Barycentric invariance and universal parameters for face rings

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Abstract. We prove that barycentric subdivision preserves a face ring as a module over its universal parameters, for every finite simplicial poset and every commutative coefficient ring. The comparison is integral, commutes with all face restrictions, and induces the canonical straightening map on associated graded; its filtered structure proves the equivariant Tor conjecture of Adams and Reiner in every characteristic. Equivariance of the comparison itself has a sharp obstruction; in rank d, the universal condition is invertibility of $(d-1)!$. The construction lifts comparisons from the boundary of each face, giving relative resolutions and extending to homogeneous straightening algebras, including arbitrary unions of Grassmannian Richardson varieties.

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1. **Introduction**

Barycentric subdivision preserves the topology of a simplicial poset while changing the multiplication in its face ring, but the two rings still carry a common system of universal parameters. Adams and Reiner conjectured that their equivariant Betti numbers over this parameter ring agree [1], while Murai asked whether the rings themselves are isomorphic as parameter modules [2].

We construct integral module isomorphisms compatible with restriction to every face, over every commutative coefficient ring and without a Cohen–Macaulay hypothesis. Their filtered structure proves the equivariant Tor formula in every characteristic, even though the comparison itself need not be equivariant. The obstruction to making the isomorphisms themselves equivariant is also determined exactly; for rank at most d, the universal condition is that $(d-1)!$ be invertible.

Let P be a finite simplicial poset with minimum 0 and maximum rank d. Thus each interval $[0, F]$ is a Boolean lattice. Let K be a commutative ring, and write

$$R_K(P) = K[P], \qquad B_K(P) = K[\operatorname{Sd} P], \qquad A_K = K[z_1, \dots, z_d], \quad \deg z_i = i.$$

The generator u_F of the face ring has degree $\operatorname{rk} F$, as does the barycentric vertex v_F, for $F > 0$. The parameter actions are

$$z_i \longmapsto \theta_i = \sum_{\operatorname{rk} F = i} u_F, \qquad z_i \longmapsto \gamma_i = \sum_{\operatorname{rk} F = i} v_F.$$

On a rank-r face, the first action is by the elementary symmetric polynomials in its r vertex variables. The second is by color sums in the order complex, colored by rank.

A comparison is *coherent* if it commutes with restriction to every face. Such comparisons give compatible maps on every lower ideal of P and on every kernel of a restriction. The *rank-weight filtration* gives a rank-i face variable weight $i(2d-i)$. Straightening lowers this weight, so chain monomials canonically identify $\operatorname{gr} R_K(P)$ with $B_K(P)$.

Theorem 1.1. *For every finite simplicial poset P there are coherent graded A_K-module isomorphisms*

$$R_K(P) \simeq B_K(P)$$

over every commutative ring K. They can be constructed over $\mathbb{Z}$ and then base-changed. They preserve the rank-weight filtration, with the canonical chain-monomial identification on associated graded. The same comparisons identify every relative restriction kernel.

Blum-Smith and Marques constructed equivariant comparisons for Cohen–Macaulay complexes when the group order is invertible, and exhibited modular obstructions [2]. Theorem 1.1 answers the nonequivariant module question in full. Its construction uses freeness only on individual faces, where the parameters are elementary symmetric polynomials. Coherence then assembles these local comparisons even when the global face ring is not free.

Let G be a finite group of automorphisms of P. For a graded representation we write $[-]$ for its class in the Grothendieck group $R_k(G)$. For $S \subset [d]$, let $(\mathrm{Sd}\, P)|_S$ denote the rank-selected order complex.

Theorem 1.2. *There is a finite G-stable filtration in each internal degree with canonical equivariant identification*

$$\operatorname{gr} \operatorname{Tor}_m^{A_K}(R_K(P), K)_j \simeq \bigoplus_{\substack{S \subset [d] \\ \sum_{s \in S} s = j}} \widetilde{H}^{|S|-m-1}((\mathrm{Sd}\, P)|_S; K).$$

It is compatible with multiplication on associated graded Tor. For a G-stable lower ideal Q, the relative version uses $\widetilde{H}^{|S|-m-1}((\mathrm{Sd}\, P)|_S, (\mathrm{Sd}\, Q)|_S; K)$. Over a field this gives the corresponding identity in $R_k(G)$.

The equivariant Tor formula holds even when an equivariant module isomorphism does not exist. A single simplex gives the exact obstruction to such isomorphisms in a fixed rank.

Theorem 1.3. *For $d \geq 1$, coherent graded equivariant parameter-module comparisons exist over K for every simplicial poset of maximum rank at most d and every automorphism group if and only if $(d-1)!$ is a unit in K. If it is not a unit, the polynomial and barycentric rings of the rank-d simplex admit no S_d-equivariant parameter-module isomorphism, even without grading or coherence.*

We also determine the integral obstruction; on a rank-d simplex, the possible constant scalars of coherent S_d-equivariant graded parameter-linear maps form exactly the ideal $(d-1)!K$.

The comparison theorem is proved locally on faces; quotienting by the top parameter leaves exactly the boundary diagram, where a comparison has already been constructed. Freeness lifts that map to the face and positive degree makes the lift invertible, while bases adapted to straightening preserve the canonical leading terms. This boundary-lifting argument is also the reason the theorem extends to relative modules and survives arbitrary base change.

The same construction controls parameter changes and their fibers. For the polynomial face parameters of Section 7, the module depends only on the degree profile, and its Betti numbers at a character over a field depend only on the highest active rank. Consequently its nontrivial rank-jump loci are nested coordinate subspaces.

Local freeness and the boundary quotients also make the lifting argument available for homogeneous straightening algebras and arbitrary unions of Grassmannian Richardson varieties.

The factorial obstruction concerns comparisons uniform over all actions; particular actions can admit integral equivariant lifts without that inversion. Compatible choices on faces and their fixed-face quotients give such lifts for free actions with cyclic Sylow subgroups, using integral templates natural for cyclic orders. The relative construction in Appendix A extends this lifting method to further actions.

2. **Boundary lifting and the coherent comparison**

2.1. **The face diagrams**

For $F \in P$ of rank r, put

$$R_F = K[x_v : v \text{ an atom below } F], \qquad B_F = K[\mathrm{Sd}[0, F]], \qquad A_r = K[z_1, \dots, z_r].$$

Restrictions kill vertex variables or face generators outside the smaller face. Include the constant stalk $R_0 = B_0 = K$. Then

$$R_K(P) = \varprojlim_{F \in P} R_F, \qquad B_K(P) = \varprojlim_{F \in P} B_F. \tag{1}$$

The constant stalk keeps the components compatible when the complex is disconnected. Both descriptions follow coefficientwise from the standard bases; weak chains of nonzero faces, together with the empty monomial. A chain coefficient is detected at its largest face; compatibility there matches that coefficient in every containing stalk.

The face-ring relations are

$$u_F u_G = u_{F \wedge G} \sum_H u_H$$

when F, G have common upper bounds, the sum being over their minimal upper bounds, with $u_0 = 1$; otherwise the product is zero. On each containing Boolean face this is the ordinary product identity in its vertex variables. Sorting their exponents gives the chain basis and proves (1) over $\mathbb{Z}$ and over every K.

On a rank-r stalk, z_r acts as $e_r = \prod_{v \text{ atom}, v \le F} x_v$ on R_F and as v_F on B_F. Both are regular, and

$$R_F/e_r R_F = \varprojlim_{E<F} R_E, \qquad B_F/v_F B_F = \varprojlim_{E<F} B_E. \tag{2}$$

Indeed, the common kernel of proper-face restrictions is spanned by the chain monomials containing the top face. On the barycentric side $B_F = B(\partial F)[v_F]$.

For a two-vertex face, this says

$$R_F = K[x, y], \qquad B_F = K[v_x, v_y, v_F]/(v_x v_y),$$

with parameters $(x + y, xy)$ and $(v_x + v_y, v_F)$. Setting the second parameter to zero identifies both rings with the boundary ring $K[x, y]/(xy) = K[x] \times_K K[y]$. The free parameter bases $\{1, y\}$ and $\{1, v_y\}$ then lift this boundary identification by $1 \mapsto 1$, $y \mapsto v_y$. This illustrates the general construction; a free basis lifts the comparison already defined on the boundary, preserving the parameter action without requiring the two multiplications to agree.

Lemma 2.1 (Boundary lifting)**.** *Let A be positively graded, let M be a finite graded-free A-module, and let N be bounded below. Suppose a homogeneous $t \in A$ of positive degree is regular on both modules. Every graded isomorphism $M/tM \to N/tN$ lifts to a graded A-module isomorphism $M \to N$.*

Proof. Lift the prescribed image of each homogeneous free generator and extend A-linearly. The resulting map is surjective by induction on degree modulo t. If an element of its kernel is tm, regularity on N puts m in the kernel too. Repeated division lowers degree, so boundedness below makes the kernel zero. □

The lemma applies to any finite boundary diagram; on each nonzero stalk require a finite graded-free source over its local parameter ring, a regular positive boundary parameter on both sides, and quotients equal to the actual inverse limits of the proper stalks. Induction constructs coherent comparisons. Every coherent family whose scalar at the constant stalk is a unit is automatically an isomorphism, by the same induction.

2.2. **Adapted bases**

Recall the rank weights $w_i = i(2d - i)$. If incomparable F, G have ranks a, b, and their meet has rank c, every straightening term has other rank $a + b - c$, so the weight drops by

$$w_a + w_b - w_c - w_{a+b-c} = 2(a - c)(b - c) > 0.$$

Thus the chain bases give a canonical identification

$$\operatorname{gr} R_K(P) = B_K(P), \qquad \operatorname{in}(\theta_i) = \gamma_i, \tag{3}$$

compatible with every restriction. All filtrations are finite in each internal degree.

Order the r atoms of a face by $[r]$. For $\sigma \in S_r$, let $D(\sigma)$ be its descent set and $T_i(\sigma)$ its first i atoms. Set

$$b_\sigma = \prod_{i \in D(\sigma)} v_{T_i(\sigma)}, \qquad a_\sigma = \prod_{i \in D(\sigma)} \prod_{v \in T_i(\sigma)} x_v.$$

Neither expression involves the top face.

Lemma 2.2. *The b_σ form an integral free basis of B_F over its color sums. The a_σ form an adapted free basis of R_F over its elementary symmetric parameters, with leading symbols b_σ.*

Proof. Fix a color multidegree with support S. Its chain monomials correspond to chains whose selected ranks are S. Sorting each block between successive selected ranks gives a unique permutation τ with $D(\tau) \subset S$. Expand the parameter monomials times b_σ in this chain basis, and order permutations lexicographically. The permutation σ is the lexicographically least one with its required descent prefixes; its diagonal coefficient is 1. The matrix is unitriangular. This proves the first assertion over $\mathbb{Z}$, without division.

Transfer each chain monomial to the source. The corresponding map from a filtered free parameter module has associated graded the first basis isomorphism, by (3). Finite filtration induction in each degree makes the map an isomorphism, proving the second assertion and its compatibility with base change. □

Proof of Theorem 1.1. Induct over faces, using the maps on proper faces as the prescribed boundary map in (2). For each a_σ, apply this map to the boundary class and lift its image by the representative containing no v_F. Extend over A_r. Lemma 2.1 makes this an isomorphism and ensures coherence. The boundary map has the canonical leading symbols by induction, and b_σ contains no top variable; hence its chosen lift still has leading symbol b_σ. Lemma 2.2 gives a filtered isomorphism with canonical associated graded. Constructing over $\mathbb{Z}$ and tensoring the maps and their inverses gives the assertion over every K. Inverse limits and restriction kernels preserve these isomorphisms. □

The correction at the boundary is necessary. The uncorrected assignment $a_\sigma \mapsto b_\sigma$ need not commute with face restrictions; the lift above imposes precisely those compatibilities.

A more separated system of weights makes triangularity automatic for every normalized coherent comparison.

Proposition 2.3. *Every coherent graded comparison normalized to* 1 *on constants is automatically triangular for the weights*

$$\nu_i = 2^d i - (2^i - 1).$$

Its associated graded map is the canonical chain-monomial map.

Proof. These weights also strictly decrease under straightening; with the ranks above, the decrease is $2^c(2^{a-c} - 1)(2^{b-c} - 1) > 0$. At fixed internal degree, decreasing weight is equivalent to increasing the cost $\sum(2^i - 1)$ of the factors. An adapted generator on a rank-r face has cost at most $\sum_{i<r}(2^i - 1) < 2^r - 1$. Inductively, any discrepancy in its leading symbol vanishes on the boundary and is therefore divisible by v_F. Such a discrepancy has cost at least $2^r - 1$, so has smaller weight. Extend over the parameters and induct on rank. □

3. **Choices and equivariant filtrations**

Although the comparison need not be unique or equivariant, its choices form an affine space whose differences are controlled by a nilpotent ideal. This structure lets us compare equivariant invariants even when no equivariant comparison exists.

Choose an integral homogeneous free basis $\mathcal{A}_F$ of each polynomial stalk over its elementary symmetric parameters. A prescribed boundary image of $a \in \mathcal{A}_F$ has a unique top-free lift, and every other lift is obtained by adding $v_F h_a$, where $h_a \in (B_F)_{\deg a - \operatorname{rk} F}$. Thus the module $E_K(P)$ of coherent degree-zero parameter-linear families has coordinates

$$E_K(P) \simeq K \oplus \bigoplus_{F>0} \bigoplus_{a \in \mathcal{A}_F} (B_F)_{\deg a - \operatorname{rk} F}. \tag{4}$$

The first coordinate is its scalar at 0; write $\epsilon : E_K(P) \to K$ for this map. This is a finite free module, and the coordinates commute with every base change. Its unit-scalar points are precisely its isomorphisms.

Let $C_K(P)$ be the algebra of degree-zero coherent endomorphisms of the source diagram, and let J be its ideal of maps zero at 0. A fixed normalized comparison identifies its underlying module with (4).

The key constraint on a difference is boundary vanishing; once it vanishes on all proper faces, its image is divisible by the top parameter. Repeated compositions force

further factors of positive degree, while the degrees of a free basis remain bounded, which is why the ambiguity in the comparison is nilpotent.

Lemma 3.1. *The ideal J is nilpotent. One may take*

$$J^{N_d} = 0, \qquad N_d = \prod_{r=1}^{d} \left\lceil \frac{r}{2} \right\rceil, \quad N_0 = 1.$$

Proof. A product of N_{r-1} elements vanishes on all proper faces of a rank-r face, by induction. Its component there has image in $e_r R_F$, and so is $e_r T$ for a parameter-linear map T lowering degree by r. A free basis has degrees at most $D_r = r(r-1)/2$. A product of $\lfloor D_r/r \rfloor + 1 = \lceil r/2 \rceil$ such blocks lowers degree past every generator, and is zero. □

Normalized comparisons form an affine space of isomorphisms; after choosing one they are $1 + J$, with inverses given by finite geometric series. Over a field $C_K(P)$ is a finite-dimensional local algebra with radical J. In ranks one and two, every free generator has degree below the rank, so the normalized coherent comparison is unique and hence equivariant. In rank three, each face already contributes one free coefficient.

Proposition 3.2. *For a finite group acting coherently on a boundary diagram, the source and target modules have finite filtrations with identical equivariant successive quotients. The same holds for restriction kernels and after applying a K-linear additive covariant or contravariant functor to the modules and their group actions. The actions may be semilinear, permuting equal-degree parameters.*

Proof. Transport the target action through a normalized comparison Φ. The two actions then satisfy

$$\rho'_g = \Phi^{-1} \rho_g^B \Phi = a_g \rho_g, \qquad a_g \in 1 + J.$$

Their ratio is parameter-linear, coherent, and 1 at the constant stalk, even when both actions are semilinear for the same parameter permutation. Conjugation by ρ_g preserves J. The filtration $J^a U$ of a limit module or restriction kernel is therefore stable under both actions, which agree on its quotients. After an additive functor use powers of the image of J in its endomorphism algebra; composition and nilpotence are preserved. For a contravariant additive functor the opposite algebra gives the same argument. □

For any finite boundary diagram the nilpotence proof is unchanged; replace D_r by the maximal generator degree at that level and r by the degree of the regular

boundary parameter. This observation will also apply to interval diagrams and parameter twists.

Finally, the affine obstruction to an equivariant coherent choice is additive. Put $V_K = \ker(E_K(P) \to K)$. For any normalized comparison, $g\Phi - \Phi$ is a cocycle in V_K. Its class

$$o_P \in H^1(G, V_K) \tag{5}$$

is independent of the choice and vanishes precisely when an equivariant normalized comparison exists. This concerns coherent maps; an arbitrary isomorphism of global modules need not preserve face restrictions.

4. **Tor and rank-selected cohomology**

The comparison reduces the Tor calculation to a properly colored face module.

Let (Λ, Λ_0) be a finite properly d-colored simplicial pair. In the absolute case Λ_0 is the void complex. Write $B = K[\Lambda, \Lambda_0]$ for the relative face module and let z_i act by the sum of the vertices of color i. We first use the color multigrading $\deg z_i = \mathbf{e}_i$.

Lemma 4.1 (Colorful Koszul calculation)**.** *The color-multigraded Koszul homology is supported on squarefree degrees. In degree $\mathbf{1}_S$ it is canonically*

$$\operatorname{Tor}_m^A(B, K)_{\mathbf{1}_S} \simeq \widetilde{H}^{|S|-m-1}(\Lambda|_S, \Lambda_0|_S; K).$$

This is equivariant under color-preserving automorphisms.

Proof. In a strand whose color-i degree is at least two, a monomial already contains the unique vertex of that color in its support. Multiplication by the color sum simply raises that vertex's exponent. It is a bijection on the relevant coefficient modules, and its inverse gives a one-coordinate Koszul contraction.

In a squarefree strand S, a basis element is a face F with colors $T \subset S$, together with the exterior factor on $S \setminus T$. The Koszul differential inserts one missing-colored vertex and sums over its possible cofaces. This is the augmented relative cochain complex of the rank selection, with degree $|T| - 1 = |S| - m - 1$. The sign change

$$[F] \otimes \epsilon_{S\setminus T} \longmapsto (-1)^{\sum_{r\in S\setminus T} |\{s\in S : s<r\}|} F^*$$

identifies its differential with the cochain differential. The construction commutes with every color-preserving automorphism. □

Proof of Theorem 1.2. Give the ith Koszul exterior generator weight w_i. The filtered comparison in Theorem 1.1 gives a filtered chain isomorphism to the barycentric

Koszul complex, whose differential is homogeneous for weight. Thus kernels and images are direct sums of their weight pieces on the barycentric side. Transporting this fact back proves strictness; every boundary of a given filtration degree has a preimage of that degree. Taking homology therefore commutes canonically with associated graded. Lemma 4.1 computes that homology. The filtration and its leading symbols are G-equivariant, even though the comparison need not be.

The source Koszul complex is a filtered differential graded algebra, with the barycentric Koszul algebra as associated graded. Leading symbols of products of cycles are products of their leading symbols, and strictness identifies the leading boundaries. Consequently the canonical homology identification is multiplicative. Coherence gives the same strict chain comparison for every relative restriction kernel. □

The same argument makes the Rees deformation constant as a parameter module; its Rees module is isomorphic to $B_K(P)\otimes_K K[t]$ over the correspondingly homogenised parameter ring. Its Koszul homology is $\operatorname{Tor}^{A_K}(B_K(P), K)\otimes_K K[t]$ and commutes with every specialization of t. This is a module assertion; the algebra multiplications need not be isomorphic.

Example 4.2. The filtration can be nonsplit. Over $k=\mathbb{F}_2$, take the two triangles abc and abd, with the involution interchanging a, b. The face ring is $k[a,b,c,d]/(cd)$. Set $u=a+b$, $v=b$ and eliminate $d=u+c$ in the universal-parameter quotient. Its relations become

$$v^2+uv+u^2=0, \qquad c^2+uc=0, \qquad u^3=0.$$

Its degree-three basis is u^2v, u^2c, uvc. The involution fixes the first two and sends the third to $uvc+u^2c$, so its difference from the identity has rank one. On the barycentric side the same degree is

$$\widetilde{H}^0((\operatorname{Sd} P)|_{\{3\}}; k)\oplus \widetilde{H}^1((\operatorname{Sd} P)|_{\{1,2\}}; k),$$

a three-dimensional trivial representation. The two modules have the same representation class but are not isomorphic.

4.1. **Permutations of the colors**

Suppose G also permutes the colors, preserving positive degrees δ_i, and give every vertex of color i degree δ_i. For $S\subset[d]$, put $\operatorname{or}(S)=\bigwedge^{|S|}K^S$; its stabilizer G_S acts by the sign of its permutation of S.

Proposition 4.3. *There is an actual KG-module isomorphism*

$$\operatorname{Tor}_m^A(B,K)_j \simeq \bigoplus_{\substack{[S]\in G\backslash 2^{[d]}\\ \sum_{s\in S}\delta_s=j}} \operatorname{Ind}_{G_S}^G \left(\widetilde{H}^{|S|-m-1}(\Lambda|_S,\Lambda_0|_S;K)\otimes_K \operatorname{or}(S)\right).$$

For a source boundary diagram coherently comparable to B, the same formula gives the class of its Tor representation over a field.

Proof. Repeated-color strands contract as before. In a squarefree strand, write T for the face colors and $U = S \setminus T$, with $h = |T|$. Use the natural map

$$\bigwedge^{|U|} K^U \longrightarrow \operatorname{Hom}_K\left(\bigwedge^h K^T, \bigwedge^{|S|} K^S\right), \qquad \eta \longmapsto \left[\xi \mapsto (-1)^{h(h-1)/2}\xi\wedge\eta\right].$$

It identifies the summand with its oriented cochain line tensored with $\operatorname{or}(S)$, naturally under color permutations. Moving r from U to T changes the shuffle sign by $(-1)^{|\{s\in S: s<r\}|+h}$; the Koszul sign and the change $(-1)^h$ of the displayed degree factor leave exactly the cochain sign. Group the resulting summands into G-orbits.

For the source, let G act on Koszul exterior generators by $g\epsilon_i = \epsilon_{gi}$. The transported actions still differ by the image of $1+J$, so Proposition 3.2 gives equal representation classes. □

The orientation line matters. For two isolated vertices of distinct colors interchanged by an involution, both reduced H^0 and $\operatorname{or}(\{1,2\})$ are sign lines. Their tensor product is trivial, as is the Tor class of the relation x_1x_2.

5. **Resolutions and coefficient diagrams**

The colorful Tor formula comes from a resolution whose differential uses only face incidences; constructing it explicitly will also give minimal resolutions and allow coefficients that vary from face to face.

5.1. **The incidence resolution**

For $S \subset [d]$, let M_S be the free K-module on faces of $\Lambda \setminus \Lambda_0$ with color set exactly S. For $S \subset T$, define

$$u_{S,T}[F] = \sum_{\substack{F'\supset F\\ \operatorname{colors}(F')=T}} [F'].$$

Proper coloring gives $u_{T,U}u_{S,T} = u_{S,U}$. The relative complement is upward closed, so this also holds for relative modules.

Adding a missing color to a face monomial can be expressed either by multiplication by its parameter or by summing the cofaces obtained by inserting a vertex of that color. Since these operations agree in B, the resolution records their difference, with one interval coordinate for each color still to be added. Its exactness will be the contraction of these intervals onto their upper endpoints.

Put

$$Q_i = \bigoplus_{\substack{S\cap U=\varnothing \\ |U|=i}} A(-\mathbf{1}_{S\cup U}) \otimes_K M_S$$

and define

$$\partial[m;S,U] = \sum_{r\in U} (-1)^{|\{u\in U : u<r\}|} \big([u_{S,S\cup\{r\}}m; S\cup\{r\}, U\setminus\{r\}] \\ - z_r[m;S,U\setminus\{r\}]\big)$$

Proposition 5.1. *The complex $(Q_\bullet, \partial)$ is a finite graded free resolution of B over every K. The augmentation sends $[[F];S,\varnothing]$ to the face monomial x_F. The resolution is equivariant under color-preserving automorphisms.*

Proof. Endpoint operations in distinct coordinates commute and their orientation signs anticommute, giving $\partial^2 = 0$. In color degree β with support T, remove the coefficient monomial $z^{\beta-\mathbf{1}_{S\cup U}}$ from each summand. The strand becomes the cellular chains of a cube; coordinates in S are fixed at 1, those in U vary, and the others are 0. The coefficient at that cell is M_S; the upper endpoint uses u and the lower endpoint uses minus the identity.

For $r \in T$, let P_r push to the upper r-face. It is the identity if $r \in S$, zero if $r \in U$, and the map u if r is absent. Define $H_r = 0$ in the first two cases and otherwise

$$H_r[m;S,U] = (-1)^{|\{u\in U : u<r\}|}[m;S,U\cup\{r\}].$$

The boundary formula gives $\partial H_r + H_r\partial = P_r - \mathrm{id}$. The commuting P_r contract the cube onto its all-one vertex M_T, which is the augmentation. The empty strand is the identity on $M_\varnothing$; negative strands vanish. Everything is integral and natural in the face incidences. □

For one color and $r \geq 1$ vertices, this resolution is simply

$$0 \longrightarrow A(-1) \xrightarrow{(-z_1,1,\ldots,1)} A \oplus A(-1)^r \longrightarrow B \longrightarrow 0.$$

Its single relation says $z_1 = x_1 + \cdots + x_r$. Cancelling one unit entry gives the free module $A \oplus A(-1)^{r-1}$. The minimal construction below performs the corresponding cancellations simultaneously in every color set.

For $\Lambda = \operatorname{Sd} P$, these bases are chains with specified rank sets. Composing the augmentation with the coherent comparison inverse gives the same resolution matrices for $R_K(P)$ and its relative modules. Its free rank in degree i is

$$\operatorname{rk}_A Q_i = \sum_{S \subset [d]} \binom{d - |S|}{i} \operatorname{rk}_K M_S.$$

An equivariant comparison gives an equivariant augmentation; no such choice is presumed in modular characteristic.

5.2. **Minimal differentials**

Over a field, we can remove the unit entries from this resolution while retaining explicit differentials. Write $\partial = d_0 + \delta$, where coface insertion d_0 preserves $T = S \cup U$ and the parameter part δ deletes one element of T. On each finite complex $C(T)$ choose cohomology splittings i, p, h with

$$pi = \mathrm{id}, \quad d_0 h + h d_0 = \mathrm{id} - ip, \quad hi = ph = h^2 = 0.$$

They preserve T. Put

$$F_i = \bigoplus_{T \subset [d]} A(-\mathbf{1}_T) \otimes_k \widetilde{H}^{|T|-i-1}(\Lambda|_T, \Lambda_0|_T; k), \tag{6}$$

To turn a chosen d_0-cohomology representative into a chain map for $d_0 + \delta$, correct it first by $-h\delta i$, then correct the new error by $h\delta h\delta i$, and continue. Every correction deletes a color, so the process terminates. This gives

$$I = (1 + h\delta)^{-1} i = \sum_{a=0}^{d} (-h\delta)^a i, \qquad D = p\delta I.$$

The sum is finite because each δ lowers $|T|$.

Proposition 5.2. *The complex $(F_\bullet, D)$, augmented through I and Q_0, is a minimal free resolution of B. Its block from T to $T' \subsetneq T$ is a scalar matrix times $z_{T \setminus T'}$. The same matrices resolve the corresponding source face module.*

Proof. The identity $I = i - h\delta I$ gives

$$(d_0 + \delta) I = iD - h\delta (d_0 + \delta) I,$$

and hence $\partial I = ID$. Since $pI = \mathrm{id}$, the equality $ID^2 = \partial^2 I = 0$ gives $D^2 = 0$. The associated graded of I for the finite $|T|$ filtration is the chosen cohomology inclusion, a quasi-isomorphism. Thus I is a quasi-isomorphism. Every term of D contains

a parameter, making the resolution minimal. Each step deletes one distinct color, giving the squarefree monomial in the stated block. □

The matrices require only face incidences and finite cochain splittings, which may be chosen equivariantly when $|G|$ is invertible. Weighting color i by i changes the shift in (6) to $\sum_{i\in T} i$.

5.3. **Coefficient diagrams**

Let $V = (V_F, \rho_{FE})$ be any contravariant diagram of K-modules on P. Define

$$R(P;V) = \varprojlim_{F}(R_F \otimes_K V_F), \qquad B(P;V) = \varprojlim_{F}(B_F \otimes_K V_F).$$

Tensoring the actual stalk isomorphisms and their inverses gives a filtered comparison natural in V. No flatness is needed; after tensoring, the two maps are still mutual inverses on each stalk, and coherence lets them pass to the inverse limits. Thus the comparison retains arbitrary local coefficient data, rather than only the constant coefficients of a face ring.

For $H \in P$, put $C_H = \varprojlim_{F\geq H} V_F$. Restriction of components gives $C_H \to C_{H'}$ for $H \leq H'$. A chain monomial with largest face H has precisely the coefficient module C_H; the empty monomial has coefficient C_0. For $S \subset [d]$, define an augmented cochain complex $C_S(V)$; its degree -1 term is C_0, and its degree-q term is the sum of C_{H_q} over chains $H_0 < \cdots < H_q$ with ranks in S. Insertion below the largest face uses the identity; insertion of a new largest face uses the displayed coefficient map. Use the usual alternating signs and the augmentation $C_0 \to C_H$.

Proposition 5.3. *With $\widetilde{H}^q_S(V) = H^q(C_S(V))$, there is a canonical identification*

$$\operatorname{gr}\operatorname{Tor}^{A_K}_m(R(P;V),K)_j \simeq \bigoplus_{\substack{S\subset[d]\\ \sum S=j}} \widetilde{H}^{|S|-m-1}_S(V).$$

It is equivariant for every compatible action on the diagram.

Proof. The barycentric color decomposition has the coefficient modules just described. Raising and lowering a repeated-color exponent is the identity on that coefficient module, so the same Koszul contraction applies. A squarefree strand is exactly $C_S(V)$, with the sign change of Lemma 4.1. The filtered comparison and its inverse give strictness and the canonical associated-graded homology identification. □

6. **Straightening algebras and interval unions**

The comparison applies whenever chain monomials form a basis, the top parameter cuts out the boundary, and each stalk is free over its parameter ring. These properties let the same boundary construction pass to straightening algebras and, through their interval quotients, to arbitrary unions of Grassmannian Richardson varieties.

6.1. **Lower ideals**

The key condition is that straightening an incomparable product produces a strict upper factor; if one original factor lies outside a lower ideal, the upper factor does too, so killing the outside generators remains compatible with multiplication. Let L be a finite graded poset with ranks in $[d]$, such that every maximal chain in $L_{\le F}$ has all ranks $1,\dots,\rho(F)$. Fix positive degrees $\delta_1,\dots,\delta_d$. Suppose a commutative graded algebra $R_C(L)$ over a coefficient ring C has generators x_a of degree $\delta_{\rho(a)}$, with the weak-chain monomials and 1 forming a C-basis. Require that every nonzero standard term in the homogeneous expansion of $x_a x_b$, for incomparable a,b, contain a factor x_e with $e > a, b$. Assume also that each $\Delta(L_{\le F})$ is Cohen–Macaulay over every field. For a lower ideal $Q \subset L$, let $R_C(Q)$ be the quotient by generators outside Q, and put $D_C(Q) = C[\Delta(Q)]$. Include a separate constant stalk C, so the empty lower ideal has ring C. Let $A_C = C[z_1,\dots,z_d]$ act by rank sums on both rings, with $\deg z_i = \delta_i$.

Theorem 6.1. *Under these hypotheses there are coherent graded A_C-module isomorphisms $R_C(Q) \simeq D_C(Q)$ for all lower ideals, filtered with canonical chain-monomial symbols. They identify every relative restriction kernel. If the source is defined integrally, the maps may be constructed over $\mathbb{Z}$ and then base-changed. For rank-preserving symmetries over a field, the equivariant Tor formula is*

$$[\mathrm{Tor}_m^{A_k}(R_k(Q),k)_j] = \sum_{\substack{S\subset[d]\\ \sum_{s\in S}\delta_s=j}} [\widetilde{H}^{|S|-m-1}(\Delta(Q)|_S;k)],$$

with the analogous relative formula and multiplicative associated graded assertion.

Proof. Choose N large enough that $W_i = N\delta_i - 3^i > 0$. In a straightening relation, homogeneity cancels the $N\delta$ contributions. If $h = \max(\rho(a),\rho(b))$, the strict upper factor gives

$$\sum_\nu 3^{\rho(c_\nu)} \ge 3^{h+1} > 3^{\rho(a)} + 3^{\rho(b)}.$$

Thus every straightening strictly lowers weight. An outside factor of a lower ideal remains outside, because it is replaced by an upper factor. Its standard-monomial span

is therefore an ideal, and the quotient has exactly the chain basis in Q. Consequently $\operatorname{gr} R_C(Q) = D_C(Q)$, compatibly with restrictions and parameter symbols.

We need local freeness on the discrete side. Work first over $\mathbb{Z}$ at a principal ideal $L_{\leq F}$ of rank r. Its color sums form a system of parameters over every field; $y_a\gamma_i = y_a^2$ for $\rho(a) = i$, so their quotient is spanned by finitely many squarefree chains. Cohen–Macaulayness makes the parameters a regular sequence. The quotient's color-multigraded Hilbert series is the Hilbert series of the chain ring times $\prod_i (1 - t_i)$, independent of the field. Each integral quotient component therefore has the same dimension over $\mathbb{Q}$ and over every $\mathbb{F}_p$, and has no torsion.

Lift a color-homogeneous integral basis of this quotient. The lifts may omit y_F, because the stalk is $D_{\mathbb{Z}}(L_{<F})[y_F]$ and $\gamma_r = y_F$. Positive-degree induction gives generation over the parameter ring; over $\mathbb{Q}$ the Hilbert series gives freeness, and an integral relation would remain a relation over $\mathbb{Q}$. Thus the lifts are an integral adapted free basis. Base-change it to C, then transfer its chain expansions to the source. The associated-graded basis isomorphism proves source freeness by finite filtration induction.

The top variables x_F, y_F append the top face to a chain, so are regular, with quotients equal to the actual proper-stalk boundary limits. Apply the corrected boundary lift used in Theorem 1.1. The bases omit the top variable, so the lift has canonical leading symbols. This proves the module and relative statements. The strict filtered Koszul argument and Lemma 4.1 prove the remaining assertions. □

For a source defined only over one field, it suffices that the principal intervals be Cohen–Macaulay over that field; ordinary homogeneous-basis lifting replaces the integral torsion argument. The dual theorem, with a strict lower factor, applies to upper filters after reversing the poset and its degrees.

Quadratic laws with both strict endpoint factors and conserved rank sums admit the smaller weights $i(2d - i)$ and preserve both ordinary degree and rank degree. With only a strict upper factor, quadratic laws still satisfy the theorem in ordinary degree, using $W_i = 3^d + 1 - 3^i$. These distinctions concern the grading; the boundary construction is the same.

6.2. **Distributive lattices**

Let T be a finite poset and $L = J(T)$ its lattice of order ideals. Give x_I degree $|I| + 1$, including the genuine generator $x_\varnothing$. The relations

$$x_I x_J = x_{I \cap J} x_{I \cup J}$$

have the required endpoints and rank conservation. On a lower-ideal quotient a relation is zero if $I \cup J$ lies outside the quotient. The monomial map

$$x_I \longmapsto t \prod_{v \in I} u_v$$

distinguishes chain monomials; the exponents recover the nested ideals, and the exponent of t recovers the empty-ideal factors. The principal order complexes are shellable by lexicographic order of the linear extensions. The theorem therefore gives coherent comparisons for every lower ideal in $J(T)$, over every K.

6.3. **Interval diagrams**

Suppose each standard term in a homogeneous straightening contains both a factor strictly below a, b and a factor strictly above them. Assume every nonempty closed interval has Cohen–Macaulay order complex over every field. The intervals keep their original rank colors; they are not renumbered.

For an interval, a chain vanishes on every proper subinterval exactly when it contains both endpoints. The product of the two endpoint parameters therefore cuts out the boundary, taking the place of the single top parameter used for lower ideals. Once the comparisons are compatible on these boundaries, they glue on arbitrary unions of intervals.

Theorem 6.2. *There are filtered parameter-module comparisons*

$$R_K([U, V]) \simeq K[\Delta([U, V])]$$

simultaneously and compatibly on all closed intervals and on the constant stalk. They have canonical chain-monomial symbols. They also give compatible comparisons for arbitrary unions of interval order complexes and for all relative restriction kernels.

Proof. An index outside $[U, V]$ remains outside during straightening; failure to lie above U persists in the strict lower factor, and failure to lie below V persists in the strict upper factor. Thus interval quotients retain precisely their chain bases. The weights of Theorem 6.1 give their discrete associated graded.

For $a = \rho(U)$ and $b = \rho(V)$, the discrete interval stalk is free over $K[z_a, \dots, z_b]$ by the same integral Cohen–Macaulay argument. Its free basis may omit both endpoints, since for $U < V$ the stalk is $K[\Delta((U, V))][y_U, y_V]$. Transfer this basis to the source. Set $q_I = z_a z_b$ for $U < V$ and $q_I = z_a$ for a singleton. Multiplication by q_I appends the endpoint or endpoints to a chain and is regular. Its image is exactly the common kernel of all proper-interval restrictions. Coefficientwise, the quotient is their inverse limit, with the constant stalk included.

Induct on interval inclusion and apply Lemma 2.1 with $t = q_I$. Lift each boundary image using the chain representative not divisible by both endpoint variables. Because the free basis omits both endpoints, the lift retains its canonical leading symbol. This proves the simultaneous interval comparisons.

For a nonempty finite family $\mathcal{I}$ of intervals, put

$$\Lambda_{\mathcal{I}} = \bigcup_{I \in \mathcal{I}} \Delta(I), \qquad R_K(\mathcal{I}) = R_K(L) \Big/ \bigcap_{I \in \mathcal{I}} J_I, \quad J_I = (x_a : a \notin I).$$

The intersection ideal is spanned by chains contained in none of the intervals. Close the family under subintervals and add the constant stalk. An allowed chain is detected at the interval between its endpoints, so both rings are the inverse limits of these diagrams. The coherent interval maps give the comparison, compatibly with subunions and relative kernels. □

The complex $\Lambda_{\mathcal{I}}$ is the union of the order complexes, not in general the order complex on the union of their vertices; the latter can introduce mixed chains. The constant stalk records the common origin of the affine cones, and the strict Koszul and colorful arguments give all the absolute and relative Tor formulas for these unions. If quadratic relations conserve rank sums, the comparisons preserve ordinary degree and rank degree simultaneously.

6.4. **Grassmannian applications**

Let $L_{r,n}$ be the componentwise ordered lattice of r-subsets $I = (i_1 < \cdots < i_r)$ of $[n]$, with

$$\rho(I) = 1 + \sum_{a=1}^{r} (i_a - a).$$

The integral Plücker algebra has a standard basis of products of comparable maximal minors [3, Theorem 11.1]. For $\mathrm{Gr}(2,4)$, the relation

$$p_{14}p_{23} = p_{13}p_{24} - p_{12}p_{34}$$

already exhibits both endpoint conditions; each term on the right has one factor strictly below $14, 23$ and one strictly above them. The rank sums agree as well, since $3 + 3 = 2 + 4 = 1 + 5$. In the straightening of incomparable $p_I p_J$, the lower factor p_C has $C \leq I, J$. Column-content conservation gives $\mathbf{1}_C + \mathbf{1}_E = \mathbf{1}_I + \mathbf{1}_J$. For every prefix $[q]$, subtracting

$$|C \cap [q]| \geq \max(|I \cap [q]|, |J \cap [q]|)$$

from the content equality yields $|E \cap [q]| \leq \min(|I \cap [q]|, |J \cap [q]|)$, so $E \geq I, J$. Incomparability makes all endpoint inequalities strict. Summing the column indices proves conservation of rank sums. Every interval in $L_{r,n}$ is distributive and shellable.

Corollary 6.3. *The coordinate algebra of every nonempty union of Grassmannian Richardson varieties is coherently isomorphic, over the global rank-sum parameters, to the face ring of the union of its interval order complexes. The comparisons are integral and preserve both ordinary Plücker degree and rank degree. They commute with every subunion restriction and give the absolute and relative colorful Tor formulas.*

Proof. The Richardson variety with allowed indices $[U, V]$ has coordinate ideal $J_{[U,V]}$ [4, Proposition 3.4.2]. This is also immediate from the sum of the Schubert and opposite Schubert coordinate ideals and the common standard basis. Intersecting the ideals defines the scheme union, so Theorem 6.2 applies. □

Schubert unions are the special case with common minimal lower endpoint. With a fixed lower bound U, one may instead use $\rho_U(I) = 1 + \sum_a (i_a - u_a)$; the quadratic rank conservation survives this constant shift. No common endpoint is needed in Corollary 6.3.

Order-reversing symmetries are allowed in ordinary degree. More generally, suppose a symmetry transports intervals and their straightening laws and permutes colors while preserving the chosen degrees. Its boundary elements q_I are transported, so the semilinear argument of Proposition 3.2 and the orientation formula of Proposition 4.3 apply even when the straightening weights are not invariant. For example, reversing the ambient coordinates of a Grassmannian sends I to $(n+1-i_r, \ldots, n+1-i_1)$ and rank i to $d+1-i$. All minors acquire the same sign, removed by a common scalar change; unsigned index reversal is therefore an integral algebra symmetry. Every invariant Richardson union and relative subunion has the ordinary-graded equivariant Tor formula including this reversal.

7. **Parameter changes and their fibers**

The boundary construction also separates the degree of a parameter from its lower coefficients. Since a polynomial vanishing at the origin has the same support as the variable it replaces, a Newton basis change can be made simultaneously on every face. This compares polynomial parameters and lets us follow the parameter module away from the origin by computing its Tor at arbitrary characters.

7.1. **Polynomial face parameters**

Fix a profile of positive integers $q = (q_1, \dots, q_d)$. For each face F of rank i, choose

$$p_F(T) = a_F T^{q_i} + \text{lower terms}, \qquad a_F \in K^\times, \qquad p_F(0) = 0.$$

Let $A_K = K[z_1, \dots, z_d]$ act on the two face rings by

$$z_i = \sum_{\operatorname{rk} F = i} p_F(u_F), \qquad z_i = \sum_{\operatorname{rk} F = i} p_F(v_F).$$

For pure powers the action is graded with $\deg z_i = iq_i$; general polynomials give filtered modules.

Theorem 7.1. *The source and barycentric polynomial-parameter modules are coherently isomorphic over every K, compatibly with all relative kernels. Their parameter-module isomorphism type depends only on q. In particular, lower coefficients and unit leading coefficients introduce no new module types. The same assertion holds for the lower-ideal straightening algebras of Theorem 6.1.*

Proof. On the barycentric ring use the support-preserving Newton basis change

$$T\left(\prod_F v_F^{q_{\operatorname{rk} F} s_F + r_F}\right) = \prod_F v_F^{r_F} p_F(v_F)^{s_F}, \qquad 0 \le r_F < q_{\operatorname{rk} F}. \tag{7}$$

Triangularity with units on the diagonal gives an inverse by finite degree induction, while preservation of each exact chain support makes the map commute with restrictions. It intertwines the power action $z_i = \gamma_i^{q_i}$ with the polynomial action. To check this, if color i is already present, all other vertices of that color annihilate the monomial; if absent, both sides sum over exactly the compatible vertices of that color. The condition $p_F(0) = 0$ ensures support preservation.

For the source, use $C_r = K[z_1, \dots, z_{r-1}, t]$ on a rank-r face, with $t = u_F$ or v_F. For the power action the target has adapted basis

$$\gamma_1^{b_1} \cdots \gamma_{r-1}^{b_{r-1}} b_\sigma, \qquad 0 \le b_i < q_i,$$

with no top variable. Apply (7) in the lower colors, leaving t unchanged, and transfer the resulting chain expansions to the source. Their initial forms give precisely the same free module over the weighted-power parameters. Positive filtration induction proves that they are an adapted free C_r-basis.

The actual boundary quotients are obtained by setting $t = 0$. Lift each prescribed boundary image by its top-free representative and extend C_r-linearly. The lift is filtered with canonical leading symbols. Its associated-graded map is the basis

isomorphism already constructed, so finite induction on filtration degree makes the lift an isomorphism, including for nonhomogeneous p_F. Commutation with t gives commutation with the actual top parameter $p_F(t)$. Induction glues the comparison. For general straightening algebras use their adapted local bases, which likewise omit the top variable. □

The pure-power colorful formula has a useful relative form. For $0 \le \beta_i \le q_i$, put

$$S_\beta = \{i : \beta_i > 0\}, \qquad T_\beta = \{i : 0 < \beta_i < q_i\}, \qquad U_\beta = \bigcup_{t \in T_\beta} \Lambda|_{S_\beta \setminus \{t\}}.$$

An empty union is the void complex.

Proposition 7.2. *For the power action on a properly colored relative face module,*

$$\operatorname{Tor}_m^A(K[\Lambda, \Lambda_0], K)_\beta \simeq \widetilde{H}^{|S_\beta|-m-1}\big(\Lambda|_{S_\beta}, U_\beta \cup \Lambda_0|_{S_\beta}; K\big)$$

for $0 \le \beta_i \le q_i$, and all other strands vanish. For the source face ring this computes canonical associated-graded Tor. In the rank grading its internal degree is $\sum_i i\beta_i$.

Proof. If $\beta_i > q_i$, the color-i factor is already present on both sides of its Koszul multiplication; raising its exponent by q_i is a bijection and contracts that strand. If $0 < \beta_i < q_i$, the exterior generator cannot occur, and every face in the strand must contain color i. If $\beta_i = q_i$, that color may instead occur in the exterior factor. Thus the surviving strand is exactly the augmented relative cochain complex displayed above, with degree $|S_\beta| - m - 1$. The sign calculation is unchanged. Theorem 7.1 and strictness give the source assertion. □

For arbitrary polynomial parameters, compose the comparison with T^{-1}. Its associated graded identifies Tor with the same power-profile cohomology groups, as filtration quotients, without an internal-grading assertion. Equal weights group summands; one may separate them by choosing positive straightening weights outside the finitely many equality hyperplanes for these β. If $p_{gF} = p_F$, every leading identification is equivariant. The coefficient-diagram construction of Section 5.3 also extends; it tensors actual coherent maps and requires no additional flatness.

7.2. **Active colors and triangular symmetry**

We now work over a field k. For $a = (a_1, \dots, a_d)$, let k_a be the character of A_k with z_i acting as a_i. These Tor groups are ungraded when $a \neq 0$. We first use the ordinary color-sum action $z_i = \gamma_i$.

For a properly colored complex Λ, inverting a color sum separates the mutually annihilating vertices of that color into distinct factors. Doing this for all colors in $S \subset [d]$ selects a face with those colors; the unused vertices form its link. Explicitly,

$$k[\Lambda][\gamma_i^{-1} : i \in S] \simeq \prod_{\substack{\sigma \in \Lambda \\ \operatorname{colors}(\sigma) = S}} k[z_i^{\pm 1} : i \in S] \otimes_k k[\operatorname{lk}_\Lambda \sigma]. \tag{8}$$

The orthogonal idempotents are $e_\sigma = x_\sigma / \prod_{i \in S} \gamma_i$. Indeed, $x_v^2 = \gamma_i x_v$ for a color-i vertex; the idempotents sum to 1. In the σ factor the chosen active-color variable equals γ_i, the others vanish, and the remaining monomials are exactly the link faces. The formula is equivariant under color-preserving automorphisms.

Writing $S = \{i : a_i \neq 0\}$ and $T = [d] \setminus S$, regularity of the active Laurent parameters and Lemma 4.1 give

$$\operatorname{Tor}_m^{A_k}(k[\Lambda], k_a) \simeq \bigoplus_{\substack{\sigma \in \Lambda \\ \operatorname{colors}(\sigma) = S}} \bigoplus_{U \subset T} \widetilde{H}^{|U|-m-1}((\operatorname{lk}_\Lambda \sigma)|_U; k).$$

There is no homological shift from the active parameters; their Koszul complex has homology only in degree zero.

Upper-triangular changes of parameters will reduce an arbitrary character to its highest active rank; if $a_r \neq 0$ and $a_j = 0$ for $j > r$, adding multiples of z_r makes all earlier coordinates equal to 1, and scaling makes $a_r = 1$. Thus it is enough to understand the initial active colors $1, \dots, r$. We must first show that these changes preserve the parameter module and keep track of equivariant representation classes.

Lemma 7.3. *Suppose finite parameter modules M, N have nonnegative filtrations, and a filtered isomorphism with filtered inverse has equivariant associated graded. Then every additive K-linear functor gives objects with common finite equivariant filtrations and identical successive quotient actions.*

Proof. Let I be the algebra of parameter-linear endomorphisms of M lowering filtration by at least one. Choose finite generators of filtration at most D. A product of $D+1$ such maps kills the generators, hence $I^{D+1} = 0$. The transported action differs from the source by $1 + I$. Use powers of I, or of its image after the functor, as in Proposition 3.2. □

Proposition 7.4. *For the power action $z_i = \gamma_i^{q_i}$ on $B_K(P)$, every invertible upper-triangular change*

$$\tau(z_i) = \sum_{j \geq i} c_{ij} z_j, \qquad c_{ii} \in K^\times,$$

gives a coherently isomorphic ungraded parameter module. Over a field, finite-dimensional additive-functor outputs have the same equivariant representation classes for rank-preserving symmetries.

Proof. Diagonal units are removed by multiplying a chain monomial of color exponents b_i by $\prod_i c_{ii}^{\lfloor b_i/q_i \rfloor}$. This map is invertible, equivariant, and support-preserving. Assume henceforth that the diagonal is 1.

Put $L = \operatorname{lcm}(q_1, \dots, q_d)$ and give a color-i vertex auxiliary degree $\delta_i = L/q_i$. On a rank-r stalk use $C_r = K[z_1, \dots, z_{r-1}, t]$, where $t = v_F$. The source action is $z_i = \gamma_i^{q_i}$ and the target is $z_i = \sum_{j=i}^{r} c_{ij}\gamma_j^{q_j}$; all have degree L. Both boundary quotients set $t = 0$. The top-free adapted power basis used in Theorem 7.1 therefore permits the same corrected boundary lift. It commutes with t and hence with the top parameter t^{q_r}.

Give a vertex of color i weight $W_i = (d + 1 - i)\delta_i$. Then $q_i W_i = L(d + 1 - i)$ strictly decreases with i, so each twisted parameter has its original leading form. The corrected lift has the canonical leading symbol; its boundary does by induction, and a remaining discrepancy would contain the top variable, excluded by the choice of lift. It is therefore a filtered isomorphism with filtered inverse.

The global power module is generated by chain monomials whose present color-i exponents are between 1 and q_i, together with 1. Thus it is finite over A_K. Lemma 7.3 proves the representation assertion. □

For $q_i = 1$, this also gives the ordinary-graded triangular symmetry of $B_K(P)$ and, by the coherent comparison, an ungraded triangular symmetry of $R_K(P)$. The original rank grading is generally not preserved by τ.

7.3. **Only the highest active rank**

For the polynomial face parameters of Section 7, assume $p_{gF} = p_F$ and let $r = \max\{i : a_i \neq 0\}$, with $r = 0$ at the origin. For a rank-r face F, put $\Psi_F = \Delta(P_{>F})$ and let $\mathrm{Fl}(F)$ be the set of full flags in $[0, F]$. For $r = 0$ use $F = 0$, its empty flag, and $\Psi_0 = \operatorname{Sd} P$.

The ordinary-parameter case explains the contribution of each face

$$V^1_{F,m} = \bigoplus_{U \subset \{r+1,\dots,d\}} \widetilde{H}^{|U|-m-1}(\Psi_F|_U; k).$$

Localization at the initial active colors selects a full flag ending at F, leaving only faces above F available for the colorful Koszul calculation in Ψ_F. For higher powers, colors of intermediate exponent are required to occur in the face; as in Proposition 7.2, this replaces cohomology by relative cohomology.

For a vector β on the colors $i > r$ with $0 \le \beta_i \le q_i$, define S_β, T_β as before and put

$$U_{F,\beta} = \bigcup_{t \in T_\beta} \Psi_F|_{S_\beta \setminus \{t\}}, \qquad V^q_{F,m} = \bigoplus_{0 \le \beta_i \le q_i,\ i > r} \widetilde{H}^{|S_\beta|-m-1}(\Psi_F|_{S_\beta}, U_{F,\beta}; k).$$

Theorem 7.5. *For every character a and every field,*

$$[\operatorname{Tor}^{A_k}_m(R_k(P), k_a)] = \left(\prod_{i=1}^{r} q_i\right) \left[\bigoplus_{\operatorname{rk} F = r} k[\mathrm{Fl}(F)] \otimes_k V^q_{F,m}\right] \quad \text{in } R_k(G).$$

In particular,

$$\dim_k \operatorname{Tor}^{A_k}_m(R_k(P), k_a) = r! \left(\prod_{i=1}^{r} q_i\right) \sum_{\operatorname{rk} F = r} \dim_k V^q_{F,m}.$$

These dimensions and classes depend on a only through r, and vanish for $m > d - r$.

Proof. The comparison, its equivariant leading symbols, and Lemma 7.3 compare Tor at every character. The Newton basis change is an actual equivariant isomorphism, so we may use pure powers. Characters with the same highest active rank are conjugate under the invertible upper triangular group. By Proposition 7.4, it therefore suffices to take $a_i = 1$ for $i \le r$ and $a_i = 0$ otherwise.

The localization (8) splits the ring into factors indexed by full flags below rank-r faces. Their links are exactly Ψ_F, since no further face can be inserted below F. Each factor has the form

$$k[t_1^{\pm 1}, \dots, t_r^{\pm 1}] \otimes_k k[\Psi_F], \qquad z_i = t_i^{q_i} \quad (i \le r).$$

The active Koszul complex has only degree-zero homology

$$E_q = k[t_1^{\pm 1}, \dots, t_r^{\pm 1}]/(t_i^{q_i} - 1),$$

of dimension $\prod_{i \le r} q_i$, with trivial stabilizer action. No separability is needed; the successive monic polynomials are nonzerodivisors with nonzero constant terms. Proposition 7.2 computes the remaining colors as $V^q_{F,m}$. There are $r!$ flags in the Boolean interval. Finally $|S_\beta| \le d - r$ and reduced cohomology vanishes below degree -1. □

The same ordinary-parameter argument applies to the locally Cohen–Macaulay straightening posets of Theorem 6.1, replacing $r!$ by the number of full flags below F. For a relative subposet Q, add $\Delta(Q_{>F})|_{S_\beta}$ to $U_{F,\beta}$ when $F \in Q$; if $F \notin Q$, keep the absolute pair. Coherence and the same localization prove this relative formula.

The class statement need not be an actual modular isomorphism. For a two-vertex simplex over $\mathbb{F}_2$, its barycentric ring is $\mathbb{F}_2[u, v, w]/(uv)$, with $z_1 = u + v$, $z_2 = w$ and

u, v interchanged. At $(z_1, z_2) = (0, 1)$ the fiber is $\mathbb{F}_2[u]/(u^2)$ with trivial action; at $(1, 1)$ it is $\mathbb{F}_2[u]/(u^2 + u)$ with $u \mapsto u + 1$, the regular two-dimensional representation. Both have highest active rank two and class $2[\mathbb{F}_2]$.

7.4. **Fitting ideals and rank jumps**

The Fitting ideals of the source parameter module equal those of its barycentric power module. They are monomial ideals in the specified parameters; a color-multigraded presentation has minors homogeneous for $\deg z_i = q_i \mathbf{e}_i$, and each such homogeneous polynomial is a scalar times one monomial. Proposition 7.4 makes these ideals invariant under every invertible upper-triangular change, over the universal coefficient ring of such a matrix. Thus the invariance holds in every characteristic. In characteristic zero it implies reversed strong stability; if $z_i \mid m \in I$, then $mz_j/z_i \in I$ for $j > i$. Apply $z_i \mapsto z_i + tz_j$ and take the coefficient of t.

Every nonempty proper ungraded Betti rank-jump locus is a tail coordinate subspace

$$V(z_{r+1}, \ldots, z_d).$$

Indeed, Theorem 7.5 makes each Betti number constant on the corresponding highest-active stratum. Its upper level sets are closed, since a finite free resolution expresses fiber homology dimension as a free rank minus the ranks of two adjacent matrices. The closures of these strata are the displayed nested subspaces, so any such closed union is one of them. The argument holds over an algebraic closure and descends; monomiality also permits testing at coordinate zero-one points. Consequently every Fitting ideal has radical a tail-coordinate ideal, the zero ideal, or the unit ideal. All these statements hold for relative kernels and for the polynomial actions above.

8. **Equivariant boundary lifting**

An equivariant boundary map need not admit an equivariant lift when the stabilizer order is not invertible. Prescribing its image on suitable fixed-face quotients removes the part that cannot be averaged, leaving a correction in permutation modules with small stabilizers.

Because boundary lifting makes every normalized coherent map invertible, we need only construct equivariant maps with scalar 1 on constants. Corrections and averages can therefore be performed additively, with invertibility supplied by the boundary construction.

Lemma 8.1 (Parameter-linear lifting)**.** *Let a finite group H act on a positively graded polynomial ring $A = K[z_1, \dots, z_r]$ trivially, on a bounded-below graded-free A-module M, and on an equivariant surjection $N \twoheadrightarrow N'$ of graded A-modules. Suppose this surjection has an equivariant graded K-linear section, and its kernel is degreewise injective with respect to K-split inclusions of $K[H]$-modules. Every equivariant graded A-linear map $M \to N'$ lifts equivariantly to N.*

Proof. The section gives a preliminary graded lift β. Suppose the lift has been made parameter-linear in degrees below j. Its values on

$$W_j = (A_+M)_j = \sum_i z_i M_{j-\deg z_i}$$

are prescribed by $L_j(\sum_i z_i m_i) = \sum_i z_i \Phi(m_i)$. This is well-defined; the first syzygies of the variables on the A-free module M are the Koszul syzygies, and the already imposed linearity in lower degrees annihilates them. The difference $L_j - \beta_j|_{W_j}$ takes values in the kernel. Since M_j/W_j is free over K, the inclusion $W_j \subset M_j$ is K-split. Extend the difference equivariantly and add the extension to β_j. Induction on degree proves the assertion. □

The required relative injectivity has an elementary source. If $J \leq H$ and $|J|$ is a unit in K, then $K[H/J]$ is injective with respect to K-split inclusions. By coinduction, extending a map to $K[H/J]$ amounts to extending a J-equivariant map to K. Extend K-linearly and average over J. In particular, sums of regular modules have this property over every K.

We shall also use a controlled form of the lemma. Let F be a face of rank r, put $u = v_F$, and let $\pi : B_F \to B'$ kill a group-invariant collection of face variables, retaining the top variable. Assume that the retained chain monomials form a face subcomplex. Suppose an equivariant boundary map and an equivariant A_r-linear map $\Lambda : R_F \to B'$ have been prescribed and agree on their common boundary. Lift the boundary by the top-free section, obtaining β. The difference $\Lambda - \pi\beta$ is divisible by the top variable in B'. Include it by the span of retained chain monomials and add it to β, obtaining β' with $\pi\beta' = \Lambda$. This section is only required to be K-linear. In degree j the correction in the preceding proof now has the form

$$L_j - \beta'_j = uh_j, \qquad h_j : W_j \longrightarrow (\ker \pi)_{j-r}. \tag{9}$$

Indeed, it vanishes on the boundary, and regularity of u on B' puts the quotient in $\ker \pi$. Thus degreewise relative injectivity of $\ker \pi$ gives a boundary-compatible lift satisfying $\pi\Phi = \Lambda$. A normalized lift is an isomorphism by Lemma 2.1.

Corollary 8.2. *If the order of the effective vertex-permutation group of every face stabilizer is a unit in K, there is a coherent graded equivariant comparison over K.*

Proof. Induct over face orbits. At a representative face, lift its prescribed boundary and average over the effective stabilizer. Its kernel fixes the entire Boolean face interval and hence both stalks, so every local map is already equivariant under that kernel. Averaging preserves the boundary; boundary lifting makes the resulting map an isomorphism. Transport it over the orbit and continue. □

8.1. **An integral orbit retraction**

To prescribe the fixed-face quotient, we need a map on the polynomial side that combines each free orbit into one variable and sends the elementary symmetric parameters to those of the orbit set. The following substitution achieves this integrally; roots of unity cancel the intermediate coefficients within each orbit, and an integral coefficient projection removes the auxiliary roots.

Let $Z = C_p$, with p prime, act freely on a finite set V, and choose a generator. Write its orbits as $O_1, \dots, O_t$, with cyclic labels $x_{l,h}$, $0 \le h < p$. Put $c = (-1)^{p-1}$ and choose a primitive pth root ζ. Define a map over $\mathbb{Z}[\zeta]$ by substituting $x_{l,h} = \zeta^h t_l$, retaining only terms whose exponent in every t_l is divisible by p, and replacing t_l^p by cX_l. Apply coefficientwise the integral functional

$$\lambda : \mathbb{Z}[\zeta] \longrightarrow \mathbb{Z}, \qquad \lambda(1) = 1, \quad \lambda(\zeta^h) = 0 \quad (1 \le h \le p-2),$$

to obtain

$$E : R_V \longrightarrow \mathbb{Z}[X_1, \dots, X_t], \qquad \deg X_l = p.$$

This is an integral map; the roots of unity only verify its properties. For a single orbit of size two, for example,

$$E(x_0) = E(x_1) = 0, \qquad E(x_0 x_1) = X, \qquad E(x_0^2) = E(x_1^2) = -X.$$

The cancellation that kills the first elementary parameter occurs without division by the orbit size, while the orbit product is retained.

Lemma 8.3. *The map E retracts the orbit-product embedding $X_l \mapsto \prod_h x_{l,h}$. It commutes with restrictions to unions of Z-orbits and with every vertex permutation centralizing Z. For elementary symmetric polynomials,*

$$E(e_i f) = \begin{cases} e_{i/p}(X) E(f) & p \mid i, \\ 0 & p \nmid i. \end{cases} \tag{10}$$

Proof. The identity $\prod_{h=0}^{p-1}(1+\zeta^h t_l T) = 1 + ct_l^p T^p$ proves (10). The retained exponent classes are preserved by the substituted symmetric polynomials, so the coefficient projection commutes with their multiplication. Also $\prod_h \zeta^h t_l = ct_l^p$ maps to X_l. A permutation centralizing Z permutes the orbits and shifts their cyclic labels. A shift multiplies a retained term by $\zeta^{spb_l} = 1$. This proves both equivariance and independence of the starting point in each orbit. Restriction is immediate. □

The construction depends on a generator, but cyclic rotations and induced cyclic orders preserve its positive choice, providing exactly the compatibility needed below.

9. **Cyclic orders and the factorial obstruction**

An ordered comparison can be symmetrized by summing over all $d!$ linear orders. To obtain the sharper factor $(d-1)!$, we construct comparisons that need only a cyclic order. The difficulty is to make them compatible with restriction to every subset, including when a rotation group has noninvertible order. The orbit retraction handles precisely those fixed subsets; the remaining chains have stabilizers whose orders are invertible.

Theorem 9.1 (Integral cyclic templates)**.** *For each d there are normalized graded parameter-module isomorphisms*

$$\Phi_{V,\omega} : \mathbb{Z}[x_v : v \in V] \longrightarrow \mathbb{Z}[\mathrm{Sd}(2^V)] \qquad (|V| \le d)$$

for all finite cyclically ordered sets (V,ω). They are natural under cyclic-order-preserving bijections and commute with restriction to every subset with its induced cyclic order.

Proof. We first construct the system over $K = \mathbb{Z}_{(p)}$, for each prime p. Induct on $n = |V|$, beginning with the identity at the empty set. The existing maps on smaller cyclic sets give a prescribed boundary map equivariant for the rotation group C_n.

If $p \nmid n$, lift this map and average over C_n. Suppose $p \mid n$, and let $Z_n \le C_n$ be the subgroup of order p, generated by rotation through n/p places. Its orbit set V/Z_n has its quotient cyclic order. Write

$$\pi_{p,V} : B_V \longrightarrow B_{V/Z_n}$$

for the projection retaining exactly the Z_n-fixed subsets; degrees in the target are multiplied by p. Impose, in addition, the identity

$$\pi_{p,V}\Phi_V = \Phi_{V/Z_n} E_{p,V}. \tag{11}$$

Here $E_{p,V}$ is the retraction of Lemma 8.3 for the positive generator. Both sides are parameter-linear after z_i is sent to zero for $p \nmid i$ and z_{pi} to the quotient parameter z_i.

The prescription agrees with the boundary. To see this precisely, identify V with $\mathbb{Z}/n\mathbb{Z}$. A Z_n-invariant subset of size ps is the union of the orbits indexed by an s-subset of $\mathbb{Z}/(n/p)\mathbb{Z}$. In its induced cyclic order, Z_n acts by rotation through s places, its positive order-p generator. Its orbit order is the induced order on that s-subset. Thus both the retraction and the smaller template restrict to the terms in (11) already prescribed there.

Apply the controlled lift (9) with $\Lambda = \Phi_{V/Z_n} E_{p,V}$. Every chain monomial in $\ker \pi_{p,V}$ contains a subset not fixed by Z_n. Its stabilizer in C_n has order prime to p; a subgroup of order divisible by p contains Z_n, and ranks force a stabilizer of a chain to fix every member of the chain. Hence each graded kernel piece is a sum of modules $K[C_n/J]$ with $|J|$ a unit. The relative injectivity above supplies the correction. This constructs the p-local system, and boundary lifting makes all its maps isomorphisms.

It remains to make the system integral. For each $n \le d$, specify images of one integral Artin basis of R_V over A_n. These images lie in finitely many finite free graded pieces. Rotation invariance, subset compatibility and normalization are a finite system of integer linear equations. Solvability over every $\mathbb{Z}_{(p)}$ implies solvability over $\mathbb{Z}$; the class of the inhomogeneous term in the finitely generated cokernel of the coefficient matrix vanishes in every localization, and therefore vanishes. □

A compatible cyclic order on a simplicial poset means a cyclic order on each face inducing that on every subface. The templates assemble into a coherent comparison natural for order-preserving isomorphisms, and more generally for nondegenerate simplicial-poset maps preserving these orders. Such a map is an isomorphism on each Boolean face interval; its pullback on face generators sums over their preimages. The local comparisons commute with these relabelings and restrictions, so the assertion follows on inverse limits. In particular, an orientation-preserving action on an oriented triangulated surface admits an integral equivariant comparison; use the oriented cyclic order on each triangle, and the unique orders on edges and vertices. The resulting equivariant isomorphisms also identify their Tor modules.

Corollary 9.2. *If $(d-1)!$ is a unit in K, there are coherent equivariant comparisons for every simplicial poset of maximum rank at most d and every automorphism group.*

Proof. For a nonempty n-set, average $\Phi_{V,\omega}$ over its $(n-1)!$ cyclic orders. If $E \subset V$ has $s \ge 1$ elements, each cyclic order on E is induced by exactly $(n-1)!/(s-1)!$ orders on V. Thus these symmetric averages remain coherent. They are normalized, so are isomorphisms, and assemble independently of face labels. □

We next determine the scalar obstruction, including its prime-power multiplicities. A suitable parameter fiber turns the polynomial ring into the regular permutation representation on root orderings, while the barycentric fiber retains an invariant evaluation. An invariant functional on a regular representation takes the same value on every coordinate, forcing a factorial divisor.

Lemma 9.3. *Suppose r is a unit in a commutative ring K. If $f : R_K(2^{[r]}) \to B_K(2^{[r]})$ is an S_r-equivariant graded parameter-linear map with $f(1) = C$, then $C \in r!K$.*

Proof. Specialize $z_1 = \cdots = z_{r-1} = 0$ and $z_r = (-1)^{r+1}$. The source fiber is the splitting algebra O of $T^r - 1$; the integral Artin basis makes it free of rank $r!$ over K, with basis containing 1. In particular there is a K-linear retraction $\lambda : O \to K$ with $\lambda(1) = 1$.

Let $\xi_1, \dots, \xi_r$ be the roots in O. They and all their pairwise differences are units; the discriminant identity gives

$$\prod_{i<j}(\xi_i - \xi_j)^2 = \pm r^r \prod_i \xi_i^{r-1} \in O^\times.$$

Give O the trivial group action and extend coefficients to it. Successive Chinese remaindering now splits the source fiber as $O^{r!}$, with S_r acting regularly on the ordered root assignments. The target fiber has an invariant evaluation to O, obtained by setting all proper-face variables to zero and the top variable to $(-1)^{r+1}$. Composition with f gives an invariant functional $O^{r!} \to O$ taking 1 to C. It has the same value a on every coordinate idempotent, whence $C = r!a$. Apply λ to conclude. □

For the rank-d simplex, abbreviate $E_K = E_K(2^{[d]})$ and write o for its comparison obstruction.

Theorem 9.4. *For the rank-d simplex and every commutative ring K,*

$$\epsilon(E_K^{S_d}) = (d-1)!K.$$

The obstruction class in (5) *satisfies $Ko \simeq K/(d-1)!K$. For every simplicial poset of maximum rank at most d and every automorphism group, $(d-1)!$ annihilates its coherent comparison obstruction.*

Proof. Put $D = (d-1)!$; the case $d = 1$ is immediate. A coherent family of scalar C restricts to an S_r-equivariant map of the same scalar on every rank-r face. Apply Lemma 9.3 with $r = d$ over $K[1/d]$, and with $r = d-1$ over $K[1/(d-1)]$. It follows that C belongs to DK after both localizations. Some powers d^aC and $(d-1)^bC$ therefore belong to DK. The two powers are coprime, so a Bezout combination gives

$C \in DK$. This argument applies without finiteness or reducedness assumptions on K.

For the opposite inclusion, use the integral cyclic templates and set on a rank-r face

$$T_F = \frac{D}{(r-1)!} \sum_{\omega \in \mathrm{CycOrd}(F)} \Phi_{F,\omega}, \qquad T_0 = D\mathrm{id}.$$

The coefficient is integral. Counting induced cyclic orders as in Corollary 9.2 shows that T is coherent; summing over all orders makes it equivariant for every automorphism. It has scalar D. Base change and scalar multiplication give all of DK.

Finally, the exact sequence $0 \to \ker \epsilon \to E_K \xrightarrow{\epsilon} K \to 0$ has a surjective last arrow by the nonequivariant comparison theorem. Its connecting map sends 1 to o and has kernel $\epsilon(E_K^{S_d})$. This proves the claimed cyclic module and, by the universal family T, the universal annihilator assertion. □

The scalar obstruction concerns coherent graded maps. To rule out arbitrary isomorphisms, we instead choose a fiber whose two group representations have different projectivity properties. A fixed root assignment gives a trivial summand on the polynomial side; on the barycentric side, sufficiently frequent active ranks force the flag components into free orbits.

Proposition 9.5 (Fixed-vertex obstruction)**.** *Let k have characteristic $p > 0$. If an order-p permutation of the vertices of a simplex has both fixed and moved vertices, its polynomial and barycentric rings are not isomorphic as equivariant parameter-modules, even without grading or coherence.*

Proof. Suppose there are $t \geq 1$ moved orbits and $r \geq 1$ fixed vertices. Over $L = k(a_1, \ldots, a_t, b_1, \ldots, b_r)$, specialize the parameters to the elementary symmetric coefficients of the list with p copies of each a_j and one copy of each b_h. Its generating polynomial is

$$\prod_{j=1}^{t} (1 + a_j^p T^p) \prod_{h=1}^{r} (1 + b_h T).$$

Algebraic independence implies that the nonzero coefficient ranks are

$$S = \{pq + u : 0 \leq q \leq t,\ 0 \leq u \leq r\} \setminus \{0\}.$$

Terms with different u cannot cancel, since their degrees in the b_h differ. Every successive gap from 0 through S to $pt + r$ is less than p.

In the barycentric fiber, the elements v_F/c_i at each rank $i \in S$ are orthogonal idempotents summing to 1. Their products decompose the fiber into flag components

of rank set S. The cyclic group acts freely on these flags. Indeed, in a fixed flag every successive difference is a union of vertex orbits, and its size is less than p; it could therefore contain only fixed vertices, never reaching the full set. The components are consequently permuted in free orbits, so the barycentric fiber is a free $L[C_p]$-module.

Evaluation at the displayed root assignment is an equivariant map from the polynomial fiber to L, split by constants. Thus its representation contains the trivial module as a direct summand. That module is not projective over $L[C_p] \simeq L[u]/(u^p)$; a finite projective over this local ring is free, whereas its dimension is 1. The polynomial fiber cannot be free. An equivariant parameter-module isomorphism would survive specialization and contradict this distinction. □

Completion of Theorem 1.3. The scalar statement is Theorem 9.4, and Corollary 9.2 gives the asserted comparisons when $(d-1)!$ is a unit. Otherwise some prime $p < d$ is a nonunit in K. Reduce a putative ungraded isomorphism modulo a maximal ideal containing p, and restrict to a p-cycle fixing the other vertices. Proposition 9.5 gives a contradiction. Thus the universal field range is characteristic zero or $p \geq d$. □

10. **Cyclic face stabilizers**

Call an action *locally semiregular* if, for every nonempty face F, the effective image of its stabilizer on the vertices of F acts freely. Equivalently, all vertex stabilizers inside G_F are the kernel of that vertex action. This kernel fixes the entire Boolean interval below F.

Theorem 10.1. *A locally semiregular action of a cyclic p-group on a simplicial poset admits an integral coherent equivariant comparison.*

Proof. Induct on the order of the group C. The trivial case is Theorem 1.1. Let $Z \leq C$ have order p. On a Z-fixed face, either Z fixes every vertex or it acts freely; the local condition makes its intersection with every vertex stabilizer the same kernel.

Let Q_0 consist of the first kind of fixed face and the minimum. It is a downset, and C/Z acts locally semiregularly on it. Let Q_1 consist of the second kind and the minimum, dividing ranks by p. This is a simplicial poset whose faces below a fixed face are the unions of its Z-orbits. The C/Z-action on its atoms is free. Indeed, a vertex in such an orbit has trivial stabilizer in C, since every nontrivial subgroup contains Z. If g stabilizes its Z-orbit, then $gv = zv$ for some $z \in Z$, forcing $g = z$. Induction therefore gives comparisons ψ_0 and ψ_1 on Q_0 and Q_1.

Construct the original maps in increasing face rank. On Q_0 use ψ_0. A face not fixed by Z has trivial stabilizer, so lift its boundary and transport the lift around its orbit. At $F \in Q_1$, let π_F retain the Z-fixed face variables and prescribe

$$\pi_F \Phi_F = \psi_{1,F} E_F.$$

This agrees with the boundary; every proper positive Z-fixed subface of F lies in Q_1, and Lemma 8.3 commutes with its restriction. The kernel of π_F has a chain basis with trivial stabilizers in C_F. A nontrivial stabilizer would contain Z and fix each member of the chain, contrary to membership in the kernel. Thus (9) has values in sums of regular $\mathbb{Z}[C_F]$-modules and extends. Transport these maps over their face orbits. The resulting family is coherent and normalized, hence consists of isomorphisms. □

Theorem 10.2. *Let k have characteristic $p > 0$, and suppose a Sylow p-subgroup of G is cyclic. A coherent graded equivariant comparison exists if and only if, for every face F, the effective action of a Sylow p-subgroup of G_F on its vertices is free.*

If every Sylow subgroup of G is cyclic, the corresponding criterion over an arbitrary ring K is that every prime for which this local condition fails be a unit in K.

Proof. For sufficiency over k, a Sylow p-subgroup C of G acts locally semiregularly; $C \cap G_F$ is contained in a Sylow p-subgroup of G_F, and semiregularity modulo the kernel passes to subgroups. Apply Theorem 10.1 and take its coset trace to G, divided by $[G : C]$. The resulting family is coherent and normalized.

For necessity, restrict a coherent comparison to the offending face and pass to the effective cyclic p-group. If this faithful action is not free, its unique subgroup of order p fixes a vertex and also moves a vertex. Proposition 9.5 contradicts the stalk isomorphism.

For the ring assertion, call a prime dividing $|G|$ *bad* if the local condition fails. A bad nonunit prime belongs to a maximal ideal of K, and reduction to its residue field proves necessity. For sufficiency, trace an ordinary integral comparison from the trivial subgroup; its scalar is $|G|$. For every good prime p, trace the integral comparison equivariant under its cyclic Sylow subgroup C_p; its scalar is $[G : C_p]$. The greatest common divisor of these integers is

$$N = \prod_{p \text{ bad}} p^{v_p(|G|)}.$$

A Bezout combination gives an integral equivariant coherent family of scalar N. Divide by its unit image in K. Boundary lifting again makes the normalized family an isomorphism. □

In particular, a free vertex action of a group with cyclic Sylow subgroups admits an integral equivariant comparison. For a faithful action $G \le S_n$ on a full simplex, the field criterion simplifies; if its Sylow p-subgroup is cyclic, an equivariant parameter-module isomorphism exists exactly when that subgroup acts freely. An isomorphism can then be graded and coherent; failure rules out even an ungraded isomorphism, by Proposition 9.5. When all Sylow subgroups are cyclic, the same statement over K requires precisely that the primes of nonfree Sylow actions be units.

Appendix A. Relative equivariant lifting

The preceding construction chooses its own compatible boundary maps; extending an arbitrary equivariant boundary instead requires information about all subgroup-fixed quotients. Throughout this section k is a field of characteristic $p > 0$.

A.1. Cofixed polynomial sources

Let a cyclic p-group C of order q act freely on a vertex set V of size r. Every C-fixed subset is a union of q-element orbits, so its rank is divisible by q. On the barycentric ring of these fixed faces, the other parameters act by zero and C acts trivially. A C-equivariant parameter-linear map from the polynomial ring to this fixed-face quotient must therefore factor through the following coinvariant module. Put

$$A_q = k[z_q, z_{2q}, \dots, z_r], \qquad M_q = k[x_v : v \in V]/(e_i : q \nmid i), \qquad \mathcal{C}_q = (M_q)_C.$$

The subscript denotes coinvariants for the linear group action. The parameters act by elementary symmetric functions, and the integral Artin basis makes M_q finite graded-free over A_q. Proving the same freeness after taking coinvariants will allow the lifting lemma to apply on fixed-face quotients.

Lemma A.1. *Over a ring of finite global dimension, a doubly infinite complex of projective modules with projective homology has projective cycles, boundaries and differential cokernels.*

Proof. Write Z_i, B_i, H_i for its cycles, boundaries and homology. The sequence $0 \to B_i \to Z_i \to H_i \to 0$ splits. From $0 \to Z_i \to P_i \to B_{i-1} \to 0$ we obtain, for $s \ge 1$,

$$\operatorname{Ext}^s(B_i, N) \simeq \operatorname{Ext}^{s+1}(B_{i-1}, N).$$

Iterate beyond the global dimension to obtain $\operatorname{Ext}^1(B_i, N) = 0$. Thus B_i and $Z_i = B_i \oplus H_i$ are projective. The differential cokernel is an extension of B_{i-1} by H_i, so it too is projective. □

Theorem A.2. *For every free cyclic p-group action, both $(M_q)_C$ and $(M_q)^C$ are finite graded-free over A_q.*

Proof. Induct on q. For $q = 1$ this is the ordinary Artin basis theorem. Let $Z \leq C$ have order p and put $\bar{C} = C/Z$. There is a characteristic-p orbit map

$$E : R = k[x_v : v \in V] \longrightarrow R' = k[X_O : O \in V/Z], \qquad \deg X_O = p.$$

Identify the variables in each orbit with u_O, retain terms whose orbit exponents are divisible by p, and replace u_O^{pb} by X_O^b. It is C-equivariant and satisfies (10), since $(1 + u_O T)^p = 1 + u_O^p T^p$. The orbit-product embedding $\iota : R' \to R$ is an equivariant k-linear section. The kernel has permutation basis

$$x^a - \iota E(x^a), \qquad x^a \text{ not fixed by } Z.$$

Every basis stabilizer is trivial; a nontrivial subgroup of C contains Z. Thus each internal-degree piece of $\ker E$ is free over $k[C]$.

Apply the complete two-periodic Tate complex for C to the bounded Koszul complex $K(e_i : q \nmid i; R)$. Its Koszul homology is M_q, because the indicated parameters form a regular sequence. Replacing R by R' through E leaves total Tate cohomology unchanged; the kernel Koszul complex is bounded and has degreewise free $k[C]$-terms, whose Tate cohomology vanishes. Both assertions also follow from the two spectral sequences. The bounded Koszul direction makes them valid for the complete Tate complex, separately in each internal degree.

On R', the parameters with $p \nmid i$ act as zero. The others form the regular sequence e'_j with $q/p \nmid j$. Hence the replacement Koszul complex is equivariantly quasi-isomorphic to

$$M' \otimes_k \bigwedge \langle \epsilon_i : p \nmid i \rangle, \qquad M' = R'/(e'_j : q/p \nmid j),$$

with zero differential on the exterior factor. All these are modules over A_q, with degrees in R' multiplied by p. By induction $(M')_{\bar{C}}$ is projective over A_q, and M' is free. The two exact sequences for $g - 1$, where g generates C, then split successively over A_q; they also make $(M')^{\bar{C}}$ projective.

The C-action on M' factors through $\bar{C}$. Its norm is p times the quotient norm and is zero. Thus the complete Tate complex of M' alternates $g - 1$ with zero, and its cohomology alternates the invariant and coinvariant modules just proved projective. The exterior decomposition shows that every Tate cohomology module of M_q is projective over A_q.

The complete Tate complex of M_q has finite free A_q-terms. Since the polynomial ring A_q has finite global dimension, Lemma A.1 makes its differential cokernel

$M_q/(g-1)M_q$ projective. The two splitting sequences for $g-1$ also make its invariant module projective.

Finally, a finite graded projective over a positively graded polynomial ring with degree-zero part k is graded-free. Lift a homogeneous basis modulo the positive-degree ideal, split the resulting graded free surjection, and apply graded Nakayama to its kernel. This proves both assertions. □

A.2. **The subgroup-fixed lifting criterion**

Proposition A.3. *Let a finite p-group H act freely on the vertices of a rank-r simplex. For every subgroup $L \leq H$, suppose*

$$\mathcal{C}_L = \big(k[x_v]/(e_i : |L| \nmid i)\big)_L$$

is graded-free over $A_L = k[z_{|L|}, z_{2|L|}, \dots, z_r]$. Then every coherent H-equivariant graded comparison prescribed on the proper faces extends equivariantly to the simplex.

Proof. Let B_L be the barycentric ring on the L-fixed faces, a Boolean poset on the L-orbits of vertices with degrees multiplied by $|L|$. Construct maps $\Lambda_L : R \to B_L$ in decreasing subgroup order, on conjugacy representatives and then by transport. Require normalizer equivariance, agreement with the prescribed boundary, and compatibility with all larger-subgroup fixed quotients.

At the L stage, take the union, inside the order complex of its fixed faces, of the proper-face boundary and the K-fixed complexes for all $K \supsetneq L$. Intersections are fixed complexes for the generated subgroups, so the maps already constructed glue coefficientwise to its face ring B'_L. This map is L-invariant, and parameters of rank not divisible by $|L|$ act as zero. It therefore factors through $\mathcal{C}_L$, equivariantly for $N_H(L)/L$.

The kernel of $B_L \to B'_L$ has a basis of chain monomials containing the top face and having stabilizer exactly L in H. Indeed, the boundary removes chains omitting the top, while the other fixed complexes remove exactly those with a larger stabilizer. Consequently $N_H(L)/L$ acts freely on this basis. Each graded kernel piece is a sum of regular modules, hence injective. The standard-monomial section is graded and normalizer-equivariant. Apply Lemma 8.1 to the free A_L-source $\mathcal{C}_L$. Normalizer equivariance makes transport well-defined; the imposed larger-subgroup compatibilities handle the intersections with conjugate subgroups.

At $L = 1$ this gives an H-equivariant parameter-linear map extending the entire prescribed boundary. Lemma 2.1 makes it an isomorphism. □

Corollary A.4. *For a free cyclic p-group action, every prescribed coherent equivariant graded boundary comparison extends to the simplex.*

Proof. Apply Theorem A.2 to every subgroup and then Proposition A.3. □

Lemma A.5. *If r is a power of p and $L \leq S_r$, then*

$$\big(k[x_1, \ldots, x_r]/(e_1, \ldots, e_{r-1})\big)_L$$

is graded-free over $k[e_r]$.

Proof. Make the finite faithfully flat base change $e_r \mapsto -t^r$ to $k[t]$. Translation of every variable by t is S_r-equivariant. Since $\binom{r}{i} = 0$ in k for $0 < i < r$, the elementary translation formula identifies the base-changed ring with

$$k[x_1, \ldots, x_r]/(e_1, \ldots, e_r) \otimes_k k[t].$$

Coinvariants commute with base change, so the resulting module is free over $k[t]$. Faithful flatness makes the original finite module torsion-free over the PID $k[e_r]$, hence projective and graded-free. □

Theorem A.6. *Let H be a finite p-group all of whose proper subgroups are cyclic, acting regularly on its own vertices. Every prescribed coherent H-equivariant graded boundary comparison extends to the simplex. This applies to $C_p \times C_p$ in characteristic p and to Q_8 in characteristic two.*

Proof. For each proper subgroup use Theorem A.2; for H itself use Lemma A.5, since there is one vertex orbit. Proposition A.3 applies. □

Corollary A.7. *Let G act on a simplicial poset over k. Suppose that, on each face F, the effective image of a Sylow p-subgroup of G_F is trivial, cyclic and free on its vertices, or a p-group with all proper subgroups cyclic acting regularly on those vertices. Then a coherent graded equivariant comparison exists. When all these effective images are cyclic, the free-or-trivial condition is also necessary.*

Proof. Induct over face orbits. The previously chosen maps prescribe a boundary invariant under G_F. Pass to the effective image of a Sylow subgroup; its kernel acts trivially on both stalks. Corollary A.4, Theorem A.6, or ordinary lifting extends this same boundary. Take the coset trace to G_F and divide by the Sylow index. The average still extends the prescribed boundary, so is an isomorphism; transport it over the face orbit. Necessity in the cyclic case is Proposition 9.5 on the offending stalk. □

Thus the fixed-quotient construction extends the earlier criterion from cyclic Sylow subgroups to actions controlled only by their effective stabilizers on individual faces.

A.3. **Integral relative lifting**

For a prime-order effective action, the relative modular theorem lifts integrally by a lattice argument.

Lemma A.8. *Let a group H of order N act on a nonempty affine torsor T under a finite free abelian group V. If T/NV has an H-fixed point, then T has an H-fixed point.*

Proof. Lift the fixed point to $\phi \in T$, and write $h\phi - \phi = Nv_h$ with $v_h \in V$. In the rational affine space,

$$\frac{1}{N}\sum_{h\in H} h\phi = \phi + \sum_{h\in H} v_h.$$

The right side lies in T, and the left side is invariant. □

Theorem A.9. *For a free C_p-action on the vertices of a simplex, every prescribed integral coherent equivariant graded boundary comparison extends to an integral equivariant comparison on the simplex.*

Proof. Choose an integral Artin basis $\mathcal{A}$ of R_F over A_r. Ordinary boundary lifting makes the set T of extensions nonempty. It is a torsor under the finite free lattice

$$V = \mathrm{Hom}^0_{A_r}(R_F, v_F B_F) \simeq \bigoplus_{a\in\mathcal{A}} (B_F)_{\deg a - r}.$$

These coordinates commute with reduction modulo p; lift the boundary image of each basis element by its top-free representative, and choose its arbitrary correction in $v_F B_F$. Thus T/pV is exactly the space of extensions of the reduced boundary. Corollary A.4 supplies a fixed point there. Lemma A.8, with $N = p$, gives an integral fixed point. Every such boundary lift is an isomorphism by Lemma 2.1. □

Theorem A.10. *Suppose that, for every face F and every prime $p \mid |G_F|$, the effective image of a Sylow p-subgroup of G_F is trivial or has order p and acts freely on the vertices of F. Then there is an integral coherent graded G-equivariant comparison, and hence one over every commutative coefficient ring. It identifies all subcomplex restrictions and relative kernels.*

Proof. Induct on face orbits and let $H = G_F$. For each Sylow subgroup P_p, Theorem A.9, or ordinary lifting in the trivial effective case, gives a P_p-equivariant lift ϕ_p of the same prescribed boundary. Its coset trace to H has boundary m_p times

that map, where $m_p = [H : P_p]$. The integers m_p have greatest common divisor one. Choose $\sum_p a_p m_p = 1$; then

$$\sum_p a_p \operatorname{Tr}^H_{P_p}(\phi_p)$$

is an integral H-equivariant lift of the prescribed boundary. For $H = 1$ use an ordinary lift. Boundary lifting makes each map an isomorphism. Transport over face orbits and pass to inverse limits. Coherence identifies restrictions and their kernels, and tensoring the maps and their inverses gives arbitrary base change. □

Within the class whose effective local Sylow images all have order 1 or p, the condition in Theorem A.10 is exact; a nonfree effective order-p action is obstructed after reduction modulo p by Proposition 9.5. In particular the theorem applies when every effective face stabilizer has squarefree order and each of its Sylow subgroups acts freely on the face vertices.

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