Monomial bases for two-fermion coinvariants

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Abstract. The $(1,2)$-bosonic–fermionic coinvariant rings in types A and B admit monomial bases indexed by two-colored lattice paths, with polynomial exponents bounded by the path heights. We prove these basis formulas by identifying the polynomial pieces with explicit complete-intersection colon quotients; rational Newton fields give rectangular bases, while an exterior Gauss identity supplies the matching bounds. This description persists in weighted moment quotients under coupled polynomial perturbations, over any commutative ring in which $n!$ and the nonempty subset sums of the weights are units. The same boxes describe local factors under monic covers and bases for opposite-character wreath products; retaining the two exterior colors gives an equivariant filtration and the polynomially graded GL_2 character in type A.

1. Introduction

The ordinary Artin basis describes a coinvariant algebra by a rectangular set of monomials, with bounds $0 \le e_i < i$ in type A. With two exterior families, the corresponding bounds depend on which exterior variables occur; a mark at one index changes the polynomial degrees available at every later index. We prove that this dependence is measured exactly by the heights of a two-colored lattice path, establishing the basis conjectures of Lentfer [1] for the $(1,2)$-bosonic–fermionic coinvariant rings of the symmetric and hyperoctahedral groups. Their dimensions are $2^{n-1}n!$ and $4^n n!$, respectively.

To state the path rule, let $m_i \in \{0,1,2\}$ count the exterior marks at index i and put $c_i = i - \sum_{j \le i} m_j$, with $c_0 = 0$. Unmarked indices give up-steps, single marks give two colors of flat steps, and double marks give down-steps, so the bases take the form

$$\begin{array}{lll} \text{type } A & c_i > 0 & 0 \le e_i < c_i \\ \text{type } B & c_i \ge 0 & 0 \le e_i < 2c_i + m_i \end{array} \qquad (1 \le i \le n).$$

These bounds arise from a common calculation with moment ideals, which also explains why the bases survive changes in the weights and lower polynomial coefficients. The two coinvariant algebras are specializations of two such families.

The proof must account for each exterior pattern separately, since highest-weight reduction gives spanning under a perturbation without preventing the dimension from

dropping. The polynomial ideal attached to a pattern is a colon ideal; rational Newton fields identify it as a complete intersection and give its rectangular basis. Creation operators then pair triangularly with these boxes, proving independence, while an exterior Gauss identity bounds every prefix and recovers exactly the same dimensions. The pairing remains nondegenerate under coupled lower-weight perturbations, and a universal construction carries the bases to the coefficient rings specified below.

Let K be a commutative ring and fix $n \geq 1$. Throughout, assume

$$n! \in K^{\times}, \qquad \beta_U := \sum_{i \in U} \beta_i \in K^{\times} \quad (\varnothing \neq U \subseteq [n]). \tag{1}$$

Put

$$\mathcal{P} = K[x_1, \ldots, x_n] \otimes \bigwedge(\theta_1, \ldots, \theta_n, \xi_1, \ldots, \xi_n).$$

The two exterior families have bidegrees $(1,0)$ and $(0,1)$.

For the first family, fix $s \geq 1$, positive integers r_i, and nonnegative integers u_i, v_i with $u_i + v_i = r_i s$. Give x_i weight $1/r_i$. Choose polynomials ϕ_i, a_i, b_i in all the x-variables with highest weighted forms

$$\operatorname{top}(\phi_i) = x_i^{r_i}, \qquad \operatorname{top}(a_i) = x_i^{u_i}, \qquad \operatorname{top}(b_i) = x_i^{v_i}. \tag{2}$$

Thus an exponent zero makes the corresponding polynomial equal to one. Let I be generated, for $k \geq 0$, by

$$P_k = \sum_i \beta_i a_i b_i \phi_i^k, \quad Q_k^{\theta} = \sum_i b_i \phi_i^k \theta_i,$$
$$Q_k^{\xi} = \sum_i \beta_i a_i \phi_i^k \xi_i, \quad E_k = \sum_i \phi_i^k \theta_i \xi_i.$$

For $T, S \subseteq [n]$, set

$$t_i = \mathbf{1}_{i \in T}, \quad z_i = \mathbf{1}_{i \in S}, \quad m_i = t_i + z_i,$$

and

$$h_i = s + c_{i-1}, \qquad d_i = r_i h_i - u_i t_i - v_i z_i.$$

At arrival height c_i, the four polynomial bounds are

marks	$\varnothing$	θ	ξ	$\theta\xi$
d_i	$r_i(c_i + s - 1)$	$r_i c_i + v_i$	$r_i c_i + u_i$	$r_i(c_i + 1).$

A zero bound contributes an empty box.

Theorem 1.1. *The following residue classes form a free K-basis of $\mathcal{P}/I$,*

$$\mathcal{B} = \{x^e\theta_T\xi_S \mid c_i \geq 0 \text{ for all } i, \quad 0 \leq e_i < d_i\}. \tag{3}$$

Its rank and total-exterior Hilbert polynomial are

$$\operatorname{rank}_K \mathcal{P}/I = 2^n \left(\prod_i r_i\right)(s)_n, \qquad \sum_j \operatorname{rank}_K(\mathcal{P}/I)_{\mathrm{ext}=j} z^j = \left(\prod_i r_i\right)(s)_n(1+z)^n,$$

where $(s)_n = s(s+1)\cdots(s+n-1)$.

Give θ_i weight u_i/r_i and ξ_i weight v_i/r_i. The associated graded algebra is the homogeneous algebra obtained by replacing (ϕ_i, a_i, b_i) by $(x_i^{r_i}, x_i^{u_i}, x_i^{v_i})$. Weighted homogenization gives a finite free degeneration over $K[t]$ with this special fiber and basis $\mathcal{B}$.

For the strict-height family, choose only ϕ_i with highest forms $x_i^{r_i}$ and let I^{str} be generated by

$$\sum_i \beta_i\phi_i^k \ (k \geq 1), \qquad \sum_i \phi_i^k\theta_i, \quad \sum_i \beta_i\phi_i^k\xi_i, \quad \sum_i \phi_i^k\theta_i\xi_i \quad (k \geq 0). \tag{4}$$

Theorem 1.2. *The quotient by* (4) *has free K-basis*

$$\{x^e\theta_T\xi_S \mid c_i \geq 1 \text{ for every } i, \quad 0 \leq e_i < r_ic_i\}.$$

Its rank is $2^{n-1}n!\prod_i r_i$, and its total-exterior Hilbert polynomial is $n!\prod_i r_i(1+z)^{n-1}$. With exterior weight zero, its associated graded algebra is the strict homogeneous power algebra.

The homogeneous specializations $r_i = 1$, $\beta_i = 1$ in Theorem 1.2 and $s = 1$, $r_i = 2$, $u_i = v_i = 1$, $\beta_i = 1$ in Theorem 1.1 recover the type-A and type-B bases displayed above.

The weighted formulation records more than persistence of dimension. For separated monic covers, the subset-sum hypothesis is exactly the condition for finiteness, and the same homogeneous algebras describe every local factor; over $\mathbb{C}$, the roots of a common polynomial G and the ramification of ϕ_i determine these factors when $a_ib_i = G(\phi_i)$ (Section 7). The exterior patterns also retain representation-theoretic information. Their path bounds give bases for opposite-character wreath products, and, in type A, retaining each entire two-color space makes the box decomposition GL_2-equivariant and determines its polynomially graded character (Section 8).

2. **Polynomial boxes from rational Newton fields**

We begin over a field F of characteristic zero or greater than n, with every $\beta_U \neq 0$. Put $S = F[y_1, \dots, y_n]$. The polynomial result needed for the exterior basis is the following. For signs $\sigma_i \in \{-1, 0, 1\}$ and a monic polynomial G of degree $s \geq 1$, set

$$g_\sigma = \prod_{i<j} (y_i - y_j)^{1-\sigma_i}, \qquad J_{G,\boldsymbol{\beta}} = \left(\sum_i \beta_i G(y_i) y_i^k : 0 \leq k < n \right).$$

Proposition 2.1. *Suppose $h_i = s + \sum_{j<i} \sigma_j > 0$ for every i. Then*

$$\{y^e \mid 0 \leq e_i < h_i\} \text{ is a basis of } S/(J_{G,\boldsymbol{\beta}} : g_\sigma). \tag{5}$$

The colon ideal is generated by a regular sequence whose highest forms have degrees $h_1, \dots, h_n$.

For $G = t^s$, we construct the colon generators by rational vector fields and obtain the rectangular basis by deletion. Deleting a variable changes its multiplicity, so the induction needs a slightly larger class in which the coordinate multiplicities may vary.

2.1. **The Newton fields**

Choose nonnegative integers δ_i and signs $\sigma_i \in \{-1, 0, 1\}$ satisfying

$$\sigma_i = 1 \Longrightarrow \delta_j \leq \delta_i \ (j > i), \qquad \sigma_i = -1 \Longrightarrow \delta_j \geq \delta_i \ (j > i). \tag{6}$$

Set

$$h_i = \delta_i + \sum_{j<i} \sigma_j, \quad Q_- = \prod_{i<j:\sigma_i=-1} (y_i - y_j), \quad Q_+ = \prod_i y_i^{\delta_i} \prod_{i<j:\sigma_i=1} (y_i - y_j).$$

Let $D_{\boldsymbol{\beta}}(\delta, \sigma) \subset Q_-^{-1} S^n$ consist of vectors with numerator $A = (A_i)$ satisfying the following conditions, with hyperplane indices always ordered $i < j$

$$y_i^{\delta_i} \mid A_i; \qquad y_i - y_j \mid A_i - A_j \quad (\sigma_i = 1);$$

and, when $\sigma_i = -1$,

$$y_i - y_j \mid \beta_i A_i + \beta_j A_j, \qquad y_i - y_j \mid A_k \quad (k \neq i, j).$$

Thus a negative hyperplane permits a simple pole only in the weighted normal direction $\beta_i^{-1}\partial_i - \beta_j^{-1}\partial_j$.

Define

$$R_i(t) = t^{\delta_i} \frac{\prod_{j<i:\sigma_j=1}(t-y_j)}{\prod_{j<i:\sigma_j=-1}(t-y_j)}, \qquad N_i = \{j < i : \sigma_j = -1\}.$$

Let b_i be the first $j > i$ with $\delta_j > \delta_i$, or $n+1$ if none exists, and put $K_i = \{i, \dots, b_i - 1\}$. Define the column ϑ_i by

$$(\vartheta_i)_k = \begin{cases} R_i(y_k), & k \in K_i, \\ \operatorname{Res}_{t=y_k} R_i(t) \displaystyle\sum_{\ell \in K_i} \frac{\beta_\ell}{\beta_k(y_k - y_\ell)}, & k \in N_i, \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

The first line is a Newton interpolation column, with a denominator for each earlier negative sign. Its active interval stops before a larger coordinate multiplicity would violate divisibility. The second line supplies the residues forced at those earlier negative hyperplanes. Thus the poles are part of the construction; they allow a negative sign to lower every subsequent degree by one.

For example, take two variables, $\delta = (2, 2)$, and $\sigma = (-1, 0)$. The columns are

$$\vartheta_1 = \begin{pmatrix} y_1^2 \\ y_2^2 \end{pmatrix}, \qquad \vartheta_2 = \frac{1}{y_2 - y_1} \begin{pmatrix} -\beta_2 y_1^2/\beta_1 \\ y_2^2 \end{pmatrix}.$$

Their determinant is $(\beta_1 + \beta_2) y_1^2 y_2^2 / [\beta_1(y_2 - y_1)]$. Weighted contraction cancels the pole and gives the polynomials $\beta_1 y_1^2 + \beta_2 y_2^2$ and $\beta_2(y_1 + y_2)$, of degrees $h_1 = 2$ and $h_2 = 1$. The determinant calculation below generalizes this subset-sum factor to every profile.

Proposition 2.2. *The columns* (7) *form a homogeneous S-basis of $D_{\boldsymbol{\beta}}(\delta, \sigma)$, of degrees h_i. Their determinant is*

$$\det(\vartheta_1, \dots, \vartheta_n) = C \frac{Q_+}{Q_-}, \tag{8}$$

where C is a product of signs and ratios of nonempty subset sums of the β_i.

Proof. An active or earlier negative index has $\delta_k \le \delta_i$, giving the coordinate divisibility. A positive edge cannot leave a finite active interval, by (6); tangency inside follows by subtracting evaluations of R_i, and an earlier positive vertex supplies a numerator factor. At a pole joining an earlier negative index to an active one, the two residues have ratio $\beta_\ell/\beta_k : -1$. At a pole between two earlier negative indices, their weighted active sums agree and their partial-fraction residues are opposite. Hence every column belongs to the module.

At a positive hyperplane, a row change makes one determinant row divisible by its equation. At a negative hyperplane, a row change puts the possible pole in one row only. The coordinate orders and homogeneity then give (8), initially with C possibly zero. If $C \neq 0$, Cramer's rule proves freeness; every replaced-column numerator has the same local divisibility, so its coefficient is polynomial.

We prove nonvanishing by deleting the first coordinate. For the all-one profile, we simultaneously prove the following auxiliary assertion; in the expansion of $\rho_d = (y_i^d)_i$, for $d \geq 1$, the coefficients at height d are one at the first such index and zero thereafter; coefficients at larger heights vanish. The coefficient at index i is homogeneous of degree $d - h_i$. The induction has two stages. The auxiliary assertion for $n-1$ variables first proves that the n-variable determinant is nonzero. Cramer's rule then gives polynomial coefficients in n variables, from which we prove the auxiliary assertion at that size. Primes below denote the suffix $2, \ldots, n$, whose indices retain their original names; in particular, $h'_i = h_i - \sigma_1$.

For a nonnegative first sign, the first row is $(y_1, 0, \ldots, 0)$, and the suffix rows differ from the suffix matrix by factors $(y_k - y_1)^{\sigma_1}$. This gives a nonzero determinant factor up to sign. For a negative first sign, put $z = y_1$, factor z from the first row and $(y_k - z)^{-1}$ from each suffix row. The matrix becomes

$$\begin{pmatrix} 1 & r \\ \rho'_2 - z\rho'_1 & M' \end{pmatrix}, \qquad r_i = \frac{R'_i(z)}{\beta_1 z} \sum_{k=i}^{n} \frac{\beta_k}{z - y_k}.$$

Write $A(d) = (M')^{-1}\rho'_d$. Since the first suffix column is ρ'_1, we have $A(1) = e_2$, so the Schur complement is $1 - rA(2) + zr_2$. Local divisibility and homogeneity make this expression constant. To evaluate it, note that

$$r_i(z) \sim \frac{\beta_{[i,n]}}{\beta_1} z^{h'_i - 2} \qquad (z \longrightarrow \infty).$$

Terms of height less than two tend to zero. Terms of height greater than two have $A_i(2) = 0$, and at height two the suffix assertion leaves only the first index i_*, if it exists. Since $h'_2 = 1$, the remaining term zr_2 tends to $\beta_{[2,n]}/\beta_1$. Thus the constant is

$$\frac{\beta_1 + \cdots + \beta_{i_*-1}}{\beta_1}, \quad \text{or} \quad \frac{\beta_1 + \cdots + \beta_n}{\beta_1}$$

if none exists, and is nonzero.

To finish the simultaneous induction, $d = 1$ is $\rho_1 = \vartheta_1$. For $d \geq 2$, the suffix coefficient vectors are, according as the first sign is $0, 1, -1$,

$$A(d) - z^{d-1}A(1), \qquad \sum_{a=1}^{d-1} z^{d-1-a}A(a),$$
$$A(d+1) - zA(d) - a_1A(2) + za_1A(1),$$

where a_1 is the first coefficient in the full expansion. Suppose $h_i = d$. If $\sigma_1 = 0$, then $h_i' = d$, so $A_i(1) = 0$ and only $A_i(d)$ remains. If $\sigma_1 = 1$, then $h_i' = d - 1$, and only the summand with $a = d - 1$ remains. If $\sigma_1 = -1$, then $h_i' = d + 1 \geq 3$; the terms $A_i(d), A_i(2), A_i(1)$ vanish, leaving only $A_i(d + 1)$. In each case the suffix assertion gives one at the first index of the required height and zero thereafter. Coefficients with $h_i > d$ vanish by their negative degree. This proves the auxiliary assertion and the all-one case. Constant profiles follow from $M(d\mathbf{1}, \sigma) = \operatorname{diag}(y_i^d)M(0, \sigma)$.

For a nonconstant profile with first sign negative, let $d = \delta_1$ and let b be the first later index with $\delta_b > d$. All later multiplicities are at least d. The same factoring gives lower-left block $(Y - zI)\eta$, where $\eta_k = y_k^d$ for $2 \leq k < b$ and zero otherwise, and

$$r_i = \frac{R_i'(z)}{\beta_1 z^d} \sum_{k \in K_i} \frac{\beta_k}{z - y_k}.$$

If $b = 2$, the lower-left block is zero and the Schur complement is one. Otherwise the initial suffix block $[2, b-1]$ has constant multiplicity d. Every sign there is nonpositive; a positive sign would force $\delta_b \leq d$, contrary to the choice of b. Each of its Newton columns is supported on that block, and its first column is η. By the constant-profile case, these columns expand $Y\eta$ with polynomial coefficients. Their degrees satisfy $h_i' \leq d$, while

$$r_i(z) = O(z^{h_i' - d - 1}).$$

The coefficients in the expansion of $Y\eta$ do not involve z. Consequently its contribution to the Schur complement tends to zero. The first-column contribution from $z\eta$ tends to $\beta_{[2,b-1]}/\beta_1$. The Schur complement is therefore $\beta_{[1,b-1]}/\beta_1$. A nonnegative first sign again strips the first row. This completes the induction and proves the stated form of C. □

2.2. **Contraction and deletion**

Weighted contraction cancels the polar parts

$$q_i = \sum_k \beta_k(\vartheta_i)_k = \sum_{k \in K_i} \beta_k \operatorname{pol}(R_i)(y_k), \qquad Q_{\boldsymbol{\beta}}(\delta, \sigma) = (q_1, \ldots, q_n).$$

If $h_i = 0$, then $q_i = \beta_{K_i}$ is a unit. If every $h_i > 0$, these generators form a finite-colength regular sequence. Indeed, the polynomial fields $\rho_b, \dots, \rho_{b+n-1}$ belong to the module for $b \geq \max(1, \delta_i)$, so $Q_{\boldsymbol{\beta}}$ contains

$$J_{b,\boldsymbol{\beta}} = \left(\sum_i \beta_i y_i^{b+k} : 0 \leq k < n \right).$$

Its only zero is the origin. Indeed, the first n Laurent coefficients of $\sum_i \beta_i y_i^b/(t - y_i)$ vanish there. Its denominator has degree at most n and its numerator has smaller degree, so this forces the rational function to be zero. A nonzero repeated value a would then give the nonzero residue $a^b \beta_{\{i:y_i=a\}}$, a contradiction.

Proposition 2.3. *If every $h_i > 0$, the monomials y^e with $0 \leq e_i < h_i$ form a basis of $S/Q_{\boldsymbol{\beta}}(\delta, \sigma)$.*

Proof. Choose the first negative vertex q with $\delta_q > 0$, or, if none exists, the last vertex with $\delta_q > 0$. Deletion lowers δ_q by one, preserving (6) and changing only h_q. Restriction sets $y_q = 0$ and removes q; an incident negative edge changes the remaining multiplicity to $\max(\delta_i - 1, 0)$, a positive edge to $\max(\delta_i, 1)$, and an absent edge leaves it unchanged. For $i < q$ no multiplicity changes, while for $i > q$ the change is σ_q, exactly compensating for the removed sign; hence all surviving heights remain unchanged.

Restriction of fields has kernel $y_q D_{\mathrm{del}}$; dividing a field that vanishes on $y_q = 0$ by y_q removes exactly one coordinate divisibility condition. The coordinate and normal conditions put its image in the stated restricted module. These two modules have the same Hilbert series, since

$$\mathrm{Hilb}(D; t) - t\,\mathrm{Hilb}(D_{\mathrm{del}}; t) = \frac{\sum_{i \neq q} t^{h_i}}{(1-t)^{n-1}} = \mathrm{Hilb}(D_{\mathrm{res}}; t).$$

Thus restriction is onto. Contraction commutes with restriction, so $Q_{\mathrm{res}} = (Q + (y_q))/(y_q)$ and $Q_{\mathrm{del}} \subseteq Q : y_q$. Apply the usual multiplication sequence for y_q to S/Q. The known complete-intersection Hilbert series of Q and Q_{res} make its kernel quotient have exactly the Hilbert series of S/Q_{del}. The displayed inclusion must therefore be equality. This also covers a deleted height zero, when $Q_{\mathrm{del}} = S$. We obtain

$$0 \longrightarrow S/Q_{\mathrm{del}}(-1) \xrightarrow{y_q} S/Q \longrightarrow S'/Q_{\mathrm{res}} \longrightarrow 0.$$

Induct on the number of variables and then on $\sum_i \delta_i$. Restriction supplies exponent zero at q, and deletion supplies exponents $1, \dots, h_q - 1$. □

For a constant profile $b \geq 1$, express $\rho_b, \dots, \rho_{b+n-1}$ in the Newton basis. The polynomial coefficient matrix C has determinant a nonzero scalar times g_σ. Contraction

compares two finite-colength regular sequences. The standard Koszul comparison gives

$$Q_{\boldsymbol{\beta}}(b\mathbf{1},\sigma) = J_{b,\boldsymbol{\beta}} : g_\sigma. \tag{9}$$

Indeed, the comparison of Koszul resolutions has top map multiplication by $\det C$. Dualizing gives an injection of the shifted complete-intersection quotients, because finite-length modules have no lower Ext over the polynomial ring. Its kernel description is exactly $J_{b,\boldsymbol{\beta}} : \det C = Q_{\boldsymbol{\beta}}$. This proves the colon identity and its rectangular basis.

2.3. **A monic polynomial in place of a power**

We will use the following filtered fact repeatedly. If the highest forms $q_1,\dots,q_n$ of polynomials $f_1,\dots,f_n$ form a regular sequence for a positive grading, then

$$\operatorname{gr} S/(f_1,\dots,f_n) = S/(q_1,\dots,q_n). \tag{10}$$

For one element, a nonzerodivisor highest form makes the highest form of every product the product of the highest forms. Induct on the sequence using the quotient filtration. This proves both (10) and regularity of the actual sequence.

Proof of Proposition 2.1. For $G = t^s$, the assertion is (9) and Proposition 2.3. For general G, multiply the zero-profile Newton matrix by $\operatorname{diag}(G(y_i))$. Its weighted contractions are

$$q_i^G = \sum_{k\ge i} \beta_k \operatorname{pol}\left(G(t)\frac{\prod_{j<i:\sigma_j=1}(t-y_j)}{\prod_{j<i:\sigma_j=-1}(t-y_j)} \right)(y_k).$$

Their highest forms are the constant-profile-s contractions, so (10) gives the regular sequence and the box basis. The coefficient matrix expressing $(G(y_i)y_i^k)_i$ is unchanged by this diagonal multiplication; its determinant remains a scalar times g_σ. The same Koszul comparison proves $(q_i^G) = J_{G,\boldsymbol{\beta}} : g_\sigma$. The nonhomogeneous comparison is valid because finite-length modules still have no lower Ext. □

3. **An exterior Gauss identity**

The polynomial boxes will give independent classes in the exterior quotient. For the upper bound, we use a factorization that constrains vectors killed by the mixed contractions. Its degree bound will become the nonnegative-height condition on every prefix of a path.

Let $d \le n$ and let M be a $d \times d$ matrix over a commutative ring with $n!$ invertible. On the exterior algebra of d pairs, use left derivatives and put

$$e_k = \partial_\xi^t M^k \partial_\theta, \qquad f_k = \theta^t M^k \xi,$$

$$h_k = \operatorname{tr}(M^k) - \sum_{j,l} (M^k)_{jl} \theta_j \partial_{\theta_l} - \sum_{j,l} (M^k)_{lj} \xi_j \partial_{\xi_l}.$$

Anticommutation gives

$$[e_k, f_l] = h_{k+l}, \qquad [h_k, e_l] = 2e_{k+l}, \qquad [h_k, f_l] = -2f_{k+l};$$

like-letter families commute. The transposition in the last summand of h_k allows nonsymmetric matrices.

Write $e_M = \partial_\xi^t M \partial_\theta$, $f_M = \theta^t M \xi$, and let $\Gamma_\theta(C)$ send θ_j to $\sum_i C_{ij}\theta_i$, with the same convention for ξ.

Lemma 3.1. *As a formal identity in u, v, with $C = (I + uvM)^{-1}$,*

$$\begin{aligned} \exp(ue_M)\exp(vf_I) = \exp(vf_C)\det(I + uvM) \\ \cdot \Gamma_\theta(C)\Gamma_\xi(C^t)\exp(ue_{MC}). \end{aligned} \tag{11}$$

If w has exterior degree k and $e_j w = 0$ for every $j \ge 0$, then $w = 0$ when $c = d - k < 0$. Otherwise

$$\det(I - zM)\Gamma_\theta((I - zM)^{-1})\Gamma_\xi(((I - zM)^{-1})^t)w = \sum_{j=0}^{c} \frac{(-z)^j}{(j!)^2} e_M^j f_I^j w. \tag{12}$$

In particular, the left side is polynomial of degree at most c.

Proof. For one pair and $M = (\lambda)$, both sides of (11) act on $\langle 1, \theta\xi \rangle$ by

$$\begin{pmatrix} 1 + uv\lambda & u\lambda \\ v & 1 \end{pmatrix},$$

and act as the identity on θ, ξ. This is the usual two-by-two Gauss factorization; its diagonal factor explains the determinant and the two exterior substitutions. Taking tensor products proves the formula for diagonal M. Conjugating by $\Gamma_\theta(S)\Gamma_\xi(S^{-t})$ replaces M by SMS^{-1} and fixes f_I, so the identity holds for complex matrices with distinct eigenvalues. Every formal coefficient is a polynomial in the entries of M over $\mathbb{Z}[1/n!]$; exterior exponentials truncate at d, and formal inversion introduces no denominators. Zariski density therefore proves the identity over that ring, and hence over every allowed coefficient ring.

If L is maximal with $f_I^L w \neq 0$, the e_0, f_0, h_0 relations give

$$0 = e_0 f_I^{L+1} w = (L+1)(c-L) f_I^L w.$$

For $k > 0$, exterior degree gives $L + 1 \leq d$ and $|c - L| \leq d$. Every nonzero integer factor is a unit, so $L = c$. For $k = 0$, directly $f_I^d w = d!$ times the top wedge with coefficient w, again giving $L = c$. Thus $c \geq 0$ and $f_I^{c+1} w = 0$. The last exponential in (11) kills w termwise; comparison of equal powers of u, v, followed by $uv = -z$, gives (12). □

3.1. **A rank-one perturbation**

The odd contractions enter the same identity through a rank-one perturbation of a diagonal matrix. Let $Y = \mathrm{diag}(Y_1, \ldots, Y_d)$ and let U, V be columns of operators. Assume that all entries of Y, U, V commute. Suppose

$$\sum_j V_j Y_j^a \partial_{\theta_j}, \qquad \sum_j U_j Y_j^a \partial_{\xi_j} \qquad (a \geq 0)$$

kill w, as do all mixed currents for Y. For $M = Y + \varepsilon U V^t$, $\varepsilon^2 = 0$, each current correction is a sum of products of these odd contractions, and also kills w.

Put $D(z) = I - zY$. The inverses factor as

$$(I - zM)^{-1} = D(z)^{-1}(I + \varepsilon z U V^t D(z)^{-1}),$$
$$((I - zM)^{-1})^t = D(z)^{-1}(I + \varepsilon z V U^t D(z)^{-1}).$$

The right exterior corrections are

$$\varepsilon z\Big(\sum_j U_j \theta_j\Big)\Big(\sum_k \frac{V_k}{1 - zY_k}\partial_{\theta_k}\Big), \qquad \varepsilon z\Big(\sum_j V_j \xi_j\Big)\Big(\sum_k \frac{U_k}{1 - zY_k}\partial_{\xi_k}\Big),$$

so they kill w before the diagonal actions. The determinant contributes

$$\det(I - zM) = \prod_j (1 - zY_j)\left(1 - \varepsilon z \sum_j \frac{U_j V_j}{1 - zY_j}\right).$$

Thus (12) supplies two polynomial identities, its scalar and ε coefficients. This is the only perturbation identity needed below.

4. **The homogeneous basis**

We first prove Theorem 1.1 over F for $\phi_i = x_i^{r_i}$, $a_i = x_i^{u_i}$, $b_i = x_i^{v_i}$. All moments then have weight $s + k$ under the weights in the theorem.

4.1. **Polynomial independence and a Frobenius dual**

Set

$$J = \left(\sum_i \beta_i x_i^{r_i(s+k)} : 0 \le k < n \right), \qquad E = F[x]/J.$$

The substitution $y_i = x_i^{r_i}$ is free with basis x^b, $0 \le b_i < r_i$. The ordinary Newton box proves that E is a finite graded complete intersection with basis $0 \le e_i < r_i(s+i-1)$.

For a valid path, put

$$g_m(y) = \prod_{i<j} (y_i - y_j)^{m_i}, \qquad F_{T,S}(x) = \prod_i x_i^{u_i t_i + v_i z_i}\, g_m(x_1^{r_1}, \dots, x_n^{r_n}). \tag{13}$$

Taking $\sigma_i = 1 - m_i$ in (9) and making the free substitution gives a basis

$$0 \le e_i < r_i h_i \quad \text{of } F[x]/(J : g_m(x^r)).$$

Multiplying the proposed small box by $\prod_i x_i^{u_i t_i + v_i z_i}$ places its distinct monomials inside this enlarged box. Therefore

$$\{x^e \mid 0 \le e_i < d_i\} \text{ is independent modulo } J : F_{T,S}. \tag{14}$$

Let $\varepsilon : E \to F$ extract the coefficient of the unique top-degree box monomial. It is a Frobenius functional; the graded complete-intersection socle pairing is perfect. Let X_i be transposed multiplication by x_i on E^*. Then

$$f(X)\varepsilon = 0 \quad \Longleftrightarrow \quad f \in J.$$

The full dual $\mathcal{H} = (\mathcal{P}/I)^*$ is the common kernel of the transposed odd and mixed moments in $E^* \otimes \bigwedge(\theta, \xi)$. This finite dual requires no polynomial differentiation. More explicitly, $f \mapsto f(X)\varepsilon$ identifies E with E^*. Under this identification, the kernel of $f \mapsto F_{T,S}(X) f(X)\varepsilon$ is exactly $J : F_{T,S}$. The polynomial independence statement (14) can therefore be used directly to construct functionals separating the proposed classes. This remains valid when some covering degree r_i is divisible by the characteristic.

4.2. **Triangular creation**

To create a mark at index j, we want support only at indices $k \le j$, with the coefficient at j supplying the difference factors contributed by that mark in (13). The polynomial

$\prod_{l>j}(t - X_l^{r_l})$ enforces this support; its evaluation at $t = X_k^{r_k}$ vanishes for every $k > j$. It also lets each commutator below be completed to a full moment sum. Define

$$D_j^\theta = \sum_{k\le j} \beta_k \theta_k X_k^{u_k} \prod_{l>j} (X_k^{r_k} - X_l^{r_l}),$$
$$D_j^\xi = \sum_{k\le j} \xi_k X_k^{v_k} \prod_{l>j} (X_k^{r_k} - X_l^{r_l}).$$

They preserve $\mathcal{H}$. For example, anticommuting D_j^θ with a θ-annihilator gives

$$\sum_{k\le j} \beta_k X_k^{r_k(s+a)} \prod_{l>j} (X_k^{r_k} - X_l^{r_l}).$$

Terms with $k > j$ vanish if included; expansion of the product expresses the result as a polynomial combination of ordinary moments, hence zero on E^*. Mixed commutators give polynomial combinations of opposite odd annihilators. The other supercommutators vanish.

Apply products of these operators to ε, in a fixed exterior order, obtaining $H_{T,S} \in \mathcal{H}$. Each created index is at most the operator index, so its exterior support lies below (T, S) in product Gale order. Its coefficient at $\theta_T \xi_S$ is

$$\pm \left(\prod_{i\in T} \beta_i \right) F_{T,S}(X)\varepsilon. \tag{15}$$

For a proposed linear relation in one exterior bidegree, choose its smallest nonzero exterior block in a total colex order refining this support order, and let $f(x)$ be that block. All remaining blocks lie above (T, S), whereas $H_{T,S}$ is supported at or below (T, S). Only the chosen block can contribute to their pairing. By (14), $fF_{T,S}$ is nonzero in E, and perfectness gives g with $\varepsilon(gfF_{T,S}) \neq 0$. Pairing the relation with $g(X)H_{T,S}$ is then nonzero, a contradiction. Thus the proposed monomials are independent.

4.3. **Prefix equations and the upper bound**

The upper bound uses the opposite end of the exterior order. Fix a bidegree and let $F \in E^*$ be the coefficient of the largest exterior term $\theta_T \xi_S$ of a nonzero harmonic, using descending interleaved bits $(\theta_n, \xi_n, \ldots, \theta_1, \xi_1)$. Extract the coefficient of its exact suffix beyond a prefix of length i, leaving a vector in the exterior algebra of that prefix. A contraction outside the prefix could contribute to this coefficient only from a source with an additional suffix mark. That source would be larger than the chosen term and is absent. Hence all prefix odd and mixed annihilators kill the extracted

vector. Its exterior degree is $\sum_{j\le i} m_j$, so its primitive weight in Lemma 3.1 is exactly c_i.

Apply Lemma 3.1 with

$$Y_j = X_j^{r_j}, \qquad U_j = \beta_j X_j^{u_j}, \qquad V_j = X_j^{v_j}.$$

The primitive bound gives $c_i \ge 0$. In the diagonal identity and its rank-one perturbation, set $z = t^{-1}$ and multiply by t^{c_i}. The scalar coefficient gives $A_i(t)F$ below; the ε coefficient gives $B_i(t)F$. Since the latter identity has zero constant term in z, reversing its powers gives the sharper degree bound $c_i - 1$. Thus

$$A_i(t)F = \prod_{j\le i}(t - Y_j)^{1-m_j} F, \qquad B_i(t)F = -A_i(t)\sum_{j\le i} \frac{\beta_j Y_j^s}{t - Y_j} F \tag{16}$$

are polynomials of degrees at most c_i and $c_i - 1$.

At a θ-only mark, apply the prefix creator $\sum_{j\le i} X_j^{v_j}\xi_j$. Its commutators with mixed annihilators are the opposite odd annihilators. Up to sign, the completed top coefficient is $X_i^{v_i}F$; a contribution with $j < i$ would require a source already containing ξ_i, hence a larger exterior term. Completing the mark changes its exponent in the diagonal product from zero to -1. Diagonal polynomiality therefore makes

$$\frac{X_i^{v_i} A_{i-1}(t)}{t - Y_i} F$$

a polynomial. Monic division of its numerator by $t - Y_i$ therefore has zero remainder, so

$$X_i^{v_i} A_{i-1}(Y_i)F = 0.$$

For a ξ-only mark, the analogous creator gives $X_i^{u_i} A_{i-1}(Y_i)F = 0$. If $c_i = 0$, the created vector has negative primitive weight and is zero, giving the same conclusion.

To bound the space of possible largest coefficients, we turn these necessary equations into a complete intersection of the required length. The rational expressions in (16) follow the path directly; an up-step introduces a factor $t - y_i$, a flat step leaves A_i unchanged, and a down-step removes one factor. Monic division records precisely the remainders that must vanish for these updates to stay polynomial.

Put $y_i = x_i^{r_i}$ and start with $A_0 = 1$, $B_0 = 0$. At an unmarked index set

$$A_i = (t - y_i)A_{i-1}, \qquad B_i = (t - y_i)B_{i-1} - \beta_i y_i^s A_{i-1}.$$

At a single mark divide $A_{i-1} = (t - y_i)q_i + \rho_i$, impose $x_i^{v_i}\rho_i$ for θ or $x_i^{u_i}\rho_i$ for ξ, and set

$$A_i = A_{i-1}, \qquad B_i = B_{i-1} - \beta_i y_i^s q_i.$$

At a double mark make the two monic divisions

$$A_{i-1} = (t - y_i)q_i + \rho_i, \qquad B_{i-1} - \beta_i y_i^s q_i = (t - y_i)B_i + \tau_i,$$

impose ρ_i, τ_i, and set $A_i = q_i$. Finally impose all c_n coefficients of B_n. Denote the resulting ideal by $K^{\mathrm{end}}_{T,S}$.

There is one generator at a flat step, two at a down-step, and c_n terminal generators, giving n altogether because c_n is the number of up-steps minus the number of down-steps. A zero bound makes its single-mark relation one; otherwise the positive weights of the generators form the multiset $\{d_i/r_i\}$, as follows. A single mark gives its corresponding bound. A double mark gives weights $c_i + 1$ and $c_i + s$. The first is its own bound. Match the second with the latest unmatched up-step reaching height $c_i + 1$; that up-step has bound $(c_i + 1) + s - 1 = c_i + s$. The unmatched up-steps reach heights $1, \ldots, c_n$, and their bounds are precisely the weights $s, \ldots, s + c_n - 1$ of the terminal coefficients.

Modulo these relations,

$$\frac{B_i(t)}{A_i(t)} = -\sum_{j \le i} \frac{\beta_j y_j^s}{t - y_j}.$$

At a single mark the imposed factor divides y_i^s, so it kills the remainder also in this rational identity. Every A_i remains monic. Consequently the identity can be evaluated at any common zero of the endpoint generators. There $B_n = 0$, while a nonzero repeated value a among the y_i would contribute the nonzero residue $a^s \beta_{\{i : y_i = a\}}$. Hence every y_i, and then every x_i, is zero. The n endpoint generators therefore form a regular sequence, with quotient length

$$\frac{\prod_i (d_i/r_i)}{\prod_i (1/r_i)} = \prod_i d_i.$$

Equations (16) and the single-mark relations show that this ideal annihilates every largest harmonic coefficient. In the filtration by exterior blocks, a graded piece of $\mathcal{H}$ injects into the space of its largest coefficients. That space is contained in the dual of $F[x]/K^{\mathrm{end}}_{T,S}$, and so has dimension at most $\prod_i d_i$. Summing over the blocks gives $\dim \mathcal{P}/I \le \sum_{c_i \ge 0} \prod_i d_i$. Together with the independent monomials, this proves the homogeneous basis theorem.

Applying the same prefix equations to the harmonic (15) gives $K^{\mathrm{end}}_{T,S} \subseteq J : F_{T,S}$. The endpoint length and (14) force equality

$$K^{\mathrm{end}}_{T,S} = J : F_{T,S}, \qquad \{x^e \mid 0 \le e_i < d_i\} \text{ is a basis of the colon quotient.}$$

If a path is inadmissible or has a zero bound, the primitive or single-mark argument makes the colon the unit ideal.

5. **The strict-height basis and the path counts**

We prove the homogeneous strict family first with $\phi_i = y_i$. The ordinary algebra is $S/J_{1,\boldsymbol{\beta}}$. In a prefix of length i, the zeroth odd contractions restrict the two exterior spaces to the kernels of

$$\sum_{j\le i}\partial_{\theta_j}, \qquad \sum_{j\le i}\beta_j\partial_{\xi_j}.$$

Both have dimension $i-1$, and the induced mixed pairing is nondegenerate; its possible radical is the line (β_j), whose coordinate sum is the unit $\beta_{[i]}$. In dual bases, mixed contraction is the standard contraction on $i-1$ pairs. Its primitive weight is now $(i-1)-\sum_{j\le i} m_j = c_i - 1$. This removal of one pair explains the strict bound $c_i \ge 1$.

In the rank-one argument take $U_j = \beta_j$, $V_j = 1$. For a largest harmonic coefficient, it gives

$$A_i(t)F = \prod_{j\le i}(t-X_j)^{1-m_j}F, \qquad B_i(t)F = -A_i(t)\sum_{j\le i}\frac{\beta_j}{t-X_j}F$$

of degrees at most $c_i, c_i - 1$. The endpoint recursion is the previous one with every $\beta_i y_i^s$ replaced by β_i and every single-mark factor replaced by one. At the end, impose the coefficients of

$$tB_n + \beta_{[n]}A_n. \tag{17}$$

The leading coefficient of B_i is $-\beta_{[i]}$. Thus the leading terms in (17) cancel. Single marks contribute degree c_i, double marks degrees c_i+1, c_i, and the terminal coefficients degrees $1,\dots,c_n$. Matching down-steps with up-steps gives exactly the multiset $\{c_i\}$.

At a common zero, the rational recursion and (17) give

$$\sum_i \frac{\beta_i}{t-y_i} = \frac{\beta_{[n]}}{t}.$$

Subset-sum residues force every $y_i = 0$. Thus the endpoint ideal is a complete intersection of length $\prod_i c_i$, and prefix polynomiality gives the corresponding upper bound.

For independence, take the constant-one profile with $\sigma_i = 1 - m_i$. Its enlarged colon bounds are $h_i = c_i + m_i$. Multiplication by $\prod_i y_i^{m_i}$ embeds the boxes $0 \le e_i < c_i$ in the enlarged boxes. The creators are

$$D_j^{\theta} = \sum_{k\le j}\beta_k\theta_k X_k\prod_{l>j}(X_k - X_l), \qquad D_j^{\xi} = \sum_{k\le j}\xi_k X_k\prod_{l>j}(X_k - X_l).$$

The extra X_k makes their ordinary anticommutators positive moments. Their top coefficient is a unit times

$$F_{T,S}(y)\varepsilon, \qquad F_{T,S}(y) = \prod_i y_i^{m_i} \prod_{i<j} (y_i - y_j)^{m_i}.$$

The triangular pairing argument proves independence. The endpoint ideal is again exactly $J_{1,\boldsymbol{\beta}} : F_{T,S}$, by inclusion and the box length. Free substitution $y_i = x_i^{r_i}$ gives the bounds $r_i c_i$ and multiplies every box size by $\prod_i r_i$. This proves the homogeneous form of Theorem 1.2.

Counting the boxes. For the paired family, divide each local bound by r_i. At arrival height c, the up, combined flat, and down weights are

$$c + s - 1, \qquad 2c + s, \qquad c + 1.$$

Let $A_i(z)$ count boxes in the first i positions by endpoint height. Then

$$A_i(z) = r_i\big((z+1)^2 A'_{i-1}(z) + s(z+1)A_{i-1}(z)\big), \qquad A_0 = 1.$$

It follows that $A_n(z) = \prod_i r_i\,(s)_n(1+z)^n$. Total exterior degree is $n - c_n$, and this polynomial is palindromic, giving the rank and exterior polynomial of Theorem 1.1.

For the strict family, first take $r_i = 1$. The first step is compulsory and $A_1(z) = z$; thereafter

$$A_n(z) = (z+1)^2 A'_{n-1}(z) + (z - z^{-1})A_{n-1}(z) = n!z(1+z)^{n-1}.$$

Reversing endpoint height to exterior degree and restoring the covering factors proves the count in Theorem 1.2.

6. **Deformation and arbitrary coefficient rings**

The homogeneous proof uses the ordinary algebra in two ways; its perfect pairing turns polynomial independence into exterior independence, and its grading controls spanning. Both survive a lower-weight perturbation, even when the new coefficients couple different coordinates. We establish these two facts before descending the construction to arbitrary coefficient rings.

6.1. **The ordinary algebra and its pairing**

Work first over F. For the paired family, put $J = (P_0, \dots, P_{n-1})$ and $E = F[x]/J$. The characteristic-polynomial recurrence for $\phi_1, \dots, \phi_n$ makes every later ordinary

moment a combination of these; the coefficients $\beta_i a_i b_i$ need only be fixed as the index varies. Their highest forms are the homogeneous ordinary regular sequence. Therefore (10) gives

$$\operatorname{gr} E = E_0,$$

where E_0 is the ordinary power algebra. Its box is an actual basis of E.

The top-box-coefficient functional $\varepsilon : E \to F$ is still perfect. Ordered by degree in one copy and complementary degree in the other, its pairing matrix is block triangular; the diagonal blocks are the graded complete-intersection socle pairings. This proves nondegeneracy without any condition on lower coefficients.

For a valid path with positive bounds, put

$$F_{T,S} = \prod_i a_i^{t_i} b_i^{z_i} \prod_{i<j} (\phi_i - \phi_j)^{m_i}.$$

Its highest form is (13). If a nonzero small-box polynomial f satisfied $F_{T,S} f \in J$, its highest part would contradict (14) in E_0. Thus the same small box is independent modulo $J : F_{T,S}$ in the actual algebra.

6.2. **Transfer to the exterior quotient**

On E^* use the creators

$$\begin{aligned} D_j^{\theta} &= \sum_{k \le j} \beta_k \theta_k a_k(X) \prod_{l>j} (\phi_k(X) - \phi_l(X)), \\ D_j^{\xi} &= \sum_{k \le j} \xi_k b_k(X) \prod_{l>j} (\phi_k(X) - \phi_l(X)). \end{aligned} \tag{18}$$

Their supercommutators are the same polynomial combinations of ordinary or opposite odd moments as before; the ordinary anticommutator contains exactly $a_k b_k$. All coefficient operators commute even when the polynomials couple different coordinates. The support remains triangular and the top coefficient is a unit times $F_{T,S}(X)\varepsilon$. The perfect pairing and small-box independence therefore prove independence of (3).

The highest form of every moment is its power-case moment. Hence the power quotient surjects onto the associated graded quotient. Its basis spans the latter and, by induction on weight, spans the actual quotient. Independence makes this surjection an isomorphism, proving Theorem 1.1 over F.

For the strict family, use $J = (\sum_i \beta_i \phi_i^k : 1 \le k \le n)$ and

$$F_{T,S} = \prod_i \phi_i^{m_i} \prod_{i<j} (\phi_i - \phi_j)^{m_i}.$$

The ordinary associated graded and Frobenius arguments are identical. Replace a_k, b_k in (18) by ϕ_k in both creators. Their odd anticommutators are positive ordinary moments, so the strict power boxes again give actual independence and filtered spanning. This proves Theorem 1.2 over F.

6.3. **Universal bases and colons**

The passage to coefficient rings uses a single universal family. Let

$$B_0 = \mathbb{Z}[1/n!, \beta_1, \dots, \beta_n, \Delta(\boldsymbol{\beta})^{-1}], \qquad \Delta(\boldsymbol{\beta}) = \prod_{\varnothing \neq U \subseteq [n]} \beta_U.$$

Adjoin independent variables for all allowed lower-weight coefficients in (2), giving a polynomial ring B. There are finitely many such coefficients. Both rings are reduced and Jacobson, and every closed residue field has characteristic greater than n.

The descent rests on the following elementary principle; specified elements of a finite module over a reduced Jacobson ring form a basis if they do so on every closed-point fiber. Indeed, the map from the free module on those elements has zero cokernel by Nakayama, while every coordinate of a kernel vector lies in all maximal ideals and is therefore zero.

The homogeneous ordinary and full quotients are finite over B_0; each weighted piece is finite, and all pieces above the uniform box bound vanish on every closed fiber, hence vanish. The principle gives the power bases over B_0. Filtered reduction then makes the coupled quotients finite over B, and the field proofs give their prescribed bases on every closed fiber. The same principle gives free B-bases. Every allowed choice over K is a base change from B, proving both main theorems, including nonreduced rings. Since reductions strictly lower weight, the exact associated graded identifications also survive base change.

The endpoint colons require one further argument; their multiplication maps must remain injective under base change. In the paired recursion replace y_i by ϕ_i, $\beta_i y_i^s$ by $\beta_i a_i b_i$, and the single-mark factors by b_i or a_i. In the strict recursion replace y_i by ϕ_i and retain (17). For either family,

$$K_{T,S}^{\mathrm{end}} = J : F_{T,S} \tag{19}$$

holds over every allowed K, with the corresponding free box basis. Over a field, the highest endpoint generators are the power-case regular sequence, so their quotient has the same length. The Gauss identity applied to the triangular top coefficient gives the endpoint-to-colon inclusion; box independence forces equality.

For the universal assertion, the endpoint quotient has its box basis over B by the same finite-fiber argument. Multiplication by $F_{T,S}$ maps it into the free ordinary

algebra; all defining products vanish on each closed fiber, hence their coordinates vanish in B. This matrix is injective on every closed fiber. Its maximal minors generate one, so a linear combination of the corresponding adjugates supplies a left inverse. It remains injective after every base change, proving (19). Inadmissible and zero-bound paths give the unit ideal by the primitive and single-mark equations.

The pairing

$$([f],[g]) \longmapsto \varepsilon(F_{T,S} f g)$$

on each colon quotient is perfect. Over a field this follows from the Frobenius pairing on E and the definition of $J : F_{T,S}$. Its universal determinant is nonzero on every closed fiber, hence is a unit; therefore perfectness survives every base change. The same argument proves the universal version of the polynomial boxes (5). In that case the coefficient matrix and its unit determinant factor also give the colon inclusion directly by the adjugate identity.

6.4. **Flat and formal versions**

Clear denominators in the weights. Weighted homogenization of the polynomials in (2) has coefficients in $K[t]$, gives the original data at $t = 1$, and the power data at $t = 0$. The universal theorem gives the same free basis over $K[t]$, proving the asserted flat degeneration.

There is also a local version in the opposite filtration direction. Over F, take formal series in $F[[x_1, \dots, x_n]]$ with

$$\phi_i = x_i^{r_i} + \text{higher-weight terms},$$
$$a_i = x_i^{u_i} + \text{higher-weight terms}, \qquad b_i = x_i^{v_i} + \text{higher-weight terms}.$$

The same paired boxes form a basis of the completed moment quotient, and its associated graded for weighted order is the power algebra. The strict version requires only the corresponding condition on ϕ_i and gives the strict boxes. The endpoint colons and pairings remain exact over F.

Indeed, the complete filtered regular-sequence lemma is (10) with lowest forms; cancel a lowest coefficient relation using the Koszul syzygies of the regular sequence, and iterate. Weights strictly increase, so the corrections converge. The finite graded quotient makes the ordinary local quotient finite; equivalently, sufficiently high filtration pieces stabilize and Nakayama makes them zero. Its top-box functional is perfect by the same triangular pairing argument. The transposed X_i are jointly nilpotent, so all formal creators are well-defined. A nonzero lowest class of $F_{T,S} f$ gives small-box independence; triangular creation gives full independence, and reduction

terminates in the finite ordinary algebra. The endpoint inclusions and equal box dimensions give the colon identities.

7. Monic covers and local factors

For separated monic covers, the preceding bases admit both a relative and a local interpretation. Assume that $G(t) \in K[t]$ is monic of degree s, each $\phi_i \in K[x_i]$ is monic of degree r_i, and a_i, b_i are monic with $a_i b_i = G(\phi_i)$, so that Theorem 1.1 applies. Monic division will describe the relative basis, and the roots of G will separate the local factors.

The relative description begins with the substitution $y_i \mapsto \phi_i(x_i)$, which makes $K[x]$ free over $K[y]$ on x^b, $0 \le b_i < r_i$. For $J = J_{G,\boldsymbol{\beta}} K[x]$, flat base change gives

$$(J_{G,\boldsymbol{\beta}} : g_\sigma) K[x] = J : g_\sigma(\phi).$$

The bases (5), followed by monic division, give the actual rectangular basis $0 \le e_i < r_i h_i$ of this covered colon. The transition from $x^b \prod_i \phi_i^{a_i}$ to monomials is unitriangular. Multiplication by any prescribed monic prefactor $\prod_i a_i^{t_i} b_i^{z_i}$ embeds the corresponding smaller box by a unitriangular maximal minor.

A monic extension of degree r also has a perfect relative pairing; extract the coefficient of x^{r-1}. Its matrix in $1, x, \dots, x^{r-1}$ has zeros below the complementary-degree antidiagonal and ones on it, so its determinant is ± 1. Composing these pairings with the ordinary moment pairing gives a perfect covered Frobenius pairing over any allowed base ring.

7.1. The exact finiteness condition

Proposition 7.1. *For the common-G monic-cover family, with $n!$ invertible but arbitrary weights,*

$$\mathcal{P}/I \text{ is finite over } K \quad \Longleftrightarrow \quad \beta_U \in K^\times \text{ for every nonempty } U \subseteq [n].$$

The same criterion holds for the separated strict-height covers.

Proof. Sufficiency is the basis theorem. If a subset sum is a nonunit, pass to a maximal-ideal residue field containing it and then to an algebraic closure L. Choose a root a of G. The ordinary y-moments vanish on the nonconstant line

$$y_i = t \ (i \in U), \qquad y_i = a \ (i \notin U),$$

because $\beta_U = 0$. Thus the ordinary quotient surjects onto $L[t]$. The monic cover is faithfully free, so the covered ordinary quotient is infinite-dimensional as well. It is

the exterior-degree-zero part of the full quotient. A finite K-module cannot have such a fiber. For the strict family use $a = 0$ and the positive ordinary moments. □

The converse uses the common-polynomial structure; Theorem 1.1 asserts the basis under (1) for arbitrary coupled products $a_i b_i$.

7.2. **Roots and ramification**

The basis theorem fixes the global rank as the lower coefficients vary. The local description explains how that rank is distributed among the roots; the same homogeneous moment algebras reappear, with covering degrees replaced by ramification orders. Work over $\mathbb{C}$ and assume (1). The ordinary support consists exactly of points $\rho = (\rho_i)$ with $G(\phi_i(\rho_i)) = 0$ for every i. Indeed, the rational function $\sum_i \beta_i G(y_i)/(t - y_i)$ has its first n Laurent coefficients zero at an ordinary zero, and hence is zero. A repeated value λ with $G(\lambda) \neq 0$ would have nonzero residue.

Fix such a point and partition indices into $C_\lambda = \{i : \phi_i(\rho_i) = \lambda\}$. Let s_λ be the root multiplicity of G at λ, and put

$$r'_i = \operatorname{ord}_{x_i=\rho_i}(\phi_i - \lambda), \quad u'_i = \operatorname{ord}_{x_i=\rho_i} a_i, \quad v'_i = \operatorname{ord}_{x_i=\rho_i} b_i \qquad (i \in C_\lambda).$$

Then $u'_i + v'_i = r'_i s_\lambda$.

Theorem 7.2. *The local factor at ρ is, as an exterior-bigraded algebra, the super tensor product over λ of the homogeneous paired algebras with data*

$$(s, r_i, u_i, v_i, \beta_i) = (s_\lambda, r'_i, u'_i, v'_i, \beta_i)_{i \in C_\lambda}.$$

The full algebra is the finite direct product of these local factors.

Proof. Choose formal coordinates z_i with $\phi_i - \lambda = z_i^{r'_i}$, by extracting a root of its leading coefficient and formal unit. Write

$$G(t) = (t - \lambda)^{s_\lambda} H_\lambda(t), \qquad a_i = z_i^{u'_i} A_i(z_i), \quad b_i = z_i^{v'_i} B_i(z_i),$$

where $A_i B_i = H_\lambda(\lambda + z_i^{r'_i})$ is a unit. Change odd coordinates to $\theta'_i = B_i \theta_i$, $\xi'_i = A_i \xi_i$.

Every old ordinary moment is a sum of homogeneous cluster moments with coefficients that are common functions of $z_i^{r'_i}$. Conversely, the old ordinary local algebra is Artinian. For a sufficiently large N, Hermite interpolation supplies a polynomial with jet $(t - \lambda)^k / H_\lambda(t)$ at a selected root and zero jets at the others, modulo their Nth powers. Inserting it as a linear combination of old moments gives the selected homogeneous cluster moment. Thus the ordinary local ideals agree.

Modulo this common ideal, functions of the nilpotent $\phi_i - \lambda_i$ are represented by finite polynomials. After the odd coordinate change, the odd coefficients are $z_i^{v_i'}$ and $\beta_i z_i^{u_i'}$, while the mixed coefficient is $H_\lambda(\phi_i)^{-1}$. Interpolation with jets $(t-\lambda)^k$ separates the odd moments; jets $H_\lambda(t)(t-\lambda)^k$ separate the mixed ones and remove the unit. The full completed ideals therefore agree. Each cluster quotient is finite and supported at zero, so completion does not change it. The final product decomposition is the ordinary Artinian idempotent decomposition; its idempotents are central even elements. □

If $n_\lambda = |C_\lambda|$, the local rank is

$$2^n \left(\prod_i r_i' \right) \prod_\lambda (s_\lambda)_{n_\lambda}.$$

For fixed λ, the ramification orders over its preimages under ϕ_i sum to r_i. Summing local ranks and using $\prod_\lambda (1-t)^{-s_\lambda} = (1-t)^{-s}$ recovers the global rising-factorial formula. The exterior polynomial follows with the same factor $(1+z)^n$.

When all roots and preimages are simple and $a_i = G(\phi_i)$, $b_i = 1$, each cluster is a strict-height weighted algebra tensored with one free odd generator. Indeed, in its homogeneous coordinates put

$$\zeta = \beta_{C_\lambda}^{-1} \sum_{i \in C_\lambda} \beta_i \xi_i, \qquad \xi_i^0 = \xi_i - \zeta.$$

Then $\sum_i \beta_i \xi_i^0 = 0$. Positive ξ moments change only by ζ times ordinary moments, and mixed moments by θ moments times ζ. The remaining equations are precisely the strict equations in the ξ_i^0, and no equation involves ζ.

8. **Coinvariants and the color action**

8.1. **Opposite-character wreath products**

Over $\mathbb{C}$, let $r \geq 2$, $1 \leq a \leq r-1$, and let $G(r,1,n)$ act with cyclic characters $1, a, -a$ on x_i, θ_i, ξ_i, while S_n permutes indices. Its positive invariant ideal is generated by the sums of the one-index blocks

$$x_i^{r(k+1)}, \qquad x_i^{r-a+rk}\theta_i, \qquad x_i^{a+rk}\xi_i, \qquad x_i^{rk}\theta_i\xi_i \qquad (k \geq 0).$$

To see this, average first over the diagonal cyclic group. Multiply the resulting one-index block sums and subtract terms with coincident indices, which have fewer occupied indices. Induction gives every nonzero permutation orbit sum. Repeated

odd blocks have zero orbit sum; all other coefficients are nonzero integers and may be divided out.

Theorem 1.1, with $s = 1$, $r_i = r$, $u_i = a$, $v_i = r - a$, and $\beta_i = 1$, gives the basis

$$x^e \theta_T \xi_S, \qquad c_i \geq 0, \qquad 0 \leq e_i < rc_i + (r-a)t_i + az_i. \tag{20}$$

Its dimension is $(2r)^n n!$, its total-exterior polynomial is $r^n n!(1+z)^n$, and its full polynomial/exterior Hilbert series is

$$\sum_{c_i \geq 0} u^{|T|} v^{|S|} \prod_i [d_i]_q, \qquad [d]_q = 1 + q + \cdots + q^{d-1}.$$

For $r = 2$, $a = 1$, both odd families carry the signed permutation representation, and writing $b_i = d_i - 1 = 2c_i + m_i - 1$ gives $b_1 = 1 - m_1$ and $b_i - b_{i-1} = 2 - m_i - m_{i-1}$. These are exactly the type-B bounds in [1, Definition 9.5], identifying the specialization of (20) with the proposed basis.

If both cyclic exterior characters are zero, the invariant ideal is instead the strict family with $\phi_i = x_i^r$ and $\beta_i = 1$. Its basis has $c_i \geq 1$ and $0 \leq e_i < rc_i$, with dimension $r^n 2^{n-1} n!$. At $r = 1$ this is the type-A basis. The type-A statement holds over $\mathbb{Z}[1/n!]$; the wreath-product statements also hold over fields of characteristic greater than n containing a primitive rth root of unity, with r invertible. These group-action conditions are separate from the moment theorem, which permits covering degrees divisible by the characteristic.

8.2. **An equivariant filtration in type A**

Although the monomial basis distinguishes the two colors, its path bounds depend only on the number of marks at each index. Repeating the prefix argument while retaining each entire color space therefore gives a filtration invariant under mixing the two exterior families, and the polynomial boxes determine its character.

Let $R_n^{(1,2)}$ be the strict quotient with $\phi_i = x_i$ and $\beta_i = 1$ over $\mathbb{C}$. Put $A = \mathbb{C}[x_1, \ldots, x_n]$, $J = (p_1, \ldots, p_n)$, and let $V = \mathbb{C}^2$ mix the two exterior colors at every index. For $m \in \{0, 1, 2\}^n$, put

$$c_i = i - \sum_{j \leq i} m_j, \qquad f_m = \prod_{i=1}^{n} \left(x_i \prod_{j>i} (x_i - x_j) \right)^{m_i}, \qquad K_m = J : f_m.$$

Call m admissible if all $c_i \geq 1$. The prefix bounds give $K_m = A$ otherwise; for admissible m,

$$C_m = A/K_m, \qquad \mathrm{Hilb}(C_m, q) = \prod_i [c_i]_q.$$

Give both colors at index i weight 3^{i-1}, and give polynomial variables weight zero. The multiplicity weight $\sum_i m_i 3^{i-1}$ distinguishes multiplicity vectors but not their colors. Since $0 \le m_i \le 2$, comparing these weights is the same as comparing multiplicities from the largest index downwards. Thus an extra mark outside a prefix still gives a strictly larger block, as required by the prefix argument.

Theorem 8.1. *For smallest-weight initial forms,*

$$\operatorname{in}_w^{\min} I_n = \bigoplus_{m\in\{0,1,2\}^n} K_m \otimes \bigotimes_i \bigwedge^{m_i} V. \tag{21}$$

Consequently there is a $\mathrm{GL}(V)$*-equivariant flat algebra degeneration with special fiber*

$$\operatorname{gr}_w R_n^{(1,2)} = \bigoplus_{m \text{ admissible}} C_m \otimes \bigotimes_i \bigwedge^{m_i} V.$$

The multiplication is induced from the polynomial and exterior products. The special fiber has an action of $\mathrm{GL}(V)^n$*.*

Proof. Work in fixed polynomial and exterior degrees. Take a harmonic whose largest multiplicity vector is m, and extract any exact colored suffix beyond a prefix. An outside contraction could come only from a larger multiplicity source, which is absent. The prefix proof therefore applies to every colored coefficient in this block and places it in $K_m^\perp$; inadmissible blocks vanish.

Let L_m be the coefficient space of smallest-weight initial relations in block m. A relation beginning at weight $w(m)$ pairs with a harmonic supported at weights at most $w(m)$ only through that block. Orthogonal-complement linear algebra therefore identifies $L_m^\perp$ with the block image of those harmonics; explicitly, $(\mathcal{H} \cap U_{\le w(m)})^\perp = I_n + U_{>w(m)}$. Hence

$$L_m^\perp \subseteq K_m^\perp \otimes \left(\bigotimes_i \bigwedge^{m_i} V \right)^*.$$

The basis theorem makes the sum of these dimension bounds an equality, in each polynomial degree; there are $2^{\#\{i:m_i=1\}}$ colorings, each with the same box. Every block bound is therefore equality, proving (21).

The right side is an ideal; adding a nonvanishing exterior factor at i multiplies f_m by $x_i \prod_{j>i}(x_i - x_j)$, so $K_m \subseteq K_{m+e_i}$. Weight scaling commutes with the diagonal color action, and the weight-Rees family is finite free over $\mathbb{C}[t]$. Each block is also stable under independent color changes at the indices, giving the larger action on the special fiber. □

Semisimplicity of rational GL_2 representations gives, in each polynomial degree, a generally noncanonical module decomposition

$$R_n^{(1,2)} \simeq \bigoplus_{m \text{ admissible}} C_m \otimes V^{\otimes \ell(m)} \otimes (\det V)^{d(m)},$$

where $\ell(m) = \#\{i : m_i = 1\}$ and $d(m) = \#\{i : m_i = 2\}$. This is a module decomposition; the algebra structure is the filtered one above.

The multiplicity of $\mathrm{Sym}^{\ell-2j} V \otimes (\det V)^j$ in $V^{\otimes \ell}$ is $\binom{\ell}{j} - \binom{\ell}{j-1}$. Hence the multiplicity polynomial of $\mathrm{Sym}^a V \otimes (\det V)^b$ is

$$\sum_{\substack{m \text{ admissible, } 0 \le j \le \ell(m)/2 \\ \ell(m)-2j=a,\ d(m)+j=b}} \left(\binom{\ell(m)}{j} - \binom{\ell(m)}{j-1} \right) \prod_i [c_i]_q.$$

Equivalently, the full polynomially graded color character is

$$\mathrm{ch}_{\mathrm{GL}_2}(R_n^{(1,2)}; q; u, v) = \sum_{m \text{ admissible}} \left(\prod_i [c_i]_q \right) (u+v)^{\ell(m)} (uv)^{d(m)}. \tag{22}$$

The index filtration does not supply a simultaneous S_n-equivariant splitting.

8.3. **The Eulerian character**

Define the homogeneous Eulerian polynomial

$$\mathcal{E}_n(a,b) = \sum_{\pi \in S_n} a^{\mathrm{asc}(\pi)} b^{\mathrm{des}(\pi)}.$$

Then the ungraded color character has the compact form

$$\mathrm{ch}_{\mathrm{GL}_2} R_n^{(1,2)}(u,v) = \mathcal{E}_n(1+u, 1+v). \tag{23}$$

Indeed, after the compulsory first up-step write the height as $k+1$. The up, colored-flat and down weights in (22) at $q=1$ are $k+2$, $(k+1)(u+v)$ and kuv. Their transfer operator is $\mathcal{L} = (z+u)(z+v)\partial_z + 2z + u + v$, so the character is $(\mathcal{L}^{n-1}1)|_{z=1}$. Inserting the largest letter in a permutation gives

$$\mathcal{E}_{n+1} = (a+b)\mathcal{E}_n + ab(\partial_a + \partial_b)\mathcal{E}_n, \qquad \mathcal{E}_1 = 1.$$

Substitution $a = z+u$, $b = z+v$ identifies this recurrence with $\mathcal{L}$, proving (23).

References

[1] J. Lentfer, *A conjectural basis for the* $(1,2)$*-bosonic-fermionic coinvariant ring*, Algebr. Comb. **8** (2025), no. 3, 711–743. DOI 10.5802/alco.424.