Singular-vector spans beyond boundary format

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Abstract. Singular-vector span defects beyond boundary format are governed by multiplication in Artinian reductions of Segre rings. We prove that the first Lefschetz map of every Segre ring has maximal rank in every characteristic, and use it to classify all defects one step beyond boundary format. The same comparison shows that a generic tensor of any format lies in the span of its singular rank-one tensors. Integral multiplication for two factors and rigidity of products of chains give the rank theorem; the rigidity argument also yields full spans at arbitrary excess, both for sufficiently many factors and for sufficiently large dimensions with the number of factors fixed.

1. Introduction

The singular rank-one tensors attached to a generic tensor form a finite set, but their span can be smaller than the critical space cut out by the singular-vector equations. Over $\mathbb{C}$, Draisma, Ottaviani and Tocino [1] proved equality when the projective factor dimensions satisfy the triangle inequalities. Immediately beyond that range, Teixeira Turatti and Ventura [4] conjectured the exceptional formats and their defects. We prove this classification in every characteristic by identifying the defect with a multiplication kernel in a Segre ring. The identification also explains why the tensor itself remains in the span, even when the span is defective, and gives full-span ranges farther beyond boundary format.

Let K be an algebraically closed field. We use the coordinate bilinear forms $(x, y) = \sum_j x_j y_j$. A tuple $([x_1], \dots, [x_r])$ is *singular* for $T \in \bigotimes_i K^{n_i+1}$ if contraction against all slots except i is proportional to x_i, for every i. Write Z_T for the singular-tuple scheme in the Segre embedding. The *critical space* H_T is cut out by these equations before restricting to rank-one tensors; thus $Z_T \subset \mathbb{P}(H_T)$ and $T \in H_T$.

One step beyond boundary format, the multiplication kernel is the kernel of the first Lefschetz map, so the classification rests on a maximal-rank theorem for every Segre ring. For a tuple $\mathbf{n} = (n_1, \dots, n_k)$ of positive integers, put

$$R(\mathbf{n}) = \bigoplus_{q \geq 0} \bigotimes_{i=1}^{k} \operatorname{Sym}^q K^{n_i+1}, \qquad d = 1 + \sum_i n_i.$$

This is the homogeneous coordinate ring of the Segre embedding of $\prod_i \mathbb{P}^{n_i}$.

Theorem 1.1. *Let $A = R(\mathbf{n})/(\theta_1, \ldots, \theta_d)$ be a general linear Artinian reduction, and let $\ell \in A_1$ be general. Multiplication*

$$\ell : A_1 \longrightarrow A_2$$

has maximal rank. For $k = 2$ and $n_1, n_2 \geq 2$, its only nonzero kernel dimensions, up to transposition, are

(n_1, n_2)	$(2,2)$	$(2,3)$	$(2,4)$
$\dim \ker \ell$	3	3	2

For $k \geq 3$, the only nonzero kernel dimensions, up to permutation, are

$\mathbf{n}$	$(1,1,1)$	$(1,1,2)$	$(1,1,3)$
$\dim \ker \ell$	3	3	1

For two factors with $\min(n_1, n_2) = 1$, one has $A_2 = 0$; for $k = 1$, one has $A = K$.

Thus every exceptional defect is forced by the positive difference $\dim A_1 - \dim A_2$. For a tensor with smaller projective factor dimensions $n_1, \ldots, n_k$, write $n = \sum_i n_i$. Boundary format has last vector-space dimension $n + 1$; one step beyond it has last dimension $n + 2$. The graph-span formula in Section 2 identifies the defect of Z_T in $\mathbb{P}(H_T)$ with the kernel in Theorem 1.1.

Corollary 1.2. *Assume either $k \geq 3$, or $k = 2$ and $n_1, n_2 \geq 2$. Put $n = \sum_i n_i$. For a generic tensor of format $(n_1 + 1, \ldots, n_k + 1, n + 2)$, the singular tuples span $\mathbb{P}(H_T)$, with precisely the following exceptions*

Formats	$\operatorname{codim}_{\mathbb{P}(H_T)} \langle Z_T \rangle$
$(3,3,6)$, $(3,4,7)$	3
$(3,5,8)$	2
$(2,2,2,5)$, $(2,2,3,6)$	3
$(2,2,4,7)$	1

Permuting factors gives the same conclusions.

The remaining three-factor tensor family $(2, b+1, b+3)$ has defect b, because its Artinian reduction has $\dim A_1 = b$ and $A_2 = 0$. This accounts for all formats with at least three factors one step beyond boundary format. For matrices, $A = K$ and the defect is zero.

The graph-span formula applies to every last dimension at least $n+1$. It also shows that the additional linear equations vanish on T itself. Together with the range satisfying the triangle inequalities, this gives the following consequence without a restriction on format.

Theorem 1.3. *A generic tensor of every format is a linear combination of its singular rank-one tensors.*

A cohomology gap in the Koszul resolution shows that the linear equations on the singular-tuple scheme are precisely those detected by multiplication in the Segre ring. We obtain this gap by applying the critical-space argument of Draisma, Ottaviani and Tocino to the graph of the last contraction map with a multiquadratic twist. Writing the contractions directly keeps this part of the proof valid in characteristic two.

The rank calculation uses two forms of the same staircase degeneration. For two Segre factors, an integral shelling proof of the descent basis of Oeding, Raicu and Sam [2] reduces multiplication to two unimodular starting cases and an induction on the last column. With more factors, degree-two multiplication becomes an infinitesimal rigidity matrix; coning and gluing propagate rigidity through products of chains, while successive completions of one hexagon supply the small starting boxes. Both arguments work without exceptional characteristics.

The rigidity argument also controls larger excess. If the last dimension is $n+e+1$, Section 6 proves full span when $k \geq 4e$, and when $k \geq 3\lceil \log_2(4e) \rceil$ for $e \geq 3$. For each fixed $k \geq 3$, it gives explicit sufficient lower bounds on the smaller dimensions. The same span–multiplication comparison therefore treats both the exact first-excess classification and uniform ranges farther beyond boundary format.

2. The graph-span formula

Let $X = \prod_{i=1}^{k} \mathbb{P}(V_i)$, $U = \bigotimes_i V_i$, $n = \dim X$, and $R = R(\mathbf{n})$. Write a tensor $T \in U \otimes K^m$ in slices $f_1, \dots, f_m \in R_1$, using the bilinear identifications. Define

$$\mu_T : R_1 \otimes K^m \longrightarrow R_2, \qquad (l_r)_r \longmapsto \sum_r f_r l_r, \qquad \Gamma_T = \operatorname{im} \mu_T^*.$$

Degree-two evaluations span R_2^*, so Γ_T is spanned by $x \otimes (f_1(x), \dots, f_m(x))$, for $x \in X$. Thus Γ_T is the linear span of the graph of the last contraction map. The singular tuples are cut out on this graph by the remaining critical equations. The following proposition says that these equations account for every linear relation on that finite scheme.

Proposition 2.1. *Suppose the slices have no common zero and Z_T is finite and reduced. Then*

$$\operatorname{span} Z_T = \Gamma_T \cap H_T, \qquad \dim \operatorname{span} Z_T = \operatorname{rank} \mu_T - \sum_i \binom{n_i+1}{2}. \tag{1}$$

In particular, $T \in \operatorname{span} Z_T$. The hypotheses hold generically whenever $m \geq n+1$.

We use the following characteristic-independent cohomology facts

$$\begin{aligned} t>0: &\quad H^q(\mathbb{P}^N, \Omega^p(t)) = 0 \ (q>0), \quad H^0 = 0 \ (t \leq p), \\ t=0: &\quad H^q(\mathbb{P}^N, \Omega^p) = 0 \ (q \neq p), \\ t<0: &\quad H^q(\mathbb{P}^N, \Omega^p(t)) = 0 \ (q<N). \end{aligned}$$

The Euler sequence and its exterior powers give the first two lines; the degree-t Koszul complex gives the additional vanishing of global sections, and Serre duality gives the last line. See [3, Section 50.11].

Proof of Proposition 2.1. Base-point-freeness makes the last singular-vector equation the graph $[y] = [f_1(x) : \cdots : f_m(x)]$, scheme-theoretically. On the graph, the remaining equations are the zero scheme Z of

$$s_i(x) = \sum_r f_r(x) c_i(f_r)(x) \pmod{\langle x_i \rangle}$$

in the rank-n bundle

$$E = \bigoplus_i \pi_i^* Q_i \otimes \mathcal{O}(2, \ldots, 2, 1, 2, \ldots, 2).$$

Here c_i contracts all slots except i, and the 1 is in position i. Finiteness of Z makes this section regular.

Put $L = \mathcal{O}(2, \ldots, 2)$. For $r \geq 2$, the summands of $\bigwedge^r E^* \otimes L$ are

$$\boxtimes_i \Omega^{r_i}_{\mathbb{P}^{n_i}}(2r_i - 2r + 2), \qquad \sum_i r_i = r.$$

A factor with nonzero degree-zero cohomology would require $r_i > 2r-2$, which is impossible. Every nonzero factor cohomology has degree at least r_i, and Künneth gives

$$H^q(X, \bigwedge^r E^* \otimes L) = 0 \qquad (r \geq 2, \ q < r). \tag{2}$$

The twisted Koszul resolution yields

$$H^0(X, I_Z \otimes L) \simeq H^0(X, E^* \otimes L) \simeq \bigoplus_i \bigwedge^2 V_i^*.$$

Pairing the section with $x_p \wedge x_q$ gives $\sum_r f_r(x_p c_i(f_r)_q - x_q c_i(f_r)_p)$, exactly the image of a small-mode critical equation under μ_T. Write $\mathcal{C}_T$ for the span of those equations. The displayed isomorphism shows that μ_T is injective on $\mathcal{C}_T$, whose dimension is $\sum_i \binom{n_i+1}{2}$.

A linear form on tensor space restricts to the graph as its image under μ_T. It vanishes on Z_T precisely when that image lies in $\mu_T(\mathcal{C}_T)$. Hence

$$(\operatorname{span} Z_T)^\perp = \ker \mu_T + \mathcal{C}_T.$$

The last-mode critical equations already belong to $\ker \mu_T$, since their images are $f_r f_s - f_s f_r$. Taking annihilators gives $\operatorname{span} Z_T = \Gamma_T \cap H_T$. Subtracting the independent $\mathcal{C}_T$ equations from $\dim \Gamma_T = \operatorname{rank} \mu_T$ gives the dimension formula in (1).

The functional $\tau \in R_2^*$ defined by

$$\tau\left(\prod_i x_{i,p_i} x_{i,q_i}\right) = \prod_i \delta_{p_i q_i}$$

satisfies $\tau(f_r l_r) = (f_r, l_r)$, hence $\mu_T^* \tau = T$. Thus $T \in \Gamma_T \cap H_T$. Genericity follows from the next lemma. □

Lemma 2.2. *The singular-tuple scheme of a generic tensor is finite and reduced. A general family of $m \geq n+1$ multilinear forms on X is base-point-free.*

Proof. Write $Y = \prod_{i=1}^r \mathbb{P}(V_i)$, $q = \dim Y$, and $D = \dim \bigotimes_i V_i$. At $x = ([v_1], \ldots, [v_r])$, the linear conditions on T span

$$C(x) = \sum_i \left(\bigotimes_{j \neq i} v_j\right) \otimes v_i^\perp.$$

If no v_i is isotropic, the summands are independent and $\dim C(x) = q$. If exactly $a \geq 1$ are isotropic, their only overlaps are the common tensor line, giving $\dim C(x) = q - a + 1$. The corresponding stratum has dimension $q - a$, so its singular incidence has dimension $D - 1$. In characteristic two the reduced isotropic locus is a hyperplane; the same calculation applies. By projectivity, the union of these incidences has closed proper image.

Over the nonisotropic open, the incidence is a smooth vector bundle of rank $D - q$, hence has dimension D. Its projection is separably dominant; at $T = e_{1,0} \otimes \cdots \otimes e_{r,0}$ and the tuple of coordinate points, the local singular section is $(-z_{i,\alpha})$, with derivative $-I_q$. Remove the isotropic image and the closure of the critical image. The remaining fibers are projective, zero-dimensional and ètale, hence finite and reduced.

At a fixed point of X, vanishing of m arbitrary slices imposes m independent conditions. The base-point incidence has dimension $m \dim R_1 + n - m < m \dim R_1$ and closed image when $m \geq n + 1$. □

2.1. **The defect as Koszul homology**

Suppose the slices are independent and span $W \subset R_1$. Their last-mode critical equations form the Koszul subspace $\bigwedge^2 W \subset \ker \mu_T$. The small-mode equations are independent of this subspace, since their images under μ_T are independent. Comparing the dimension of H_T with the graph-span formula therefore leaves exactly the multiplication relations modulo the Koszul relations

$$\operatorname{codim}_{H_T} \operatorname{span} Z_T = \dim \ker \mu_T - \binom{m}{2} = \dim H_1(K_\bullet(W; R))_2. \tag{3}$$

For $m = n + 2$, choose parameters $H \subset W$ of dimension $n + 1$, put $A = R/(H)$, and let ℓ generate W/H. Resolve first by the Koszul complex on H, which is exact because the Segre ring is Cohen–Macaulay. The remaining complex is multiplication by ℓ on A. Its homological-degree-one part in internal degree two is precisely the map $A_1 \to A_2$, giving

$$\operatorname{codim}_{H_T} \operatorname{span} Z_T = \dim \ker(\ell : A_1 \to A_2). \tag{4}$$

Canonical-module duality also identifies this kernel with the dual of the cokernel of $W \otimes (\omega_R)_{n-1} \to (\omega_R)_n$, recovering the cohomological map of [4, Corollary 4.5].

2.2. **All tensor formats**

To prove Theorem 1.3, omit one-dimensional factors; order-one tensors are immediate. If one factor has dimension at least one plus the sum of the other projective dimensions, apply Proposition 2.1. Otherwise $n_i \leq \sum_{j \neq i} n_j$ for every i. On the full product, the singular section belongs to

$$F = \bigoplus_i \pi_i^* Q_i \otimes \mathcal{O}(1, \ldots, 1, 0, 1, \ldots, 1).$$

For $L = \mathcal{O}(1, \ldots, 1)$, the summands of $\bigwedge^r F^* \otimes L$ are $\boxtimes_i \Omega^{r_i}(1 + 2r_i - r)$, with $\sum r_i = r$. For $r \geq 2$, a degree-zero factor requires $r_i = r$. All other factors then have negative twist, and nonzero total cohomology has degree $\sum_{j \neq i} n_j \geq n_i \geq r$. If no factor has degree-zero cohomology, the total degree is again at least r. Thus the analogue of (2) holds. Lemma 2.2 and the Koszul resolution identify $H^0(I_Z \otimes L)$ with all critical equations. Hence $\operatorname{span} Z_T = H_T$ in this range. Since $T \in H_T$, Theorem 1.3 follows.

3. **Two Segre factors**

Put

$$R_{a,b} = K[x_{ij} : 0 \le i \le a,\ 0 \le j \le b]/I_2(x),$$
$$L_s = \sum_{i+j=s} x_{ij}, \quad A_{a,b} = R_{a,b}/(L_0, \dots, L_{a+b}).$$

At a rank-one point $x_{ij} = u_i v_j$, vanishing of all L_s would give

$$\left(\sum_{i=0}^{a} u_i t^i\right)\left(\sum_{j=0}^{b} v_j t^j\right) = 0.$$

Thus the L_s are parameters, and Cohen–Macaulayness makes them a regular sequence.

Lemma 3.1. *Over every field, bases of the first two graded pieces are*

$$(A_{a,b})_1 : \quad x_{ij}, \qquad 0 \le i < a, \quad 1 \le j \le b,$$
$$(A_{a,b})_2 : \quad x_{ij}x_{i'j'}, \quad 0 \le i < i' < a, \quad 1 \le j < j' \le b.$$

In particular, $h_1 = ab$ *and* $h_2 = \binom{a}{2}\binom{b}{2}$.

Proof. In the staircase degeneration, variables of rank $i + j$ have the same weight, so the rank-sum parameters degenerate to the same sums in the order complex. A maximal chain is a lattice path, encoded by a word with a letters 1 and b letters 2. Order these words lexicographically, with $1 < 2$. This is a shelling, whose restriction face consists of the vertices at descents 21.

Here is why the restriction faces give an integral basis after quotienting by the rank sums. When a chain is attached, its new faces are exactly those containing its restriction face. The corresponding relative module is the product of the restriction variables times the polynomial ring on that chain. There is one chain vertex of each rank, so the rank-sum parameters restrict to those individual variables, with coefficient matrix the identity. The successive quotients of the shelling filtration are therefore free rank-one modules over the parameter ring, generated by the restriction monomials. The filtration splits as a parameter-module filtration. Elimination along the shelling has diagonal coefficients one and works over the integers.

A descent 21 occurs at (i, j) with $i < a$ and $j > 0$; two descents have both coordinates strictly increasing. The restriction monomials in degrees one and two are therefore exactly those in the statement.

The parameters remain regular in the flat family. The monomials are therefore a basis near the initial fiber, and torus rescaling identifies all nonzero fibers while scaling

the parameters and the monomials. This proves the assertion in every characteristic, and recovers the relevant degrees of [2, Theorem 5.2]. □

Lemma 3.2. *If $a \geq 2$, $b \geq 3$, and some linear form acts injectively from degree one to degree two in $A_{a,b-1}$, then the same is true in $A_{a,b}$.*

Proof. Setting the last column to zero gives $\pi : A_{a,b} \twoheadrightarrow A_{a,b-1}$, with kernel J satisfying $J_1 = \langle x_{0b}, \ldots, x_{a-1,b} \rangle$. Multiplication by x_{02} is injective on J_1. For $i > 0$, the products $x_{02}x_{ib}$ are distinct basis monomials; for $i = 0$, the minors and the relations $L_2 = 0$, $x_{00} = 0$ give

$$x_{02}x_{0b} = -x_{11}x_{0b} - x_{20}x_{0b} = -x_{01}x_{1b},$$

a further basis monomial. The open set of forms injective on J_1 meets the inverse image of the open set of injective forms in the smaller algebra. Any form in the intersection is injective on $(A_{a,b})_1$. □

The two necessary starting points can be read from the dimensions; for $a, b \geq 2$,

$$\frac{\dim(A_{a,b})_2}{\dim(A_{a,b})_1} = \frac{(a-1)(b-1)}{4}.$$

Up to transposition, the minimal pairs where injectivity is possible are $(3, 3)$ and $(2, 5)$. The column lemma will propagate these two cases, while their quotients will handle the smaller targets.

In $A_{3,3}$, multiplication by x_{21} is an isomorphism. Write the source coefficients in the order

$$(x_{01}, x_{02}, x_{03}, x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23})$$

as $(a, b, c, d, e, f, g, h, i)$. In the target order

$$x_{01}x_{12}, x_{01}x_{13}, x_{02}x_{13}, x_{01}x_{22}, x_{01}x_{23},$$

$$x_{02}x_{23}, x_{11}x_{22}, x_{11}x_{23}, x_{12}x_{23},$$

the image has coefficients

$$(-a, d, e, -b - 3d, g, h, c - 3e - 2g, -f - 3h, -i).$$

They recover the source, successively in the order $a, d, e, b, g, h, c, f, i$, using only additions and sign changes.

In $A_{2,5}$, write $u_j = x_{0j}$, $v_j = x_{1j}$, and $w_{ij} = u_i v_j$ for $i < j$, and take $\ell = u_2 - u_5 - v_3$. The defining relations give

u_1	$-w_{13}$	v_1	$w_{12} + w_{14} - w_{15} + w_{23}$
u_2	$-w_{12} + w_{15} - w_{23}$	v_2	$2w_{13} - 2w_{15} + 3w_{24} - w_{25}$
u_3	$-w_{13} + 2w_{15} - 2w_{24} + w_{25}$	v_3	$w_{23} + 2w_{34} - w_{35}$
u_4	$-w_{14} - w_{34} + w_{35}$	v_4	$w_{24} + 3w_{35} - w_{45}$
u_5	$-w_{15} - w_{35} + w_{45}$	v_5	$w_{25} + w_{45}$.

If $(a, b, c, d, e, f, g, h, i, j)$ lies in its kernel, the rows for $w_{12}, w_{14}, w_{23}, w_{34}$ give successively $f = b = d$, $h = 0$, and $d = 2h = 0$. Thus $b = d = f = h = 0$. The other rows give

$$a = -c + 2g, \quad e = 2c - 2g, \quad i = 2c - 3g, \quad j = -c + g, \qquad c = 2g, \quad 4c = 7g.$$

Hence $g = 0$ and all coefficients vanish. Thus multiplication is invertible over every field; its integral determinant is 1.

Lemma 3.2, in both directions, gives injectivity for $a, b \geq 3$ and for $a = 2$, $b \geq 5$, together with their transposes. The quotient $A_{2,5} \twoheadrightarrow A_{2,b}$ transfers surjectivity to $b = 2, 3, 4$, where the kernel dimensions are $2b - \binom{b}{2} = 3, 3, 2$. Nonvanishing of the required minors is open in the parameter coefficients, so this proves the two-factor part of Theorem 1.1.

4. **Multiplication and rigidity**

Because degree-two multiplication in a Stanley–Reisner ring depends only on vertices and edges, its relations can be read from a graph. The square coefficients determine one component at each vertex; eliminating those components leaves exactly the equations of infinitesimal rigidity. Let Δ have N vertices and graph G. For a general placement $p_v \in K^r$, write $\rho_r(G)$ for the rank of the rigidity matrix, whose row at uv has $p_u - p_v$ in the u columns, $p_v - p_u$ in the v columns, and zero elsewhere. For rigidity in dimension r, assume that the graph has at least $r + 1$ vertices, so a general placement affinely spans K^r. The graph is *generically r-rigid* if $\rho_r(G) = rN - \binom{r+1}{2}$, and *generically stress-free* if its edge rows are independent.

Lemma 4.1. *Let $B = K[\Delta]$, and let $W \subset B_1$ be a general $(r+1)$-dimensional subspace. Then*

$$\dim H_1(K_\bullet(W; B))_2 = rN - \binom{r+1}{2} - \rho_r(G). \tag{5}$$

If Δ is Cohen–Macaulay of dimension $d-1$, A is a general linear Artinian reduction, and ℓ is general, then

$$\dim \ker(\ell : A_1 \to A_2) = dN - \binom{d+1}{2} - \rho_d(G).$$

Thus rigidity gives injectivity, and stress-freeness gives surjectivity.

Proof. Rescale variables so that W has basis $\sum_v y_v$ and $\sum_v p_{v,j} y_v$, $1 \le j \le r$. A vector in the multiplication kernel has coefficients (c_v, u_v) with $u_v \in K^r$. The square coefficient gives $c_v = -p_v \cdot u_v$; the edge coefficient becomes

$$(p_v - p_w) \cdot (u_v - u_w) = 0.$$

Thus the kernel has dimension $rN - \rho_r(G)$. Quotienting by the injective Koszul subspace $\bigwedge^2 W$ proves (5). For an Artinian reduction, use the same Koszul identification as in (4). If E is the number of edges, then

$$\dim A_1 = N - d, \qquad \dim A_2 = E - (d-1)N + \binom{d}{2},$$

so stress-freeness gives surjectivity. No division by two occurs. □

Trivial infinitesimal motions are $u_v = Ap_v + b$, where A is alternating; $A_{ii} = 0$ and $A_{ij} = -A_{ji}$. Their dimension on an affinely spanning placement in K^r is $\binom{r+1}{2}$, also in characteristic two.

Lemma 4.2. *Over K, the join of an r-rigid graph and an s-rigid graph is $(r+s+1)$-rigid. In particular, coning raises rigidity dimension by one. Coning preserves stress-freeness as well, and is an equivalence for rigidity.*

Two r-rigid graphs with at least $r+1$ common vertices have r-rigid union.

Adding a vertex with at least r neighbors preserves r-rigidity; adding one with at most r neighbors preserves stress-freeness in dimension r. A one-extension, replacing uw by a new vertex joined to u, w and $r-1$ other vertices, preserves both properties in dimension r.

Proof. For the join, place the frameworks in the skew affine subspaces $(p, 0, 0)$ and $(0, q, 1)$. Choose an affine anchor basis in each part; together they form an ambient affine basis. Internal rigidity implies all pairwise edge constraints within each part, since those constraints involve only the tangential velocity components. The cross edges therefore make the anchor vertices a rigid simplex. Subtract its ambient rigid motion. Internal rigidity now kills the tangential components at every vertex, and the cross edges to the opposite anchors kill the remaining normal components.

For the converse cone assertion, put the apex at zero and generic base positions at $t_v(q_v, 1)$. After fixing the apex velocity, the cone edges determine the last coordinates. Substituting $u_v = t_v U_v$ in the remaining coordinates turns each base-edge equation into the original one multiplied by $t_v t_w$. Thus, for a base with N vertices,

$$\rho_{r+1}(\operatorname{cone} G) = N + \rho_r(G).$$

This also proves the stress assertion. The gluing assertion follows because rigid motions agreeing on an affine basis agree everywhere.

The vertex-addition assertions follow from independence of the incident directions. For a one-extension, place the new vertex on the old edge, away from its endpoints, with the other $r-1$ directions independent modulo that line. Eliminating its velocity from the two collinear edge equations recovers the old edge constraint. The remaining directions then determine its whole velocity. Dually, equilibrium at the new vertex kills the transverse stresses and combines the two remaining stresses into a stress on the old edge. This proves the required rank at one placement, hence at a general placement. □

4.1. **The staircase degeneration**

Put $P = \prod_i \{0, \dots, n_i\}$ and $n = \sum_i n_i$. Let $G(P)$ join distinct comparable elements. The Segre straightening relations are

$$y_u y_v - y_{u \wedge v} y_{u \vee v} \qquad (u, v \text{ incomparable}).$$

Give y_u weight $|u|(n - |u|)$. If $|u \wedge v| = s$, $|u| = s + p$, and $|v| = s + q$, then the first monomial has weight $2pq > 0$ more than the second. Chain monomials form a basis by sorting the indices in each factor. Thus the initial ring is the Stanley–Reisner ring of the order complex of P. Its maximal chains have a lexicographic shelling, so it is Cohen–Macaulay of dimension $n + 1$.

A maximal multiplication rank in this initial ring passes to the Segre ring; parameter regularity and nonvanishing of a minor are open in the flat family. For the first Lefschetz map, the Artinian Hilbert function is constant on the parameter locus. For arbitrary W, the universal rank bound from $\bigwedge^2 W$ is the same on every fiber. Consequently the graph statements below give the desired multiplication ranks.

5. **Products of chains**

Proposition 5.1. *For $k \geq 3$ and $n_i \geq 1$, the graph $G(P)$ is generically $(n+1)$-rigid except at $(1,1,1)$, $(1,1,2)$, $(1,1,3)$ and their permutations. At these three tuples it is generically stress-free in dimension $n+1$.*

It is enough to construct a few rigid cores and then lengthen the chains. Boolean products supply the cores with at least four factors, and two three-chain boxes account for the remaining tuples; we begin with the propagation step shared by both constructions.

5.1. **Increasing a chain length**

Suppose $G(\{0,\dots,a\}\times Q)$ is rigid in dimension $a+s+e$, where s is the sum of the other chain lengths. After increasing a by one, take the lower old box coned by the new maximum, and the translated upper old box coned by the global minimum. They are rigid in dimension $r=a+s+e+1$, cover every vertex, and share $a|Q|+2$ vertices. Gluing applies exactly when

$$a(|Q|-1)\ge s+e. \tag{6}$$

For $e=1$, the other product has at least two positive chain lengths, so $|Q|\ge s+2$ and the inequality holds.

5.2. **Boolean products**

Lemma 5.2. *For $r\ge 3$, the graph of the barycentric subdivision of the boundary of an r-simplex is generically r-rigid over every field.*

Proof. Start with K_{r+1} and perform stellar subdivisions of faces in decreasing dimension. Subdivision of a face with at least three vertices preserves the old graph and adds a vertex with at least r neighbors. For an edge, its link in the current triangulated $(r-1)$-sphere has at least $r-1$ vertices. The subdivision contains a one-extension using the endpoints and any $r-1$ link vertices. Additional edges preserve rigidity, and subdividing a vertex merely renames it. Apply Lemma 4.2. □

Let Q_k be the comparability graph of the nonempty proper subsets of $[k]$. It is the graph in Lemma 5.2 with $r=k-1$. The full Boolean graph is its double cone, hence is $(k+1)$-rigid for $k\ge 4$. Increasing chain lengths handles every tuple with at least four factors.

5.3. **Three chains**

Every sorted positive triple outside the three exceptions dominates $(1,1,4)$ or $(1,2,2)$. We construct these two bases from one sequence of minimally rigid completions. Write H_j for the proper-part graph of $\{0,1\}^2\times\{0,\dots,j\}$, and name its vertices

$$A_i=(0,0,i),\quad B_i=(0,1,i),\quad C_i=(1,0,i),\quad D_i=(1,1,i).$$

A minimally r-rigid graph is both r-rigid and stress-free. Coning, adding a vertex on exactly r neighbors, and one-extension preserve minimal rigidity in the appropriate dimension.

The temporary edges in these completions explain both sides of the classification. Deleting them proves stress-freeness for the exceptional boxes, whereas consuming them in successive one-extensions produces the two rigid starting boxes. We keep track of the edges so that the same construction proves both assertions.

The graph H_1 is the hexagon with cyclic order $A_1, B_1, B_0, D_0, C_0, C_1$. Add the triangle on the alternate vertices A_1, B_0, C_0. Starting with this triangle, each remaining vertex has two neighbors, so the completion is minimally 2-rigid.

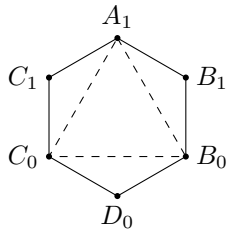


FIGURE 1. The hexagon H_1. The three added edges form the central triangle of its minimally rigid completion.

Cone this completion by D_1. Add C_2 by a 3-dimensional one-extension removing B_0C_0, with neighbors A_1, B_0, C_0, C_1. The edge B_0C_2 is temporary.

Add B_2 on A_1, B_0, B_1, then add A_2 by a one-extension removing B_0C_2, with neighbors A_1, B_0, B_2, C_2. The result is the minimally 3-rigid completion

$$H_2 + \{A_1B_0,\ A_1C_0,\ A_2B_0\}.$$

Cone it by D_2. A one-extension at B_3, on A_1, A_2, B_0, B_1, B_2, removes A_1B_0. The symmetric extension at C_3, on A_1, A_2, C_0, C_1, C_2, removes A_1C_0. Finally add A_3 on A_1, A_2, B_3, C_3. This gives the minimally 4-rigid completion

$$H_3 + \{A_2B_0\}.$$

Deleting the added edges shows that H_1, H_2, H_3 are stress-free in dimensions $2, 3, 4$, respectively.

To obtain H_4, cone the last completion by D_3. Add B_4 by a 5-dimensional one-extension removing the remaining temporary edge A_2B_0. Choose any six of its seven old neighbors $A_1, A_2, A_3, B_0, B_1, B_2, B_3$ that contain the two endpoints. Vertex C_4 also has seven old neighbors, and A_4 has the five neighbors A_1, A_2, A_3, B_4, C_4. Add

these two vertices and restore any omitted edges. Every edge now belongs to H_4, proving its 5-rigidity.

For the proper-part graph of $(1,2,2)$, return to the completed H_2 and cone it by D_2. Write $E_i = (0,2,i)$ and $F_i = (1,2,i)$ for the new vertices. The three added edges are removed by one-extensions at F_1, E_2, E_1, in that order

new vertex	edge removed	neighbors
F_1	A_1C_0	A_1, B_0, B_1, C_0, C_1
E_2	A_2B_0	A_1, A_2, B_0, B_1, B_2
E_1	A_1B_0	A_1, B_0, B_1, E_2, F_1.

The two further edges F_1D_0, F_1D_1 may be restored. Add F_0 on its four old neighbors B_0, C_0, D_0, F_1, then E_0 on its five old neighbors B_0, E_1, E_2, F_0, F_1. The resulting graph is precisely the proper-part graph of $(1,2,2)$, and is 4-rigid. Double coning restores the full graphs; increasing chain lengths now proves Proposition 5.1.

5.4. **The first Lefschetz map**

Write

$$N = \prod_i (n_i + 1), \qquad Q = \prod_i \binom{n_i + 2}{2}, \qquad d = 1 + \sum_i n_i.$$

The first Artinian Hilbert values are $h_1 = N - d$ and $h_2 = Q - dN + \binom{d}{2}$. At the exceptional triples they are $(4,1)$, $(7,4)$, and $(10,9)$. Stress-freeness gives surjectivity and kernel dimensions $3, 3, 1$; all other tuples give injectivity. The staircase degeneration proves Theorem 1.1.

The parameter and maximal-rank loci pull back to nonempty open loci in tensor space, because a basis of $H + \langle \ell \rangle$ is an arbitrary array of slices. Intersect with the open from Lemma 2.2 and apply (4). This proves Corollary 1.2.

6. **Arbitrary excess**

The graph comparison is not confined to the first Lefschetz map. If $m = n + e + 1$ with $e \geq 1$, equations (3) and (5) identify the additional relations with the failure of rigidity in dimension $n+e$. Thus the staircase degeneration shows that $(n+e)$-rigidity of $G(P)$ gives full singular-vector span for a general tensor of format

$$(n_1 + 1, \ldots, n_k + 1, n + e + 1).$$

There are two useful regimes. The number of factors may grow with the excess while their dimensions remain small, or the number of factors may stay fixed while their

dimensions grow. The constructions in both regimes contain at least $n+e+1$ vertices, so the slices can be independent.

Theorem 6.1. *Assume $n_i \geq 1$ for all i. The generic singular tuples span $\mathbb{P}(H_T)$ whenever $k \geq 4e$. They also span $\mathbb{P}(H_T)$ if $k \geq 3b$ and $e \leq 2^b - 3b + 1$ for some integer $b \geq 4$. In particular, $k \geq 3\lceil \log_2(4e) \rceil$ suffices when $e \geq 3$.*

Both bounds come from the same family of interval cores in the Boolean lattice. Fix ordered block sizes $b_1 + \cdots + b_t = k$ and a partition $[k] = B_1 \sqcup \cdots \sqcup B_t$. Put $C_i = B_1 \cup \cdots \cup B_i$ and take the core

$$\left(\bigcup_{i=1}^{t} [C_{i-1}, C_i] \right) \setminus \{\varnothing, [k]\}.$$

It is the join of $Q_{b_1}, \ldots, Q_{b_t}$ and the clique on the $t-1$ intermediate prefixes.

Two partitions related by swapping one element between adjacent blocks of sizes p, q have exactly

$$\sum_{j \neq i, i+1} 2^{b_j} + 2^{p-1} + 2^{q-1} - t \tag{7}$$

common core vertices. Indeed, in the changed window the common intervals are the lower interval of size 2^{p-1} and the upper interval of size 2^{q-1}; the other intervals are unchanged. Subtract the repeated endpoints and the two global endpoints. Adjacent-block swaps connect all partitions of the prescribed block sizes. The cores cover every proper nonempty subset; order its elements first, then cut at the prescribed sizes.

Proof of the bound $k \geq 4e$. Take $t = e$ and $b_i \geq 4$. Each Q_{b_i} is $(b_i - 1)$-rigid by Lemma 5.2. Joining these e graphs gives rigidity dimension $k-1$; the $e-1$ intermediate prefix vertices are cones, raising it to $k + e - 2$. This explains the choice of e blocks. For $e = 1$ there is one core. For $e \geq 2$, when every $b_i = 4$, the overlap (7) is $15e - 16 \geq 5e - 1 = k + e - 1$. Increasing a block size increases the overlap by more than it increases the required dimension. Thus all cores glue, giving $(k+e-2)$-rigidity of Q_k, and double coning gives $(k + e)$-rigidity of the full Boolean graph.

To increase a chain length, use (6). The other $k - 1$ positive lengths give

$$|Q| - 1 \geq s + \binom{k-1}{2} \geq s + e,$$

so propagation holds. This proves $(n + e)$-rigidity for every positive tuple in the stated range. □

The same cores give a sharper bound if we use the complete joins between their interval interiors. With three blocks of size b, each interior has $2^b - 2$ vertices. The

following tripartite calculation converts that number of vertices into a much larger rigidity dimension.

Lemma 6.2. *If $u, v, w \geq r+1$, the complete tripartite graph $K_{u,v,w}$ is generically r-rigid over every field. Every placement in which each part affinely spans K^r is infinitesimally rigid.*

Proof. Let A, B, C have rows $(p, 1)$ for the three parts. Let U, V, W have rows $(\dot{p}, -p \cdot \dot{p})$. All three position matrices have full column rank. The first two parts satisfy

$$AV^t + UB^t = 0.$$

Applying left annihilators of A gives $U = AQ$ for a square matrix Q, and hence $V = -BQ^t$. Similarly $W = -CQ^t$. The remaining edges give $B(Q + Q^t)C^t = 0$, so $Q + Q^t = 0$ and $U = AQ$, $V = BQ$, $W = CQ$.

The velocity encoding also gives $aQa^t = 0$ for every row a of A. Outside characteristic two, skew-symmetry already makes the diagonal zero. In characteristic two, this last identity is $\sum_j Q_{jj} a_j^2 = 0$. Choose an invertible square submatrix of A; its entrywise square is invertible, so again every $Q_{jj} = 0$. Thus Q is alternating in every characteristic. Splitting its last row and column shows that all velocities are one common translation plus one common alternating linear motion. □

Proof of the three-block bound. First take $k = 3b$ and three blocks of size b. The three interval interiors have $2^b - 2$ vertices each, are completely joined, and have two universal prefix vertices. Put $r = 3b + e - 2$. The hypothesis gives $2^b - 2 \geq (r-2) + 1$, so Lemma 6.2 and two cones make every core r-rigid. Adjacent cores intersect in $2^{b+1} - 3 \geq r + 1$ vertices. The preceding connectivity and coverage argument proves rigidity of Q_{3b}.

To add a Boolean factor, cover Q_{k+1} by the subgraphs of subsets containing a fixed element. Each is a cone over Q_k, and two intersect in $2^{k-1} - 1$ vertices. The required inequality $2^{k-1} - 1 \geq k + e$ holds at $k = 3b$ and persists thereafter. Thus Q_k is $(k + e - 2)$-rigid for every $k \geq 3b$.

For arbitrary positive chain lengths,

$$|Q| - 1 - s \geq 2^{k-1} - k \geq e.$$

The first inequality is minimized when every other length is one. It implies (6), so double coning and coordinate increments finish the proof.

Finally take $b = \lceil \log_2(4e) \rceil$. For $e \geq 3$, one has $b \geq 4$, $e \leq 2^{b-2}$, and $3b - 1 \leq 3 \cdot 2^{b-2}$; hence $e \leq 2^b - 3b + 1$. □

6.1. **A fixed number of large factors**

The same use of joins permits a fixed number of chains, provided they are sufficiently long. We arrange diagonal boxes whose interiors are pairwise joined, using the intervening boxes to supply additional rigidity dimensions. Enlarging the first and last boxes ensures that every outside vertex has enough neighbors to attach to this core. For fixed k, put $c_0 = 2$ if $k = 3$ and $c_0 = 1$ if $k \geq 4$.

Theorem 6.3. *Let $e \geq 2$, $k \geq 3$, and choose $L \geq c_0$. Put $N = 2L + c_0(e-2)$. If*

$$(L+1)^{\lfloor k/2 \rfloor} + (L+1)^{\lceil k/2 \rceil} \geq kN + e, \tag{8}$$

then the generic singular-vector span is $\mathbb{P}(H_T)$ whenever every $n_i \geq N$.

Proof. In $\{0, \dots, N\}^k$, take e consecutive diagonal boxes with side lengths

$$L, \underbrace{c_0, \dots, c_0}_{e-2}, L.$$

By Proposition 5.1, the proper interval graph for a box of side λ is $(k\lambda - 1)$-rigid. These e graphs are disjoint and pairwise joined. Together with the $e+1$ diagonal prefix vertices they form a core rigid in dimension $kN + e$, by the join and cone assertions of Lemma 4.2. An outside vertex v has at least

$$\prod_i (\min(v_i, L) + 1) + \prod_i (\min(N - v_i, L) + 1)$$

neighbors in the first and last boxes. These two sets are disjoint. Since $N \geq 2L$, at least one factor in each paired coordinate equals $L+1$, and the other is at least one. Decreasing the smaller factors to one gives a sum $(L+1)^a + (L+1)^{k-a}$, minimized at the two integers nearest $k/2$. Thus (8) permits every outside vertex to be added to the rigid core.

Finally $N \geq c_0 e \geq e$. In every coordinate increment, $a \geq e$ and $|Q| - 1 \geq s + 1$, so $a(|Q| - 1) \geq s + e$. Equation (6) extends the result to all larger tuples. □

For three smaller factors, the criterion is $L^2 - 3L + 14 - 7e \geq 0$; the bound $n_i \geq 2e + 2\lceil\sqrt{7e}\rceil$ suffices. For fixed $k \geq 4$, put $q = \lceil k/2 \rceil$ and $t = \lceil ((k+1)e)^{1/q} \rceil$. Taking $L = t + 2k$ gives the sufficient bound $n_i \geq e + 2t + 4k - 2$. Indeed, with $h = 2k + 1$, the inequality $(t+h)^q \geq t^q + 2th + h^2$ makes the left side of (8) at least its right side. Thus the thresholds are $2e + O(\sqrt{e})$ for $k = 3$ and $e + O_k(e^{1/\lceil k/2 \rceil})$ for fixed $k \geq 4$.

Taking $L = 3e$ also gives the uniform sufficient bound $n_i \geq 3e^2$ for $k \geq 3$. Substitution in (8) proves this for $k = 3, 4$, and the inequality persists when k increases by two. For $e = 2$, taking $L = 3$ improves this bound to $n_i \geq 6$.

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