Hilbert series of exterior algebras modulo two quadrics

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Abstract. We determine the Hilbert series of an exterior algebra over $\mathbb{C}$ modulo two general quadratic forms in every dimension by identifying its graded pieces with Plücker sections on the common isotropic Grassmannians. The same geometric calculation gives every nonnegative Plücker twist and identifies these Hilbert functions with those of explicit Richardson varieties. In odd dimension, a degeneration realizes this correspondence and yields monomial bases and a toric degeneration of the Plücker ring. We also determine the rational cohomology and Chow rings in this case, presenting them as complete intersections in the tautological Chern classes.

1. Introduction

Two alternating forms define both a quotient of the exterior algebra and a family of common isotropic Grassmannians. We show that the graded pieces of the quotient are the Plücker sections of these varieties, so their geometry determines the Hilbert series and explains its combinatorial form. The calculation proves the formula conjectured by Crispin Quiñonez, Lundqvist and Nenashev [5], who expressed it by walks in a strip and established it as a coefficientwise upper bound.

Let V be a complex vector space of dimension n. For $1 \le s \le \lceil n/2 \rceil$, put $w = n + 2 - 2s$, and let $a(n, s)$ count walks of length $n + 2$ from 0 to w, with steps ± 1, staying in $[0, w]$.

Theorem 1.1. *For general $f, g \in \bigwedge^2 V^*$,*

$$\operatorname{Hilb}\Big(\bigwedge V^*/(f, g), t\Big) = 1 + \sum_{s=1}^{\lceil n/2 \rceil} a(n, s) t^s,$$

where

$$a(n, s) = \det \left[\binom{w + i - j}{j - i + 1} \right]_{0 \le i, j < s}.$$

The quotient vanishes above degree $\lceil n/2 \rceil$.

Binomial coefficients outside their ordinary range are zero, and a determinant of size zero is one. Boij and Lundqvist [4] have also determined the exterior Hilbert series for even n.

The formula follows from a uniform calculation on the common isotropic Grassmannians of f and g. For $1 \le s \le n$, let

$$X_s = \{U \in \mathrm{Gr}(s, V) : f|_U = g|_U = 0\}.$$

This is the zero scheme of a section of $(\bigwedge^2 \mathcal{U}^*)^{\oplus 2}$, where $\mathcal{U}$ is the tautological bundle. For $s \ge 2$, the forms $f \wedge \eta$ and $g \wedge \eta$, with $\eta \in \bigwedge^{s-2} V^*$, are the linear Plücker equations for common isotropy. A cohomology gap in the Koszul resolution shows that restriction of Plücker sections is surjective with precisely these relations. Hence the degree-s exterior quotient is the space of Plücker sections of X_s. The same calculation determines all higher Plücker twists. Write $w = n + 2 - 2s$ and $D = s(w-1)$. When $w \ge 1$, define the interval of s-subsets

$$\alpha = (1, 3, \ldots, 2s-1), \qquad \beta = (n-2s+2, n-2s+4, \ldots, n),$$

ordered componentwise.

Theorem 1.2. *Suppose $n \ge 2s-1$ and X_s is a nonempty regular zero scheme. For every $r \ge 0$,*

$$h^0(X_s, \mathcal{O}_{X_s}(r)) = \det\left[\binom{w+r-1+i-j}{r-i+j}\right]_{0 \le i,j < s}, \qquad H^{>0}(X_s, \mathcal{O}_{X_s}(r)) = 0. \tag{1}$$

This Hilbert function counts length-r multichains in $[\alpha, \beta]$. The Plücker section ring is generated in degree one, is Cohen–Macaulay and Gorenstein, and has a-invariant $-w$. For $s \ge 2$, its ideal inside the Grassmannian coordinate ring is generated by

$$f \wedge \bigwedge^{s-2} V^* + g \wedge \bigwedge^{s-2} V^*.$$

If X_s is normal, this embedding is projectively normal. A general pair has a smooth nonempty zero scheme of dimension D in this range, and has no common isotropic s-plane when $n < 2s-1$.

The class of X_s in K-theory is the square of a staircase Schubert class, and hence the class of the Richardson variety for $[\alpha, \beta]$. Standard monomials on that interval then give the determinant above. Section 2 carries out this calculation.

In odd dimension the comparison is realized by a degeneration, and the exterior quotient itself has an explicit basis. For $n = 2k + 1$, a general pencil can be written, after changing basis and the two generators, as

$$F = \sum_{i=1}^{k} z_{2i-1} z_{2i}, \qquad G = \sum_{i=1}^{k} z_{2i} z_{2i+1}. \tag{2}$$

Theorem 1.3. *For the pencil* (2)*, the degree-s exterior quotient has basis*

$$z_{i_1} \cdots z_{i_s}, \qquad 2j - 1 \le i_j \le n - 2s + 2j \quad (1 \le j \le s).$$

For $1 \le s \le k + 1$, the Plücker ring of X_s has a flat weight degeneration to the Richardson ring for $[\alpha, \beta]$, and then to its Hibi ring. It has a quadratic Gröbner basis in the good Plücker coordinates and is Koszul.

The odd-dimensional geometry also determines the topology of X_s. A regular Borel action gives the Betti numbers, and an explicit integration pairing presents $H^*(X_s; \mathbb{Q})$ and $\mathrm{CH}^*(X_s) \otimes \mathbb{Q}$ as complete intersections in tautological Chern classes, with Hilbert series $\begin{bmatrix} 2k+2-s \\ s \end{bmatrix}_q$. A dense vector-bundle chart proves rationality (Section 4). Returning to the exterior algebra, the monomial basis refines its Hilbert function by the two halves of the pencil, and the perfect exterior pairing recovers its two-generator Koszul homology.

The colored-edge estimate behind the cohomology gap continues to apply to several alternating forms and gives an exact range of comparison with Plücker sections. The explicit straightening rules, on the other hand, expose the characteristic dependence; their denominators give a range in which the same bases survive, while Frobenius supplies new Koszul cycles and additional quotient classes in larger dimensions.

2. The Koszul complex and a staircase class

The Koszul complex relates the exterior quotient to Plücker sections. A cohomology gap makes that comparison exact; a staircase identity in K-theory then computes the section dimensions. The case $s = 1$ is immediate, so assume $s \ge 2$. Write $G_s = \mathrm{Gr}(s, V)$, $\mathcal{O}_{G_s}(1) = \det \mathcal{U}^*$, and let W have dimension two. Set

$$K_j(r) = \bigwedge^{j} (\bigwedge^{2} \mathcal{U} \otimes W) \otimes \mathcal{O}_{G_s}(r).$$

Lemma 2.1. *If $q > 0$ and $H^q(G_s, K_j(r)) \neq 0$, then for some $\ell \ge 1$,*

$$q = \ell(n - s), \qquad j \ge q + (r + 1)\ell. \tag{3}$$

Proof. A Schur constituent $S_\lambda \mathbb{C}^s$ of $\bigwedge^j(\bigwedge^2 \mathbb{C}^s \otimes W)$ has $|\lambda| = 2j$. Put $d = n - s$. The first block of its Bott sequence, read in reverse, is

$$A_i = d + i - 1 + r - \lambda_i \qquad (1 \le i \le s);$$

the other block is $0, 1, \dots, d-1$. These A_i are strictly increasing. For nonzero cohomology each is negative or at least d. If exactly ℓ are negative, Bott's theorem gives $q = \ell d$ and $\lambda_\ell \ge d + \ell + r$; see [12, Corollary 4.1.9].

A weight of the exterior representation selects j distinct edges from two copies of the complete graph on s vertices. Count the incidences on the first ℓ vertices. Each selected edge contributes at most one, except for an edge entirely inside these vertices, which contributes one more. There are at most $2\binom{\ell}{2}$ such internal edges. The highest weight is an actual weight, so

$$\ell(d + \ell + r) \le \sum_{i=1}^{\ell} \lambda_i \le j + \ell(\ell - 1).$$

This is (3). □

The evaluation of pairs of alternating forms onto $(\bigwedge^2 \mathcal{U}^*)^{\oplus 2}$ is surjective. Its incidence variety is therefore a smooth vector bundle over G_s, of dimension

$$\dim(\bigwedge^2 V^*)^2 + s(n - 2s + 1).$$

Properness and generic smoothness show that a general zero scheme is smooth of the expected dimension or empty, and is empty if $n < 2s - 1$.

For a regular or nowhere-zero section, the Koszul complex resolves $\mathcal{O}_{X_s}$ or zero. After twisting by $\mathcal{O}(1)$, its terms are $K_j(1)$ in position $-j$. In its cohomology spectral sequence, an entry $H^q(K_j(1))$ has total degree $q - j$. Lemma 2.1 puts every entry with $q > 0$ in total degree at most -2. An incoming differential to $(0, 0)$ would start in total degree -1, so only the ordinary H^0 Koszul map can contribute. There are also no entries of positive total degree. Consequently,

$$H^0(X_s, \mathcal{O}(1)) = \frac{\bigwedge^s V^*}{f \wedge \bigwedge^{s-2} V^* + g \wedge \bigwedge^{s-2} V^*}, \qquad H^{>0}(X_s, \mathcal{O}(1)) = 0. \tag{4}$$

Here $H^0(K_1(1)) = \bigwedge^{s-2} V^* \otimes W$, and the Koszul map is wedge multiplication by the two forms. The same argument gives $H^{>0}(X_s, \mathcal{O}(r)) = 0$ for every $r \ge 0$; for $r \ge 1$ it also makes restriction of Plücker sections surjective.

2.1. **The staircase factorization**

The vanishing just proved lets us recover section dimensions from Euler characteristics. We therefore seek the K-class of X_s. The Euler class of a single alternating form turns out to be the class of a staircase Schubert variety. Let $\delta = (s-1, s-2, \dots, 0)$. If z_i are the formal line-bundle roots of $\mathcal{U}$ and $x_i = 1 - z_i$, the Grassmannian Grothendieck formula [10, Theorem 7.3] is

$$[\mathcal{O}_{\Omega_\lambda}] = \frac{\det[x_i^{\lambda_j+s-j}(1-x_i)^{j-1}]_{i,j=1}^{s}}{\prod_{i<j}(x_i - x_j)};$$

For $\lambda = \delta$, set $v_i = x_i^2/(1-x_i)$ and factor $(1-x_i)^{s-1}$ from row i. The remaining determinant is Vandermonde. The identity

$$v_i - v_j = \frac{(x_i - x_j)(x_i + x_j - x_i x_j)}{(1-x_i)(1-x_j)}$$

cancels the denominator and row factors, giving

$$[\mathcal{O}_{\Omega_\delta}] = \prod_{i<j}(x_i + x_j - x_i x_j) = \prod_{i<j}(1 - z_i z_j) = \lambda_{-1}(\bigwedge^2 \mathcal{U}).$$

The Schubert varieties for δ in opposite flags meet in the Richardson variety $R_{\alpha,\beta}$. They have no higher Tor; apply the homological Kleiman–Bertini theorem [9] to a general translate; opposite flags form one orbit of the Borel group preserving the first Schubert variety. Consequently

$$[\mathcal{O}_{R_{\alpha,\beta}}] = [\mathcal{O}_{\Omega_\delta}]^2 = \lambda_{-1}((\bigwedge^2 \mathcal{U})^{\oplus 2}) = [\mathcal{O}_{X_s}].$$

The last identity is the Koszul resolution. It also proves nonemptiness for a general section in the stated range; the Richardson class has Euler characteristic one, whereas an empty Koszul zero scheme would have class zero.

2.2. **Intervals, multichains and determinants**

We recall an elementary form of the interval standard-monomial theorem [8].

Lemma 2.2. *For s-subsets $\alpha \le \beta$, the ideal*

$$P_{\alpha,\beta} = P_{\mathrm{Gr}} + (p_I : I \not\in [\alpha, \beta])$$

is prime. Its degree-r quotient has basis $p_{I_1} \cdots p_{I_r}$ with $\alpha \le I_1 \le \cdots \le I_r \le \beta$, and has Krull dimension $1 + \sum_i (\beta_i - \alpha_i)$.

Proof. Take an $s \times n$ matrix with independent entries in positions $\alpha_i \le j \le \beta_i$ and zero elsewhere, and send p_I to its maximal minor. A nonzero matching can be uncrossed; if rows $a < b$ use columns $p > q$, then q is also allowed in row a and p in row b, because both interval endpoints increase with the row. Repeating these exchanges gives the diagonal matching. Thus a nonzero minor must have $\alpha \le I \le \beta$, and every forbidden minor vanishes.

Give entry (i, j) weight ij. Uncrossing a pair raises its weight by $(b-a)(p-q) > 0$, so the diagonal is the unique greatest-weight term of every allowed minor. Products indexed by multichains have distinct diagonal monomials; each row records the sorted list of indices and recovers the chain.

Plücker straightening spans the quotient by these standard products, and the preceding independence embeds it in the polynomial ring of matrix entries. Thus it is a domain. If $D = \sum_i (\beta_i - \alpha_i)$, the interval has maximal strict chains of $D+1$ elements, obtained by increasing coordinates from right to left. Counting multichains by their strict supports gives a polynomial in r of degree D, proving the dimension. □

Richardson varieties have no higher cohomology in nonnegative Plücker twists [8, Theorem 5.0.6]. The class identity and the preceding vanishing therefore identify the Hilbert function of X_s with the multichain count. Put $c = w - 1$ and

$$\lambda_i = c + s - i, \qquad \mu_i = s - i \qquad (1 \le i \le s).$$

The interval $[\alpha, \beta]$ is the lattice of ideals of the cell poset of λ/μ. Its length-r multichains count plane partitions of this shape with entries in $\{0, \dots, r\}$. Kreweras's determinant, in the form of [6, Proposition 2.2], is

$$\det \left[\binom{r + c - i + j}{c - 2i + 2j} \right]_{1 \le i,j \le s}.$$

Transposing and complementing the lower binomial entries gives (1).

For $r \ge 1$, the ideal of X_s is the image of $H^0(K_1(r))$. Since

$$K_1(r) = \bigwedge^{s-2} \mathcal{U}^* \otimes \mathcal{O}(r-1) \otimes W,$$

the Cartan-product map from $H^0(K_1(1)) \otimes H^0(\mathcal{O}(r-1))$ is onto; its irreducible target has highest weight $(r^{s-2}, (r-1)^2)$, and the product of highest-weight sections is nonzero. Thus the ideal is generated in degree one, as claimed. The complete section ring of a normal variety is integrally closed, which proves the projective-normality assertion; it applies in particular to the general smooth zero scheme.

Taking leading coefficients of the multichain determinant gives

$$\deg X_s = D! \det\left[\frac{1}{(c+2i-2j)!}\right]_{0\le i,j<s}, \qquad D = sc,$$

where a negative factorial contributes zero. This is the normalized volume of the order polytope of λ/μ. For $c = 2$, the cell poset is a fence and the degree is the Euler secant number E_{2s}; the Hessenberg determinant has generating function $\sec\sqrt{u}$ after division by $(2s)!$.

2.3. **Duality and the negative twists**

Adjunction gives $\omega_{X_s} = \mathcal{O}_{X_s}(-w)$. To prove Cohen–Macaulayness, it remains to treat $-w < r < 0$. Write $r = -a$, with $0 < a < w$. For a nonzero Schur cohomology term, let ℓ be the number of negative Bott entries and $T = q - j$. The weight argument gives $T \le (a-1)\ell$, also when $\ell = 0$.

To obtain the opposite bound, use $\det(\bigwedge^2 \mathcal{U} \otimes W) = \mathcal{O}(-2(s-1))$. Exterior duality replaces degree j by $s(s-1) - j$, and Serre duality on the Grassmannian replaces twist $-a$ by $a - w$. The complementary constituent is

$$\lambda^c = (2(s-1) - \lambda_s, \dots, 2(s-1) - \lambda_1)$$

with $s - \ell$ crossed Bott entries. The total cohomological degree changes from T to $D - T$. Applying the same weight bound to this dual term gives

$$D - T \le (w-a-1)(s-\ell), \qquad T \ge sa + (w-a-1)\ell.$$

The two bounds differ by $sa + (w-2a)\ell \ge s\min(a, w-a) > 0$. Thus every Koszul term is acyclic in this window. Serre duality and the nonnegative-twist result now give vanishing of all intermediate cohomology in every twist. The coordinate ring is the complete section ring, so it is Cohen–Macaulay, with canonical module its shift by $-w$. This proves the Gorenstein assertion of Theorem 1.2. If $D = 0$, the Hilbert function is identically one, so X_s is a reduced point and the assertion is immediate.

2.4. **The exterior formula**

The $r = 1$ determinant in (1) is unchanged by transposition and equals the determinant in Theorem 1.1. To identify it with walks, put

$$e_j = \binom{w+1-j}{j}, \qquad Q_w(u) = \sum_{j\ge 0} (-1)^j e_j u^j.$$

Expanding the upper Hessenberg determinant $D_s = \det[e_{j-i+1}]_{0\le i,j<s}$ along its first row gives $\sum_{s\ge 0} D_s u^s = 1/Q_w(u)$. For the adjacency matrix A of the path on $0,\dots,w$, the tridiagonal recurrence and corner cofactor give

$$\det(1-tA) = Q_w(t^2), \qquad ((1-tA)^{-1})_{0,w} = \frac{t^w}{Q_w(t^2)}.$$

Thus D_s counts walks of length $w+2s = n+2$.

For $s > \lceil n/2 \rceil$ and $s < n$, a general zero scheme is empty, so (4) gives vanishing. In degree $s = n \ge 2$, multiplication of a nonzero coefficient term of f by its complementary monomial gives the top exterior line. The case $n = 1$ is immediate. Intersecting the finitely many open conditions proves Theorem 1.1 in all degrees simultaneously.

3. **The odd pencil and its degenerations**

For the odd pencil, the Hilbert-function comparison becomes a straightening procedure. A local inverse removes every monomial outside the interval $[\alpha, \beta]$; primality of the interval ring then shows that no further relations occur. Assume $n = 2k+1$. The canonical-form theorem [5, Theorem 3] puts a general pencil in the form (2). Call $I = (i_1 < \cdots < i_s)$ *good* if

$$2j-1 \le i_j \le n-2s+2j \qquad (1 \le j \le s).$$

Failure of a lower or upper inequality is called left or right failure.

3.1. **An inverse on a prefix**

To straighten a left-failing monomial, we first divide its initial segment by the restriction of F to that segment. Set $y_i = z_{2i-1}z_{2i}$; these elements commute and square to zero, so the required division becomes an inclusion-map calculation on subsets.

Lemma 3.1. *For a squarefree monomial a of degree $h+1$ in $z_1,\dots,z_{2h}$, there is an explicit $\eta(a)$ of degree $h-1$ in the same variables with*

$$(y_1 + \cdots + y_h)\eta(a) = a.$$

Proof. Write $a = Q\prod_{i\in P} y_i$, where Q contains its singleton variables. Let M be the pairs not occupied by a singleton and put $r = |P|$. Then $|M| = 2r-1$. After the singleton factors are fixed, multiplication by $\sum_{i\in M} y_i$ sends $(r-1)$-subsets to their

r-element extensions. Symmetry under permutations preserving P suggests looking for an inverse of the form

$$\eta(a) = Q \sum_{\substack{T\subset M\\ |T|=r-1}} b_{|T\cap P|} \prod_{i\in T} y_i.$$

For an r-subset $U \subset M$, put $q = |U \cap P| \geq 1$. Removing one of its q elements in P contributes b_{q-1}; removing one of its other $r-q$ elements contributes b_q. Thus the required coefficient equations are

$$qb_{q-1} + (r-q)b_q = 0 \quad (1 \leq q < r), \qquad rb_{r-1} = 1.$$

Starting with $b_{r-1} = 1/r$ and solving backwards gives

$$b_j = \frac{(-1)^{r-1-j}}{r\binom{r-1}{j}} \qquad (0 \leq j < r).$$

Singleton pairs vanish on multiplication, and complete pairs have even degree, so no extra signs occur. □

For instance, the first nontrivial inverse is

$$(y_1 + y_2 + y_3)\frac{y_1 + y_2 - y_3}{2} = y_1y_2.$$

Prescribe costs on the path edges by

$$e_{2i-1} = 2^i, \qquad e_{2i} = 2^{k+1-i}.$$

Choose integers u_j with $u_j + u_{j+1} = e_j$, and put $v_j = C - u_j > 0$. We take largest v-weight as initial weight.

Lemma 3.2. *Every left-failing monomial is the unique greatest-weight term of an element of (F), and every right-failing monomial is such a term of an element of (G), for this single weight.*

Proof. For a left failure, start at height one, moving down at selected indices and up otherwise. Its first exit below zero is at position $2h$, with $h+1$ selected variables in that prefix. Write $z_I = ab$, with a in the prefix. Then

$$F\eta(a)b = z_I + (y_{h+1} + \cdots + y_k)\eta(a)b. \tag{5}$$

Every other nonzero term introduces a pair $j > h$. Its cost 2^j exceeds the sum of all prefix-pair costs, so its total u-cost increases and its v-weight decreases. Reflection $z_j \mapsto z_{n+1-j}$ interchanges the matchings and makes the even-edge costs increasing powers of two. It gives the right-failure assertion. □

3.2. **Primality forces equality**

Let $S = \mathbb{C}[p_I : |I| = s]$ and

$$J = P_{\mathrm{Gr}} + (F \wedge \bigwedge^{s-2} V^* + G \wedge \bigwedge^{s-2} V^*).$$

Give p_I weight $v(I) = \sum_{i \in I} v_i$. Plücker relations are homogeneous for the original vertex multigrading, hence for this weight. Lemma 3.2 gives

$$\mathrm{in}_v J \supset P_{\alpha,\beta}.$$

By Lemma 2.2, the latter quotient is a domain of dimension $D+1$, so this containment gives an upper bound of $D+1$ for $\dim S/\mathrm{in}_v J$. For the reverse bound, on a Grassmannian chart $X_s = \mathrm{Proj}(S/J)$ is defined by the $s(s-1)$ pairings of its basis vectors with F, G. It is nonempty because the odd-indexed span is common isotropic. The height theorem therefore gives $\dim S/J \geq D+1$. Initial ideals preserve dimension, so both bounds are equalities. A proper further quotient of the finite-type domain $S/P_{\alpha,\beta}$ would lower dimension. There can be no additional initial relation, and hence

$$\mathrm{in}_v J = P_{\alpha,\beta}. \tag{6}$$

This is the required Richardson degeneration. Its degree-one basis gives the good exterior monomials; its full standard basis lifts to the actual products of good Plücker coordinates on X_s. The rules (5) and their reflections strictly decrease weight and give unique exterior normal forms.

The path bijection is particularly direct; put down-steps at the indices of I and add a fixed up-step at each end. The good inequalities are exactly the two walls of the strip. At $s = k+1$ there is one good monomial, $z_1 z_3 \cdots z_{2k+1}$; above that degree every monomial has a left failure and the triangular relations span the whole exterior degree.

3.3. **Toric and quadratic degenerations**

Embed the Richardson ring into the band-matrix ring of Lemma 2.2. A diagonal term order gives the good minor p_I initial monomial

$$d_I = \prod_{a=1}^{s} m_{a,i_a}.$$

Distinct standard products have distinct leading monomials, so the good minors form a SAGBI basis. The initial algebra has presentation

$$\mathbb{C}[u_I : I \in [\alpha, \beta]]/(u_I u_J - u_{I \wedge J} u_{I \vee J} : I, J \text{ incomparable}).$$

Indeed, these relations sort every product into a chain, whose diagonal monomial determines it uniquely.

For termination, put $\rho(I) = \sum_a (i_a - \alpha_a)$ and give u_I weight $C - \rho(I)^2$. Replacing an incomparable pair by its meet and join increases the separation of the two ranks without changing their sum, and strictly lowers this weight. Thus the displayed quadratics are a Gröbner basis with initial ideal $(u_I u_J : I, J \text{ incomparable})$.

The SAGBI step is realized by weights on the finitely many quadratic straightening relations; their greatest diagonal terms give precisely the meet-and-join binomials. Equality of standard-monomial Hilbert functions gives equality of the initial ideals. Combining these weights with (6) at separated scales gives an initial ideal generated by forbidden variables and these squarefree quadratics. Eliminating the forbidden variables proves Theorem 1.3; the quadratic Gröbner criterion gives Koszulness [11, Theorem 5.3].

4. **Cohomology and rationality in odd dimension**

The canonical odd pencil has a regular Borel action whose tangent weights determine the Betti numbers. Its integration pairing then recovers the ring structure, while projection onto one half of the pencil supplies a rational chart.

Throughout this section, $1 \le s \le k+1$. Write $n = 2k+1$ and $V = A \oplus B$, with bases $a_1, \dots, a_k$ and $b_0, \dots, b_k$, so that

$$f = \sum_{i=1}^{k} a_i^* \wedge b_i^*, \qquad g = \sum_{i=1}^{k} a_i^* \wedge b_{i-1}^*.$$

The canonical pair represents the general orbit, so X_s is smooth. Put

$$K = k+1, \qquad c = 2(K-s), \qquad w = c+1, \qquad D = sc.$$

Let $c_i = c_i(\mathcal{U}^*|_{X_s})$ and let $x_1, \dots, x_s$ be formal Chern roots. In half the cohomological grading, $\deg x_i = 1$ and $\deg c_i = i$. In the integration pairing, the Grassmannian factor $\prod_{i<j}(x_i - x_j)^2$ combines with the Euler factor $\prod_{i<j}(x_i + x_j)^2$ of the two isotropy conditions to give $\Delta(x_1^2, \dots, x_s^2)^2$. Its annihilators come in two families; complete homogeneous polynomials in these squares, and their first variations under $x_i^2 \mapsto x_i^2 + \varepsilon x_i$. Define

$$R_{2m} = h_m(x_1^2, \dots, x_s^2), \qquad R_{2m+1} = \sum_{i=1}^{s} x_i \frac{\partial h_{m+1}(y_1, \dots, y_s)}{\partial y_i}\bigg|_{y_i = x_i^2}.$$

These are symmetric in the x_i and thus polynomials in the c_i.

Theorem 4.1. *Evaluation in the tautological Chern classes gives*

$$H^*(X_s;\mathbb{Q}) \simeq \mathbb{Q}[c_1,\dots,c_s]/(R_w,R_{w+1},\dots,R_{w+s-1}), \qquad \deg_H c_i = 2i.$$

The relations form a homogeneous regular sequence. The same presentation gives $\mathrm{CH}^*(X_s)\otimes\mathbb{Q}$*, with* $\deg c_i = i$*. Moreover,*

$$\sum_i \dim H^{2i}(X_s;\mathbb{Q})q^i = \prod_{i=1}^{s}\frac{1-q^{c+i}}{1-q^i} = \begin{bmatrix} 2k+2-s \\ s \end{bmatrix}_q, \qquad H^{\mathrm{odd}}(X_s;\mathbb{Q}) = 0,$$

and X_s *is rational.*

4.1. **A regular Borel action**

For $k \geq 1$, define

$$\begin{aligned} Na_i &= (k-i)a_{i+1} \quad (i<k), & Na_k &= b_k, \\ Nb_j &= -(k-j+1)b_{j-1} \quad (j\geq 1), & Nb_0 &= 0. \end{aligned}$$

This is one Jordan block. On basis pairs,

$$f(Nu,v)+f(u,Nv)=0, \qquad g(Nu,v)+g(u,Nv)=-f(u,v).$$

Define $Ha_i = (2i-2k-2)a_i$ and $Hb_j = (2k-2j)b_j$. Then $[H,N]=2N$, so $\lambda(t)=t^H$ and $\varphi(u)=\exp(uN)$ define a Borel action, with

$$\varphi(u)^*f = f, \quad \varphi(u)^*g = g-uf, \qquad \lambda(t)^*f = t^{-2}f, \quad \lambda(t)^*g = g.$$

It preserves X_s. A single Jordan block has a unique invariant subspace of each dimension, so the unipotent subgroup fixes only

$$o = \ker N^s = \langle b_0,\dots,b_{s-1}\rangle.$$

Thus X_s is a smooth regular Borel variety in the sense of Brion–Carrell [3].

Write an attracting coordinate of weight $-2a$ as having positive weight a. At o, the Grassmannian tangent space is $\mathrm{Hom}(o,V/o)$, so these weights are $\{j-i : 0\leq i< s,\ s\leq j\leq 2k\}$. The differential of the two tautological sections gives the normal weights

$$\{2k-i-j,\ 2k-i-j+1 : 0\leq i<j<s\}.$$

They equal the ambient tail with $2k+2-s\leq j\leq 2k$. After reflection about $2k$, this is the elementary multiset identity

$$\{i+j, i+j-1 : 0\leq i<j<s\} = \{i+h : 0\leq i<s,\ 0\leq h<s-1\}.$$

The remaining tangent weights are therefore

$$\{i+j : 1 \le i \le s,\ 0 \le j < c\}.$$

Brion–Carrell's zero-scheme description [3, Proposition 2 and (15)] gives one coordinate of degree $2a$ and one equation of degree $2(a+1)$ for each weight a. Hence

$$\sum_i \dim H^{2i}(X_s;\mathbb{C})q^i = \prod_{i=1}^{s}\prod_{j=0}^{c-1}\frac{1-q^{i+j+1}}{1-q^{i+j}} = \prod_{i=1}^{s}\frac{1-q^{c+i}}{1-q^i}.$$

The torus has isolated fixed points. Its Białynicki–Birula decomposition [1, 2] is a filtrable affine-cell decomposition; consequently integral cohomology is torsion-free and even, and the cycle map from Chow groups is an isomorphism. The case $k=0$ is a point.

4.2. **The integration pairing**

To determine the ring, we compute its integration pairing on tautological classes. The candidate relations annihilate this pairing, and their complete-intersection Hilbert series agrees with the Betti numbers just obtained. This will prove both generation and the relations. Put $S=\mathbb{Q}[c_1,\dots,c_s]$ and, for $P\in S_D$, let $L(P)=\int_{X_s}P(c)$. The Grassmannian coefficient formula and the Euler class $\prod_{i<j}(x_i+x_j)^2$ give

$$L(P)=\frac{(-1)^{\binom{s}{2}}}{s!}[x_1^{2k}\cdots x_s^{2k}]P(x)\Delta(x^2)^2.$$

For completeness, the Grassmannian coefficient formula follows by evaluating on Schur functions of top degree; only the rectangle contributes, with coefficient $(-1)^{\binom{s}{2}}s!$.

Let $\mathcal{E}$ average over the independent sign changes of the x_i, and put $y_i=x_i^2$. Then

$$L(P)=\frac{(-1)^{\binom{s}{2}}}{s!}[y_1^{K-1}\cdots y_s^{K-1}]\mathcal{E}(P)(y)\Delta(y)^2.$$

Lemma 4.2. *For $j\ge w$ and $P\in S_{D-j}$, one has $L(PR_j)=0$.*

Proof. Put $D_i(y)=\prod_{h\neq i}(y_i-y_h)$. Partial fractions give

$$h_l(y)=\sum_i\frac{y_i^{l+s-1}}{D_i(y)}.$$

If $l\ge K-s+1$, the numerator exponent is at least K, and D_i divides $\Delta(y)^2$. Thus $\Delta(y)^2h_l\in(y_1^K,\dots,y_s^K)$, proving the even case.

For $j = 2m+1 \geq w$, let $l = m+1 \geq K-s+1$. Write $\mathcal{E}(x_i P) = y_i Q_i(y)$, since each surviving exponent of x_i is positive and even. Differentiate the partial-fraction formula. In

$$\frac{\partial}{\partial y_j}\frac{y_i^a}{D_i} = \delta_{ij}\frac{a y_i^{a-1}}{D_i} - \frac{y_i^a\,\partial D_i/\partial y_j}{D_i^2}, \qquad a = l+s-1 \geq K,$$

multiplication by $\Delta(y)^2 y_j Q_j$ gives a polynomial divisible by y_i^K; in the first term $y_j = y_i$ restores the lost power; in the second $D_i^2 \mid \Delta(y)^2$. Therefore the required coefficient vanishes in the odd case as well. □

4.3. **A recurrence for the relations**

It remains to show that the first s annihilators already have the required complete-intersection size. Both parity families come from one Chern polynomial, which gives a recurrence of length s. Define

$$C(t) = \prod_i (1 - x_i t) = \sum_{i=0}^{s} p_i t^i, \qquad p_i = (-1)^i c_i,$$

and

$$B(t) = \frac{1}{C(t)C(-t)} = \sum_{m\geq 0} R_{2m} t^{2m},$$
$$U(t) = tB(t)\sum_i \frac{x_i}{1 - x_i^2 t^2} = \sum_{m\geq 0} R_{2m+1} t^{2m+1}.$$

With $\mathcal{D} = t\,d/dt$, logarithmic differentiation gives $C(\mathcal{D}B + 2U) + 2(\mathcal{D}C)B = 0$. Its coefficient of t^m is

$$\sum_{i=0}^{\min(s,m)} p_i A_{m,i} R_{m-i} = 0, \qquad A_{m,i} = \begin{cases} m+i, & m-i \text{ even}, \\ 2, & m-i \text{ odd}. \end{cases} \tag{7}$$

The leading coefficient is m for even m and 2 for odd m.

At a common zero of $R_w, \ldots, R_{w+s-1}$, recurrence (7) forces every later R_m to vanish. Then $B(t)$ is a polynomial, and $B(t)C(t)C(-t) = 1$ forces $C(t) = 1$. The common zero is therefore the origin. The s relations form a regular sequence in S, and

$$A = S/(R_w, \ldots, R_{w+s-1})$$

is Artinian Gorenstein of socle degree $D = s(w-1)$ and Hilbert series $\prod_{i=1}^{s}(1 - q^{c+i})/(1-q^i)$.

By Lemma 4.2, L factors through A_D. It is nonzero, since $L(c_1^D) = \deg X_s > 0$. The perfect pairings of A imply that $(P, Q) \mapsto L(PQ)$ on $S_i \times S_{D-i}$ has rank $\dim A_i$. This pairing factors through tautological evaluation $\tau_i : S_i \to H^{2i}(X_s; \mathbb{Q})$, so

$$\dim A_i \leq \operatorname{rank} \tau_i \leq \dim H^{2i}(X_s; \mathbb{Q}).$$

The endpoints agree by the Betti formula, so every τ_i is onto. Now let P lie in the relation ideal. It pairs to zero with every polynomial of complementary degree; surjectivity in that degree and Poincaré duality imply $\tau_i(P) = 0$. Conversely, if $\tau_i(P) = 0$, then $L(PQ) = 0$ for every complementary Q, and the perfect pairing of A forces P into the relation ideal. This proves both tautological generation and the precise kernel. The affine-cell cycle isomorphism gives the Chow assertion.

The first relations are

$$R_1 = c_1, \qquad R_2 = c_1^2 - 2c_2,$$
$$R_3 = 2c_1^3 - 5c_1c_2 + 3c_3, \quad R_4 = c_1^4 - 4c_1^2c_2 + 3c_2^2 + 2c_1c_3 - 2c_4.$$

For $(k, s) = (2, 2)$ the presentation becomes $\mathbb{Q}[c_1, c_2]/(2c_1^3 - 5c_1c_2, c_1^2c_2 - 2c_2^2)$. For $s = 1$ it is $\mathbb{Q}[c_1]/(c_1^{2k+1})$, and for $s = k + 1$ it is $\mathbb{Q}$.

4.4. **A dense vector-bundle chart**

On the open set $U \cap A = 0$, projection to B writes U as the graph of $T : C \to A$ for $C \in \operatorname{Gr}(s, B)$. Both forms vanish on A and on B, so graph isotropy contains only the cross terms, linear in T. This gives the bundle map

$$\Phi : \mathcal{S}^* \otimes A \longrightarrow (\bigwedge^2 \mathcal{S}^*)^{\oplus 2}$$

on $\operatorname{Gr}(s, B)$. The full-space map $\Phi_B : \operatorname{Hom}(B, A) \to (\bigwedge^2 B^*)^{\oplus 2}$ is an isomorphism. Indeed, writing $T(b_j) = \sum_i t_{ij} a_i$ and setting $t_{0j} = t_{k+1,j} = 0$, its kernel equations are

$$t_{lj} = t_{jl}, \qquad t_{l+1,j} = t_{j+1,l} \qquad (0 \leq j, l \leq k).$$

They give $t_{i0} = t_{ik} = 0$ and $t_{ij} = t_{i-1,j+1}$; iteration reaches a zero boundary. Both spaces have dimension $k(k + 1)$.

Every pair of alternating forms on C extends to B, so every fiber of Φ is onto. Its kernel is a vector bundle of rank $s(k + 1 - s)$. The corresponding open of X_s is its total space, of dimension $2s(k + 1 - s) = D$. It is dense by the prime degeneration (6). A vector bundle over a rational Grassmannian is rational, completing Theorem 4.1.

5. **The exterior quotient and its homology**

The good-monomial basis retains the two gradings of the odd pencil, so the geometric calculation also refines the exterior Hilbert series. The perfect pairing in the ambient exterior algebra converts the Hilbert function into all the Betti polynomials for the two-generator action.

5.1. **Bigraded dimensions**

For the odd pencil, give a_i bidegree $(1,0)$ and b_j bidegree $(0,1)$. Both relations have bidegree $(1,1)$. Let $A = \bigwedge V^*/(f,g)$. For $0 \le s \le k+1$, put $m = k+2-s$ and define

$$D_0 = 1, \quad D_1 = 1 - vz, \quad D_m = (1-(u+v)z)D_{m-1} - uvz^2 D_{m-2}.$$

Proposition 5.1. *The bigraded Hilbert function is*

$$\sum_{p+q=s} \dim A_{p,q} u^p v^q = [z^s] \frac{1}{D_m(z;u,v)}.$$

Proof. Use the path order $b_0, a_1, b_1, \dots, a_k, b_k$, which gives the canonical pencil up to interchanging its generators and changing a sign. In the good-monomial path, pair the first fixed up-step with the b_0 step, then pair the a_i, b_i steps; leave the last fixed up-step unpaired. Halving the heights gives a walk on $0, \dots, m-1$, of length $k+1$, from 0 to $m-1$. An up-step has weight 1, a down-step uv, a loop at zero v, and a loop above zero either u or v. Its tridiagonal transfer matrix T has $\det(1-zT) = D_m$, and the corner cofactor is z^{m-1}. The required coefficient is therefore $[z^{k+1}]z^{m-1}/D_m = [z^s]1/D_m$. □

There is a compatible SL_2 action; for its standard module U, identify the a-span with $(\mathrm{Sym}^{k-1} U)^*$ when $k \ge 1$ and with zero when $k = 0$, and the b-span with $\mathrm{Sym}^k U$. The pencil is multiplication by the two basis vectors of U, and the torus weights are

$$\mathrm{wt}(a_i) = 2i - k - 1, \qquad \mathrm{wt}(b_j) = k - 2j.$$

Corollary 5.2. *If $p+q = k$, then*

$$A_{p,q} \simeq \mathrm{Sym}^{2p+1}(\mathrm{Sym}^q U), \qquad \dim A_{p,q} = \binom{2p+q+1}{q}.$$

Proof. Write the selected a-indices as $i_1 < \cdots < i_p$, and the complement of the selected b-indices as $c_1 < \cdots < c_{p+1}$. The good inequalities say $c_1 < i_1 \le c_2 < \cdots \le c_p < i_p \le c_{p+1}$. They correspond to weakly increasing lists in $\{0, \dots, q\}$ of length $2p+1$

$$c_1, i_1 - 1, c_2 - 1, i_2 - 2, \dots, c_p - (p-1), i_p - p, c_{p+1} - p.$$

The original torus weight is twice the sum minus $q(2p+1)$. This is the character of the stated symmetric power, with weights reversed; SL_2 characters are invariant under reversal and determine the representation. □

5.2. **Koszulness and the quadratic dual**

The Plücker ring in Theorem 1.3 and the exterior quotient have different homological behavior.

Proposition 5.3. *For two general quadrics, $A_n = \bigwedge \mathbb{C}^n/(f,g)$ is Koszul exactly when $n \le 4$. For $n \ge 6$, its quadratic dual is $U(\mathfrak{l})$, where $\mathfrak{l}$ is two-step nilpotent with $\dim \mathfrak{l}_1 = n$ and $\dim \mathfrak{l}_2 = 2$. Hence*

$$\mathrm{Hilb}(A_n^!, t) = \frac{1}{(1-t)^n(1-t^2)^2}.$$

Proof. The quadratic dual of an exterior quotient is the enveloping algebra of the ordinary graded Lie algebra generated by V with relation space annihilating $\langle f, g\rangle$ in $\bigwedge^2 V$; see [7, Section 10]. The dual of $\mathfrak{l}_3$ is the kernel of

$$V^* \oplus V^* \longrightarrow \bigwedge^3 V^*, \qquad (u,v) \longmapsto u \wedge f + v \wedge g.$$

For $n \ge 6$, Theorem 1.1 gives $\dim(A_n)_3 = \binom{n}{3} - 2n$, so this map is injective. Thus $\mathfrak{l}_3 = 0$, and Jacobi induction gives $\mathfrak{l}_j = [\mathfrak{l}_1, \mathfrak{l}_{j-1}] = 0$ thereafter. PBW gives the series. Koszulness would force $\mathrm{Hilb}(A_n, t) = (1+t)^n(1-t^2)^2$, impossible in degrees above n.

For $n = 5$, the series is $1 + 5t + 8t^2 + t^3$. The reciprocal of $1 - 5t + 8t^2 - t^3$ begins

$$1 + 5t + 17t^2 + 46t^3 + 99t^4 + 144t^5 - 26t^6 + \cdots,$$

contradicting the nonnegative coefficients required by the Koszul identity. For $n \le 4$, the generic ideal can be made an exterior quadratic monomial ideal; in dimension four it is (x_1x_2, x_3x_4), and in dimension three the path pencil gives (z_1z_2, z_2z_3). These have quadratic PBW presentations. □

5.3. **The two-generator Koszul homology**

For arbitrary $f, g \in \bigwedge^2 \mathbb{C}^n$, put

$$E = \bigwedge \mathbb{C}^n, \qquad H(t) = \mathrm{Hilb}(E/(f,g), t).$$

Let $\mathbb{C}[u,v]$ act on E by $u \mapsto f$, $v \mapsto g$, with $\deg u = \deg v = 2$.

Proposition 5.4. *The graded Betti polynomials of E over $\mathbb{C}[u,v]$ are*

$$\begin{aligned} B_0(t) &= H(t), \\ B_2(t) &= t^{n+4}H(t^{-1}), \\ B_1(t) &= H(t) + t^{n+4}H(t^{-1}) - (1-t^2)^2(1+t)^n. \end{aligned}$$

There are no later terms.

Proof. These are the Hilbert series of the Koszul homology on multiplication by f, g. The zeroth is $E/(f,g)$ and the second is $\operatorname{ann}_E(f,g)(-4)$. The perfect exterior pairing identifies $\operatorname{ann}(f,g)_j$ with $(E/(f,g))^*_{n-j}$, giving B_2. The Euler characteristic gives B_1. The polynomial ring has global dimension two. □

For the two-step Lie algebra $\mathfrak{n}_{f,g} = V \oplus \mathbb{C}z_1 \oplus \mathbb{C}z_2$ with $[x,y] = f(x,y)z_1 + g(x,y)z_2$, its Chevalley–Eilenberg complex is $E \otimes \bigwedge(\epsilon_1, \epsilon_2)$ with $d\epsilon_1 = f$, $d\epsilon_2 = g$. Assign internal weights one to V^* and two to the ϵ_i. Koszul position i and internal weight q have cohomological degree $q - i$, so

$$\sum_j \dim H^j(\mathfrak{n}_{f,g}; \mathbb{C}) t^j = H(t) + t^{-1}B_1(t) + t^{n+2}H(t^{-1}),$$

$$\sum_j \dim H^j(\mathfrak{n}_{f,g}; \mathbb{C}) = 4H(1).$$

Theorem 1.1 therefore determines the full graded Betti sequence for a general pair in either parity.

6. More alternating forms

The cohomology gap also gives an exactness range for more than two forms. The staircase class is special to a pair, but the colored-edge bound on Schur weights still controls the Koszul complex. Let W have dimension $m \geq 3$ and let $f : W \to \bigwedge^2 V^*$ define a section of $\bigwedge^2 \mathcal{U}^* \otimes W^*$ on $\mathrm{Gr}(s,n)$. Assume its Koszul complex is exact, as for a regular zero scheme X or a nowhere-zero section. Put

$$d = n - s, \qquad N = m\binom{s}{2}, \qquad D = sd - N, \qquad \iota = n - m(s-1).$$

For nonempty regular X, adjunction gives $\omega_X = \mathcal{O}_X(-\iota)$.

Theorem 6.1. *If $\iota \geq 0$, the degree-s exterior Koszul complex is exact before its final quotient, and*

$$\dim \Big(\bigwedge V^*/(f(W))\Big)_s = h^0(X, \mathcal{O}_X(1)) = \sum_{j=0}^{\min(m,\lfloor s/2 \rfloor)} (-1)^j \binom{m}{j}\binom{n}{s-2j}.$$

Moreover $H^{>0}(X, \mathcal{O}_X(1)) = 0$.

Proof. For a Schur constituent λ of $\bigwedge^j(\bigwedge^2 \mathbb{C}^s \otimes W)$, the colored-edge count gives

$$|\lambda| = 2j, \quad \lambda_1 \leq m(s-1), \quad \sum_{i=1}^{\ell} \lambda_i \leq j + m\binom{\ell}{2}.$$

For twist t, a nonzero cohomology term with ℓ negative Bott entries has $q = \ell d$ and, writing $T = q - j$,

$$T \leq -(t+1)\ell + (m-2)\binom{\ell}{2}.$$

The complementary partition with parts $m(s-1) - \lambda_{s+1-i}$, paired by Serre duality at twist $-t-\iota$, has $s-\ell$ crossed entries and total degree $D - T$. Combining the two inequalities gives the necessary condition

$$-st + (2t+\iota)\ell + (m-2)\ell(s-\ell) \leq 0. \tag{8}$$

At $t = 1$, the left side is concave in ℓ, with positive endpoint values $\iota+1+(m-3)(s-1)$ and $s(1+\iota)$ on $[1, s]$. Thus every positive cohomology group of every Koszul term vanishes.

In degree zero Bott requires $\lambda_1 \leq 1$. The constituent is (1^{2j}), with multiplicity $\bigwedge^j W$; permutations of the two-form blocks act symmetrically on their exterior product, leaving the alternating type on W. Hence the global-section complex has terms $\bigwedge^j W \otimes \bigwedge^{s-2j} V^*$ and is exactly the exterior Koszul complex. Acyclicity of its sheaf terms gives the assertion and the displayed Euler characteristic. □

Proposition 6.2. *If $\iota > 0$, then $H^0(X, \mathcal{O}_X) = \mathbb{C}$ and $H^{>0}(X, \mathcal{O}_X) = 0$. If $\iota = 0$ and $D > 0$, then*

$$H^0(X, \mathcal{O}_X) = H^D(X, \mathcal{O}_X) = \mathbb{C}, \qquad H^i(X, \mathcal{O}_X) = 0 \quad (0 < i < D).$$

If $\iota = D = 0$, then $h^0(X, \mathcal{O}_X) = 2$. In all these cases X is nonempty.

Proof. At $t = 0$, condition (8) is $\ell(\iota + (m-2)(s-\ell)) \leq 0$. For $\iota > 0$, only the constant term survives. For $\iota = 0$, only $\ell = 0, s$ are possible. The second case forces $\lambda = (n^s)$ and $j = N$, giving the top Koszul term ω_{Gr}, up to a constant line. The spectral

sequence therefore has just the entries $\mathbb{C}$ at $(0,0)$ and $(-N, sd)$. Their total degrees are $0, D$, and no differential connects them when $D \geq 0$. This proves the assertions, including nonemptiness. □

For three general alternating forms on $\mathbb{C}^9$, the common isotropic Grassmannian of four-planes has $\iota = 0$ and $D = 2$. Generic smoothness, adjunction, and Proposition 6.2 show that it is a K3 surface. Theorem 6.1 gives

$$h^0(\mathcal{O}_X(1)) = \binom{9}{4} - 3\binom{9}{2} + 3 = 21,$$

so its polarization has genus 20 and degree 38. Its nonzero $H^{2,0}$ shows why the full cohomology of such zero schemes cannot always be generated by tautological Chern classes.

7. Positive characteristic

The integer denominators in the local inverse determine a range in which straightening survives in positive characteristic. In larger dimensions, Frobenius supplies new Koszul cycles that force extra quotient classes. Let K now be algebraically closed of characteristic $p > 0$, and retain the explicit odd path pencil (2) in $\bigwedge K^{2k+1}$.

Theorem 7.1. *For $s < 2p$ and $s \leq k+1$, the good monomials form a basis of the degree-s exterior quotient, and the corresponding Plücker ring has the Richardson initial ideal* (6)*. In particular, the full characteristic-zero Hilbert series holds whenever $n \leq 4p - 3$.*

Proof. The denominators in Lemma 3.1 are $r\binom{r-1}{j}$, with $r \leq \lfloor s/2 \rfloor < p$, and have no prime factor above r. All local inverses therefore exist. The weights, the prime interval ideal, and the dimension argument apply over K.

If $k + 1 < 2p$, this covers all degrees through $k + 1$. The unique top monomial is $b_0 \cdots b_k$, which every b_j kills. Wedge F with the product of all b_j except the one paired with a_i; only that term survives, showing that a_i also kills the top class. Thus every higher quotient degree vanishes. The inequality is exactly $n \leq 4p - 3$. □

The argument is integral. Put $M = \lfloor (k+1)/2 \rfloor$ and give the successive path edges independent invertible coefficients $c_1, \dots, c_{2k}$. Over

$$\mathbb{Z}[1/M!][c_1^{\pm 1}, \dots, c_{2k}^{\pm 1}],$$

the good monomials form a free basis of the full exterior quotient. A diagonal change of generators removes the edge coefficients by solving successively along the path. The

local rules span, top-degree annihilation gives the ceiling, and independence follows over the fraction field. The basis consequently survives arbitrary base change from this ring.

Theorem 7.2. *For every prime p and odd $n \geq 6p-3$, the canonical pair satisfies*

$$\dim_K \Big(\bigwedge K^n/(F,G)\Big)_{2p} \geq a(n,2p)+2.$$

Proof. Since commuting square-zero pair monomials satisfy $F^p = G^p = 0$, the kernel of

$$\Phi : E_{2p-2} \oplus E_{2p-2} \longrightarrow E_{2p}, \qquad (u,v) \longmapsto Fu + Gv$$

contains the Koszul vectors $(Gh, -Fh)$ and the two vectors $(F^{p-1}, 0)$, $(0, G^{p-1})$.

Multiplication by F is injective on E_{2p-4}. Decompose by the singleton states of its disjoint pairs and by its unpaired variable. Every remaining block is the Boolean inclusion map from r-subsets to $(r+1)$-subsets of an m-set, with $r \leq p-2$ and $m \geq 2r+1$; the latter follows from $k \geq 3p-2$. On any $(2r+1)$-subset the central map is invertible by Lemma 3.1, whose denominator primes are at most $r+1 < p$. Restrict a putative kernel vector to the r-subsets of this smaller set. Its image is still zero, since an $(r+1)$-subset contained there receives contributions only from its own r-subsets. Hence all restricted coefficients vanish. Every r-subset lies in one of these smaller sets, proving injectivity. The ordinary Koszul vectors therefore span a subspace of dimension $\binom{n}{2p-4}$.

Choose $p-1$ pairwise nonconsecutive F-pairs and project all other variables to zero. Then G becomes zero and F^{p-1} becomes $(p-1)!$ times the retained product, which is nonzero. Thus a relation $\alpha F^{p-1} = Gh$ forces $\alpha = 0$. The reflected projection does the same for G, so the two Frobenius vectors are independent modulo the Koszul image. Therefore

$$\dim E_{2p}/(F,G)_{2p} \geq \binom{n}{2p} - 2\binom{n}{2p-2} + \binom{n}{2p-4} + 2.$$

For $n \geq 3s-3$, the two-wall reflection formula reduces to

$$a(n,s) = \binom{n}{s} - 2\binom{n}{s-2} + \binom{n}{s-4}.$$

Indeed, with $N = n+2$ and $H = n+4-2s$, the reflection sum is $\sum_{j\in\mathbb{Z}}(\binom{N}{s-jH} - \binom{N}{s-1-jH})$; only $j = 0, -1$ contribute. Their sum is $\binom{N}{s} - 2\binom{N}{s-1} + \binom{N}{s-2}$, equal to the displayed formula by $(1-t)^2(1+t)^{n+2} = (1-t^2)^2(1+t)^n$. Set $s = 2p$. □

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