squarefree, $\Omega$ is the only possible fine Betti degree of ordinary degree 90. □

The construction depends only on the incidence between vertices and minimal nonfaces. Replacing the ten listed nonfaces by those of any finite complex gives the following general form; the same top-colon summand survives in every power.

**Corollary 10.3** (Realization of finite-complex homology)**.** *Let $K$ be a finite simplicial complex on its nonempty vertex set $R$, with no cone vertices, and let $\mathcal{N}$ be its set of minimal nonfaces. Put $\ell = |\mathcal{N}|$ and $M = \sum_{F\in\mathcal{N}} |F|$. There is a sublattice $E \subseteq \{0,1,2\}^M$ with $2^\ell + 2^{|R|} - 1$ elements such that, over $S_\mathbb{Z} = \mathbb{Z}[x_{e,j} : 1 \le e \le M,\ 0 \le j \le 2]$,*

$$\operatorname{Tor}_i^{S_\mathbb{Z}}(I_E^m,\mathbb{Z})_{\alpha_m} \cong \widetilde{H}_{i-\ell-2}(K;\mathbb{Z}), \qquad \alpha_m = \prod_{e=1}^{M} x_{e,0}x_{e,1}x_{e,2}^m,$$

*for every $m \ge 1$. In particular $\alpha_1$ is the full squarefree degree. Over every field,*

$$\beta_{i,\alpha_m}(I_E^m;\Bbbk) = \dim_\Bbbk \widetilde{H}_{i-\ell-2}(K;\Bbbk).$$

*Proof.* Use $L = \mathcal{N}$ and $N(F) = F$. The graph has no isolated vertices; each minimal nonface is nonempty, and a vertex absent from every minimal nonface would be a cone vertex. Apply the same two-stage sublattice construction, free substitution, and integral prefix argument as in Proposition 10.2, using Lemma 10.1. No coefficient colon contains any $x_{e,2}$. Hence the exponents $m$ of all these variables force every factor of the contributing multichain to be the top element. The integral chain formula of Theorem 7.1 therefore consists of the single top-colon summand in degree $\alpha_m$. The field assertion follows by universal coefficients, or by the coefficient-ring version of that theorem. □

For every $m \ge 1$, Proposition 10.2 therefore gives a characteristic-2 excess of at least one at each of $(i,q) = (13, 30m+60)$ and $(14, 30m+60)$. In fact the fine degree $\alpha_m = (\prod_e x_{e,0}x_{e,1})(\prod_e x_{e,2}^m)$ has Betti number one in these two homological degrees in characteristic 2 and zero otherwise. Integrally, its Tor group is exactly $\mathbb{Z}/2$ in homological degree 13 and zero in every other degree.

Characteristic dependence thus persists under passage to higher powers. Conversely, the crossing blocks cannot create torsion when all first-power coefficient quotients are torsion-free. These two facts give a second first-power criterion, this time for independence of the characteristic.

**Corollary 10.4.** *For a finite sublattice $E$, characteristic independence of the Betti table of one positive power of $I_E$ is equivalent to characteristic independence of all its powers. More precisely, for a prime $p$, fixed $i, j$, and row size $d$, the difference*

$$D_{i,j,p}(m) = \beta_{i,dm+j}(I_E^m; \mathbb{F}_p) - \beta_{i,dm+j}(I_E^m; \mathbb{Q}) = \sum_{r \geq 1} A_{i,j,p,r} \binom{m-1}{r-1}$$

*has nonnegative integer coefficients $A_{i,j,p,r}$. In particular all its forward differences are nonnegative for $m \geq 1$.*

*Proof.* Every fine component of the integral Koszul complex of a monomial quotient is free over $\mathbb{Z}$, so universal coefficients make its characteristic-$p$ Betti excess nonnegative, both finely and ordinarily. Grouping the chain formula by strict supports proves the displayed expansion; $A_{i,j,p,r}$ is the sum of the excesses of $S/C_\tau$ over strict chains $\tau$ of length $r$. Pascal's identity gives the forward-difference assertion.

If one power is independent, its constant-chain summands force every $S/C_a$ to be independent, by nonnegativity. Ordinary independence implies fine independence by the same argument degree by degree. Conversely, the crossing decomposition has $R_s = C_w \cap \Bbbk[X_s] = C_{a_s} \cap \Bbbk[X_s]$. The fine Koszul strands supported on $X_s$ are precisely those of the corresponding variable restriction of $S/C_{a_s}$. Hence independence of all $S/C_a$, Künneth on the disjoint blocks, and the chain formula give independence of every power. □

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