The Hivert–Thiéry conjecture and differential normal forms

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Abstract. We prove the Hivert–Thiéry coinvariant conjecture by showing that, throughout its proposed parameter range, the deformed quotient is the ordinary coinvariant algebra as a graded symmetric-group module. The Artin monomials form a basis, with a sharp parameter range for $n \geq 2$. The proof combines deformed Koszul relations with a coordinate filtration that applies to weighted tails over arbitrary fields. The same Koszul mechanism gives normal forms for flat lower-order differential perturbations of a regular sequence, allowing noncommutative coefficients. For weighted homogeneous Jacobian systems with isolated singularities, an Euler identity and the Brieskorn module determine both the resonant parameters and the graded defect at each of them.

Contents

1. Introduction 2
2. Weighted power operators 3
2.1. A general tail theorem 4
2.2. The Hivert–Thiéry quotient 8
2.3. The exceptional parameters and the Artin basis 9
2.4. Arithmetic progressions of operator indices 10
2.5. Zero intercept and exact support decomposition 11
2.6. Coefficient fields and equivariant families 12
3. Universal resolutions and differential forms 13
3.1. The divergence–Koszul construction 13
3.2. The universal parameter resolution 15
3.3. Differential-form quotients 19
3.4. Parameter shifts and a finite Witt complex 21
4. Monomial reflection groups and residues 25
4.1. Flat braid residues 29
4.2. Cyclotomic residues 33
5. Filtered logarithmic operators 35
5.1. Free arrangements and scalar residues 38
5.2. Matrix and universal residues 41
5.3. Connections in the potential parameters 43
5.4. Constant matrix curvature 45
6. Jacobian operators and resonant fibers 47
6.1. Hypersurfaces 49
6.2. The exact defect at every exceptional parameter 49
6.3. Complete intersections 52
References 56

1. **Introduction**

In 2004, Hivert and Thiéry introduced a differential deformation of the symmetric power sums and conjectured that its coinvariants retain the graded S_n-module structure of the ordinary coinvariant algebra [1]. We prove this throughout their proposed parameter range and determine exactly when the classical Artin monomials span the deformed quotient.

Let $P_n = \mathbb{C}[z_1, \dots, z_n]$, let $E_i = z_i \partial_{z_i}$, and put

$$D_{\alpha,k} = \sum_{i=1}^{n} z_i^k (1 + \alpha E_i), \qquad H_\alpha = \sum_{k \geq 1} D_{\alpha,k}(P_n).$$

Here and throughout, a sum of operator images is a vector-space sum. Let e_i be the elementary symmetric polynomials and define

$$C_n = P_n/(e_1, \dots, e_n), \qquad \mathcal{E}_n = \left\{ -\frac{r}{s} : 1 \leq r \leq n,\ s \geq r,\ r, s \in \mathbb{Z} \right\}.$$

Theorem 1.1. *For $\alpha \notin \mathcal{E}_n$, there is a graded S_n-module isomorphism*

$$P_n/H_\alpha \cong C_n.$$

The classes of the Artin monomials

$$\mathcal{A}_n = \{ z_1^{a_1} \cdots z_n^{a_n} : 0 \leq a_i < i \}$$

form a basis. If $n \geq 2$, these monomials span the quotient only when $\alpha \notin \mathcal{E}_n$.

The first assertion settles the conjecture, while the basis statement and its converse identify the exact range of the Artin normal forms for every $n \geq 2$. Dotsenko [4, Theorem 25 and Remark 26] proved the graded-character assertion outside the larger exceptional set obtained by dropping $r \leq s$. The proof here obtains the full stated range and the monomial basis together. Malyshev's positive-weight simplex argument [3, Theorem 5 and Section 3] is a precedent for the coordinate filtration.

The difficulty is that H_α is a sum of operator images, rather than an ideal. We compare its quotient with a complete intersection by two bounds; the first two Koszul maps deform with the operators, and their zero composition, together with exactness at the undeformed point, gives a lower bound for the quotient. For the upper bound, we separate monomials of full support from those on coordinate hyperplanes. The former reduce the starting index of the operator family, and the latter reduce the number of variables. The required coefficients are invertible precisely under the stated nonresonance conditions. For $n \geq 2$, an explicit harmonic polynomial gives the obstruction to the Artin span at each excluded parameter.

This induction naturally includes unequal coefficients and every tail of the operator family. Section 2 proves that weighted theorem over an arbitrary field before specializing to Theorem 1.1. The deformed Koszul relations also assemble into a finite Lie–Rinehart resolution. Over the universal parameter ring, it is a minimal linear free resolution in each polynomial degree, so specialization and parameter shifts are controlled by finite complexes (Section 3). The same coordinate decomposition survives for monomial reflection groups and flat matrix residues; these forms are developed in Section 4.

A complementary use of Koszul exactness treats differential perturbations that lower a positive grading. Lifting the full complex preserves a homogeneous normal form. If $X_1, \dots, X_r$ form a free Lie module of polynomial vector fields and $X_1\eta, \dots, X_r\eta$ is a regular sequence, every fixed homogeneous complement to their ideal remains a complement after a flat lower-order perturbation. The result permits noncommutative coefficient algebras and applies to free arrangements; constant matrix curvature is handled by a quotient of the coefficient module (Section 5).

In weighted homogeneous Jacobian systems, the differential and multiplication terms have equal degree, so exceptional parameters can occur. For isolated singularities, an Euler identity isolates the possible resonant degree, and the Brieskorn module measures its defect exactly. Section 6 carries this out for hypersurfaces and complete intersections, including graded symmetry representations.

We write $p_j = \sum_i z_i^j$, $[j]_q = 1 + q + \cdots + q^{j-1}$, and $[n]_q! = \prod_{j=1}^{n} [j]_q$. Hilbert-series inequalities are coefficientwise. Multiplication by f is denoted by M_f; its position in a composition will matter.

2. **Weighted power operators**

The lower bound comes from a general observation about a polynomial family of complexes. It will also be used for matrix coefficients.

Lemma 2.1. *Let U, V, W be finite-dimensional vector spaces over a field K, and let*

$$G(t) : U \longrightarrow V, \qquad F(t) : V \longrightarrow W$$

be polynomial families of linear maps in one or finitely many parameters. Suppose $F(t)G(t) = 0$ identically. If $\operatorname{im} G(t_0) = \ker F(t_0)$ at one parameter value t_0, then

$$\operatorname{rank} F(t) \le \operatorname{rank} F(t_0)$$

at every parameter value. The same assertion applies after taking multiplicity spaces for a finite group whose order is invertible in K, provided the maps are equivariant.

Proof. Let r_F, r_G be the ranks over the rational function field of the parameter space. A minor nonzero at t_0 is a nonzero polynomial, so

$$r_F \geq \operatorname{rank} F(t_0), \qquad r_G \geq \operatorname{rank} G(t_0).$$

Since $FG = 0$, we have $r_F + r_G \leq \dim V$. Exactness at t_0 gives $\operatorname{rank} F(t_0) + \operatorname{rank} G(t_0) = \dim V$. Thus both inequalities are equalities. Every minor of size greater than r_F vanishes identically, proving the claim at every specialization.

For such a finite group, the functor $\mathrm{Hom}_G(L, -)$, for any representation L, is exact. Applying the preceding argument to these finite-dimensional multiplicity spaces proves the last assertion. □

2.1. **A general tail theorem**

Let K be a field and put $P_n = K[z_1, \ldots, z_n]$. For $c = (c_1, \ldots, c_n) \in K^n$, $\lambda \in K$, and $k \geq 1$, put

$$A_k(c, \lambda) = \sum_{i=1}^{n} z_i^k (c_i + \lambda E_i), \qquad B_k(c, \lambda) = \sum_{i=1}^{n} (c_i + \lambda E_i) M_{z_i^k}.$$

The original operators are $A_k(\mathbf{1}, \alpha)$. We work first with B_k because the reversed order turns the characteristic-polynomial recurrence into a relation among operator images. For a nonempty subset $I \subseteq [n]$, write $c_I = \sum_{i \in I} c_i$. Fix an integer $m \geq 1$, and consider the condition NR_m

$$c_I + \lambda s \neq 0 \quad \text{for every } \varnothing \neq I \subseteq [n] \text{ and every integer } s \geq m|I|. \tag{2.1}$$

Theorem 2.2. *Under* (2.1)*, the quotient*

$$Q^B_{n,m}(c,\lambda) = P_n \Big/ \sum_{k\geq m} B_k(c,\lambda)(P_n)$$

has the monomial basis

$$\mathcal{A}_{n,m} = \{z^a : 0 \leq a_i < m+i-1 \text{ for } 1 \leq i \leq n\}. \tag{2.2}$$

Its Hilbert series is

$$F_{n,m}(q) = \prod_{j=m}^{m+n-1} [j]_q.$$

For every subgroup $G \leq S_n$ fixing c, it is isomorphic as a graded $K[G]$-module to

$$P_n/(h_m, h_{m+1}, \ldots, h_{m+n-1}),$$

where h_j denotes the complete homogeneous symmetric polynomial. If all c_i coincide, the comparison may equivalently use the power sums $p_m, \ldots, p_{m+n-1}$. The same assertions hold with A_k in place of B_k if additionally

$$c_i + \lambda a \neq 0 \quad (1 \leq i \leq n, \quad 0 \leq a < m). \tag{2.3}$$

Lower bound. Set $J = \{m, \ldots, m+n-1\}$. The characteristic polynomial gives symmetric homogeneous polynomials $R_{\ell j}$, for $\ell \geq m$ and $j \in J$, with

$$z_i^\ell = \sum_{j\in J} R_{\ell j} z_i^j, \qquad B_\ell = \sum_{j\in J} B_j M_{R_{\ell j}}.$$

The first identity is initialized by $R_{\ell j} = \delta_{\ell j}$ for $\ell \in J$, and extended by the monic recurrence of length n. The order of multiplication in the second identity is essential. In particular, the first n tail operators generate the joint image. Direct calculation also gives

$$[B_i, B_j] = \lambda(j-i)B_{i+j}.$$

Thus the usual Koszul relation between B_i and B_j has a controlled error; their commutator is another tail operator. Expressing that operator in the first n generators supplies the correction below.

On the graded spaces

$$C_1 = \bigoplus_{j\in J} P_n(-j)\epsilon_j, \qquad C_2 = \bigoplus_{\substack{i,j\in J\\ i<j}} P_n(-i-j)\epsilon_{ij},$$

define

$$F(f\epsilon_j) = B_j f,$$
$$G(f\epsilon_{ij}) = B_j f\epsilon_i - B_i f\epsilon_j - \lambda(j-i)\sum_{k\in J} R_{i+j,k} f\epsilon_k.$$

Then $FG = 0$ identically. At $\lambda = 0$ over $K(c_1, \dots, c_n)$, these are the first two Koszul maps for the weighted powers $\sum_i c_i z_i^j$, $j \in J$. This sequence is regular in every characteristic. At a common zero, group the nonzero coordinate values as $y_1, \dots, y_r$, with index blocks $I_1, \dots, I_r$. The first r equations give

$$\sum_{a=1}^{r} c_{I_a} y_a^m y_a^k = 0 \qquad (0 \le k < r).$$

The Vandermonde matrix is invertible, whereas every c_{I_a} is a nonzero sum of independent indeterminates. Thus the only common zero is the origin, proving regularity.

Lemma 2.1 makes every larger polynomial minor vanish. Thus at every parameter value over K,

$$\mathrm{Hilb}(Q^B_{n,m}; q) \ge F_{n,m}(q). \qquad \square$$

Upper bound and basis. The induction separates monomials with full support from those on a coordinate hyperplane. The first part lowers the tail index; the second lowers the number of variables. To track nonresonance, put

$$\beta_i = c_i + m\lambda, \qquad \beta_I = \sum_{i\in I} \beta_i.$$

Condition (2.1) is equivalent to $\beta_I + \lambda t \neq 0$ for every nonempty $I \subseteq [n]$ and every integer $t \ge 0$. Write $\pi = z_1 \cdots z_n$. When $m = 1$, the identity

$$\sum_{k=1}^{n} (-1)^{k-1} B_k M_{e_{n-k}} = M_\pi\big(c_{[n]} + \lambda(E+n)\big)$$

shows that πP_n is hit, since the last factor acts in degree d by $\beta_{[n]} + \lambda d \neq 0$. Filter $R = P_n/\pi P_n$ by the stable subspaces

$$F_i = (z_{i+1} \cdots z_n)R, \qquad 0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = R.$$

This filtration sorts boundary monomials by their largest zero index. Indeed, if $\mu_i = \prod_{j>i} z_j$, the map

$$K[z_1, \dots, \widehat{z_i}, \dots, z_n] \longrightarrow F_i/F_{i-1}, \qquad f \longmapsto \mu_i f,$$

is a vector-space isomorphism shifted in degree by $n - i$. In that layer the i-th operator summand vanishes because it introduces a positive power of z_i, placing the result in F_{i-1}. For $j > i$, the extracted factor z_j adds one to the Euler operator. The remaining summands are therefore the same operators in $n - 1$ variables, with coefficients

$$c'_j = c_j \ (j < i), \qquad c'_j = c_j + \lambda \ (j > i). \tag{2.4}$$

Each surviving β_j is unchanged or replaced by $\beta_j + \lambda$, so every new nonresonance scalar is an old one with a larger nonnegative index. Taking a joint-image quotient is right exact, so each layer contributes at most its lower-variable quotient. Summing the degree shifts gives the upper bound $[n]_q F_{n-1,1} = F_{n,1}$, and the lifted lower-variable bases partition $\mathcal{A}_{n,1}$.

For $m \geq 2$, the same characteristic identity gives

$$\begin{aligned} B_\ell(c, \lambda) M_\pi &= \sum_{k=1}^{n} (-1)^{k-1} B_{\ell+k}(c, \lambda) M_{e_{n-k}}, \\ B_\ell(c, \lambda) M_\pi &= M_\pi B_\ell(c + \lambda \mathbf{1}, \lambda). \end{aligned} \tag{2.5}$$

For $\ell \geq m - 1$, every index on the right of (2.5) is at least m. Consequently the image of πP_n in $Q^B_{n,m}$ is a quotient of the degree-shifted space $q^n Q^B_{n,m-1}(c + \lambda \mathbf{1}, \lambda)$. This reduction leaves the shifted coefficients unchanged, since

$$(c_i + \lambda) + (m - 1)\lambda = \beta_i,$$

so the lower-tail nonresonance condition follows. The remaining quotient is the tail quotient of $P_n/\pi P_n$. Its same boundary filtration has shifts (2.4), preserving NR_m. Induction on $n + m$, with the preceding $m = 1$ case and the convention $F_{0,m} = 1$, therefore gives

$$\begin{aligned} \mathrm{Hilb}(Q^B_{n,m}; q) &\leq q^n F_{n,m-1} + [n]_q F_{n-1,m} \\ &= \big(q^n [m-1]_q + [n]_q\big) F_{n-1,m} = F_{n,m}. \end{aligned}$$

For the spanning assertion, the monomials in $\mathcal{A}_{n,m}$ with all exponents positive are $\pi \mathcal{A}_{n,m-1}$. Those with a zero coordinate partition according to the largest zero index, and are exactly the lifts of the lower-variable bases through the boundary filtration. Thus $\mathcal{A}_{n,m}$ spans, and equality with the lower bound makes it a basis.

Finally, (2.3) and the singleton cases of NR_m make $T = \prod_i (c_i + \lambda E_i)$ invertible. On every monomial, $TA_kT^{-1} = B_k$. This diagonal map preserves the stated monomial span and is equivariant for every permutation fixing c. It transfers the basis and Hilbert series to the A family. □

Equivariance. The nonresonance hypothesis itself makes $K[G]$ semisimple. In characteristic $p > 0$, no value among the c_i occurs p times; choose p equal coefficients and an integer $s \geq mp$ divisible by p to contradict $c_I + \lambda s \neq 0$. The stabilizer of c is therefore a product of symmetric groups on fewer than p letters, and its order is prime to p.

Choose an independent coefficient for each G-orbit, and repeat the lower-bound argument on each G-multiplicity space. Its weighted power sequence at $\lambda = 0$ is regular by the same Vandermonde argument; each nonempty block sum is a nonzero linear combination of the orbit coefficients. In characteristic p, its coefficients are at most the orbit sizes, all smaller than p.

The comparison sequence $h_m, \dots, h_{m+n-1}$ is regular over every field. At a common zero, the recurrence for complete homogeneous functions forces every later h_j to vanish. Then $H(t) = \sum_j h_j t^j$ is a polynomial satisfying $\prod_i (1 - z_i t) H(t) = 1$, so every $z_i = 0$. Its Koszul resolution and that of the generic weighted powers have identical invariant generator degrees, and hence identical graded G-multiplicities. The specialized multiplicity lower bounds, together with the proved dimension equality, give the comparison. This argument does not require K to split G; for a simple module L, its multiplicity is $\dim_K \mathrm{Hom}_G(L, -)/\dim_K \mathrm{End}_G(L)$.

If the coefficients all coincide, the characteristic is zero or greater than n. The ordinary power sequence is then regular too, since its Vandermonde block sums are nonzero integers at most n. Its Koszul resolution gives the same character. Finally, the diagonal conjugation above transfers every assertion to the A family. □

2.2. **The Hivert–Thiéry quotient**

For $c_i = 1$, $\lambda = \alpha$, and $m = 1$, the nonresonance condition of Theorem 2.2 is exactly $\alpha \notin \mathcal{E}_n$. Thus its B-quotient has the Artin basis and the graded character of C_n. The diagonal operator

$$T_\alpha = \prod_i (1 + \alpha E_i)$$

is invertible in this range and satisfies

$$T_\alpha D_{\alpha,k} T_\alpha^{-1} = \sum_i (1 + \alpha E_i) M_{z_i^k}.$$

Indeed, on a monomial the coefficient $1 + \alpha a_i$ is changed to $1 + \alpha(a_i + k)$. Since T_α preserves each monomial line and commutes with S_n, this proves the sufficiency assertion of Theorem 1.1. Over $\mathbb{C}$, for the scalar family $c_i = 1$, $\lambda = \alpha$, the sufficient excluded parameters for the tail $k \geq m$ are

$$\left\{-\frac{r}{s} : 1 \leq r \leq n,\ s \geq mr\right\} \cup \left\{-\frac{1}{a} : 1 \leq a < m\right\}.$$

At $m = 1$ this is exactly the original exceptional set; the second list is empty.

2.3. **The exceptional parameters and the Artin basis**

The obstruction is detected on a subset of the variables; a harmonic copy of the standard representation occurs beyond its usual coinvariant degrees.

Necessity in Theorem 1.1. When $n = 1$, the identity $D_{\alpha,k}(1) = z_1^k$ shows directly that the singleton $\{1\}$ is a basis for every complex parameter.

Suppose now $n \geq 2$ and $\alpha \in \mathcal{E}_n$. We may write $\alpha = -r/s$ with $2 \leq r \leq n$ and $s \geq r$; if an initial representation has numerator one, replace $1/s$ by $2/(2s)$. Setting $z_{r+1} = \cdots = z_n = 0$ commutes with every $D_{\alpha,k}$, and therefore induces a surjection from the n-variable quotient to the r-variable quotient. It also maps the Artin span onto the r-variable Artin span, so it is enough to show that the latter does not span.

Use the Fischer bilinear pairing

$$\langle z^a, z^b \rangle = a!\, \delta_{a,b}.$$

The adjoint of the original operator on r variables is

$$D^*_{\alpha,k} = \sum_{i=1}^{r} \big(\partial_i^k + \alpha z_i \partial_i^{k+1}\big).$$

Its common kernel is the graded dual of the hit quotient.

Put $S = z_1 + \cdots + z_r$ and $L = z_1 - z_2$. The harmonic vector $f = S^{s-1}L$ was found by D'Adderio and Moci [2, Theorem 6.3]. Its resonance is visible in the following one-variable calculation. For every univariate polynomial ϕ, Leibniz's rule gives

$$D^*_{\alpha,k}\big(\phi(S)L\big) = \big((r + \alpha(k+1))\phi^{(k)}(S) + \alpha S\phi^{(k+1)}(S)\big)L. \tag{2.6}$$

Indeed, $\sum_i \partial_i L = 0$ and $\sum_i z_i \partial_i L = L$. For $\phi(S) = S^{s-1}$, homogeneity reduces the right side to $(r + \alpha s)\phi^{(k)}(S)L = 0$ for every $k \geq 1$.

The S_r-orbit of f spans a copy of the standard representation in degree s. The classical coinvariant ring C_r contains that representation only in degrees $1, \ldots, r-1$,

once in each. Here is a short justification of this familiar fact. The permutation representation is the sum of the trivial and standard representations, and its multiplicity space in C_r is $C_r^{S_{r-1}}$. Exactness of invariants and the elementary-symmetric relations give

$$C_r^{S_{r-1}} \cong \mathbb{C}[z_r, e'_1, \dots, e'_{r-1}]/(e_1, \dots, e_r) \cong \mathbb{C}[z_r]/(z_r^r),$$

where e'_j is elementary symmetric in the first $r-1$ variables. Indeed $e_j = e'_j + z_r e'_{j-1}$ successively gives $e'_j = (-z_r)^j$, followed by $z_r^r = 0$. The trivial multiplicity space is $C_r^{S_r} = \mathbb{C}$, leaving standard multiplicity series $q + q^2 + \cdots + q^{r-1}$.

The comparison of all other multiplicities does not assume the desired conclusion at this exceptional parameter. In each degree, the original hit matrix has polynomial entries in α. The already proved theorem at nonexceptional parameters identifies its generic graded character with C_r. A specialization can only decrease matrix rank, also on every irreducible multiplicity space. Thus every multiplicity in the harmonic kernel is at least its generic value. Since $s \geq r$, the standard copy just constructed is additional. Consequently, in degree s, the harmonic kernel has dimension strictly greater than $\dim(C_r)_s$, the number of degree-s Artin monomials in r variables.

The degree-s quotient therefore has larger dimension than the Artin span. The r-variable Artin classes do not span, and coordinate restriction gives the same conclusion in n variables. □

Remark 2.3. For $n = 5$ and $\alpha = -3/4$, the polynomial $(z_1 + z_2 + z_3)^3(z_1 - z_2)$ is harmonic and annihilates every standard Artin monomial in five variables; its degree is four, while an Artin monomial supported on the first three variables has degree at most three. The Artin classes therefore do not span. D'Adderio and Moci [2] predict the usual graded coinvariant character at this parameter.

2.4. **Arithmetic progressions of operator indices**

Restricting the indices to multiples of h preserves each exponent class modulo h. On these residue blocks the operators are again weighted tails, which explains both the enlarged monomial ranges and the appearance of monomial reflection groups.

Corollary 2.4. *Let $h, m \geq 1$, and retain only the operators $B_{hk}(c, \lambda)$ with $k \geq m$. Suppose*

$$c_I + \lambda s \neq 0 \quad (\varnothing \neq I \subseteq [n], \quad s \in \mathbb{Z}, \quad s \geq hm|I|).$$

Their joint-image quotient has basis

$$\{z^a : 0 \leq a_i < h(m + i - 1)\}$$

and Hilbert series $\prod_{j=m}^{m+n-1}[hj]_q$. Over $\mathbb{C}$, for equal coefficients it is isomorphic, as a graded $G(h, 1, n)$-module, to

$$P_n/(p_{hm}, p_{h(m+1)}, \ldots, p_{h(m+n-1)}).$$

The corresponding A_{hk} assertion holds if additionally $c_i + \lambda a \neq 0$ for $0 \leq a < hm$.

Proof. Decompose the polynomial space into its stable residue blocks

$$P_n = \bigoplus_{r \in \{0,\ldots,h-1\}^n} z^r K[x_1, \ldots, x_n], \qquad x_i = z_i^h.$$

On the block indexed by r, the operators are the tail operators in the x variables with coefficients $c_i' = c_i + \lambda r_i$ and slope $\lambda' = h\lambda$. For $s \geq m|I|$, their required scalar is

$$c_I' + \lambda' s = c_I + \lambda\Big(\sum_{i \in I} r_i + hs\Big),$$

which is nonzero by hypothesis. Theorem 2.2 gives the block basis $0 \leq b_i < m + i - 1$. The lifted exponents $a_i = r_i + hb_i$ give precisely the displayed basis and Hilbert series.

Over $\mathbb{C}$, the two-map comparison in the preceding proof works with indices $hm, \ldots, h(m + n - 1)$, recurrence coefficients symmetric in z_i^h, and the comparison sequence $p_{hm}, \ldots, p_{h(m+n-1)}$. This sequence is regular by the same Vandermonde argument applied to $x_i = z_i^h$. All maps are $G(h, 1, n)$-equivariant, so the lower bound on every multiplicity and the already established equality of dimensions prove the assertion. The extra hypothesis for A, together with the singleton nonresonance condition for $a \geq hm$, again makes the diagonal conjugation invertible. □

When $m = 1$, this gives the ordinary $G(h, 1, n)$ coinvariant module.

2.5. **Zero intercept and exact support decomposition**

When the multiplication intercept vanishes, the diagonal conjugation used above is no longer invertible. Exact support provides a different decomposition; over $\mathbb{C}$, consider the vector fields

$$L_{hk} = \sum_{i=1}^{n} z_i^{hk+1} \partial_{z_i}, \qquad k \geq m.$$

Their coefficients vanish on monomials whose relevant exponent is zero, so they preserve each exact-support summand

$$P_n = \bigoplus_{I\subseteq[n]} z_I\mathbb{C}[z_i : i \in I], \qquad z_I = \prod_{i\in I} z_i.$$

On the summand indexed by I, multiplication by z_I identifies their action with $A_{hk}(\mathbf{1}, 1)$ in $|I|$ variables. All nonresonance conditions above are then automatic. Consequently

$$\operatorname{Hilb}\left(P_n\Big/\sum_{k\geq m} L_{hk}(P_n); q\right) = \sum_{r=0}^{n}\binom{n}{r}q^r\prod_{j=m}^{m+r-1}[hj]_q.$$

A basis is obtained by choosing $I = \{i_1 < \cdots < i_r\}$ and taking

$$\prod_{j=1}^{r} z_{i_j}^{a_j}, \qquad 1 \leq a_j \leq h(m + j - 1),$$

including the constant for $I = \varnothing$. For $h = m = 1$, the dimension is $\sum_{r=0}^{n}\binom{n}{r}r! = \lfloor e\, n!\rfloor$, and the full-support summand has the usual shifted Artin basis.

2.6. **Coefficient fields and equivariant families**

The coordinate reductions depend only on the displayed nonresonance scalars, so they also describe which coefficient fields and parameter rings support the same basis. Over a field of characteristic $p > n$, the original Hivert–Thiéry conclusion holds for $\alpha = 0$ and for every $\alpha \notin \mathbb{F}_p$. For such a parameter all the pivots $r + \alpha s$, $1 \leq r \leq n$, $s \geq r$, are nonzero. Here $(h_1, \ldots, h_n) = (e_1, \ldots, e_n)$. For the arithmetic-progression representation statement one may use $h_m(z_1^h, \ldots, z_n^h), \ldots, h_{m+n-1}(z_1^h, \ldots, z_n^h)$. The corresponding monomial group acts when the field contains the needed roots of unity; semisimplicity requires the characteristic not to divide the order of its diagonal subgroup.

There is also a family-level strengthening in characteristic zero. Put

$$R_n = \mathbb{C}\big[\alpha, (r + s\alpha)^{-1} : 1 \leq r \leq n,\ s \geq r\big].$$

Define the operators and their joint image over $R_n[z]$. The coordinate filtration uses only units of R_n, so the Artin monomials span the universal quotient over R_n. They are independent over $\mathbb{C}(\alpha)$ by the proved generic comparison. A relation over R_n would lie in the kernel of a map from a free module and become zero over the fraction field; this kernel is both torsion and torsion free, hence zero. Thus the universal quotient is a free graded R_n-module with the Artin basis.

Moreover there is a single, noncanonical graded equivariant isomorphism

$$R_n[z]/H_{R_n} \cong C_n \otimes_{\mathbb{C}} R_n.$$

Indeed R_n is a localization of the principal ideal domain $\mathbb{C}[\alpha]$, and each isotypic multiplicity module is a finitely generated projective R_n-module, hence free. Its rank is the classical multiplicity. Choosing bases for these multiplicity modules gives the isomorphism, which specializes at every $\alpha \notin \mathcal{E}_n$.

3. **Universal resolutions and differential forms**

The dimension comparison used only the first two Koszul maps. Retaining the whole complex explains how the quotients vary with their parameters and how they interact with differential forms. We begin with the divergence identity that supplies the differential, then use the coordinate calculation to control its specializations.

3.1. **The divergence–Koszul construction**

The first two Koszul maps can be deformed whenever the vector fields form a Lie module and their multiplication terms satisfy a cocycle identity. Divergence supplies the correction that makes the composition zero. Let $P = \mathbb{C}[x_1, \dots, x_N]$, with ordinary grading or positive integer weights, and let $L \subseteq \mathrm{Der}_{\mathbb{C}}(P)$ be a free graded P-module with homogeneous basis $X_1, \dots, X_r$ of positive degrees $d_1, \dots, d_r$. Assume that L is closed under brackets. Write div for divergence relative to $dx_1 \wedge \cdots \wedge dx_N$. Suppose that a homogeneous P-linear map $\omega : L \to P$ of degree zero satisfies

$$\omega([X, Y]) = X(\omega(Y)) - Y(\omega(X)),$$

and that $f_i = \omega(X_i)$ form a regular sequence. A closed logarithmic one-form restricted to L is one source of such a map, but the displayed algebraic hypotheses suffice.

For a parameter t, define

$$\mathscr{B}_t(X)(g) = tX(g) + \big(t \operatorname{div} X + (1-t)\omega(X)\big)g, \qquad \mathscr{H}_t = \sum_{i=1}^{r} \mathscr{B}_t(X_i)(P).$$

The divergence identities imply

$$\mathscr{B}_t(aX) = \mathscr{B}_t(X)M_a, \qquad [\mathscr{B}_t(X), \mathscr{B}_t(Y)] = t\mathscr{B}_t([X, Y]).$$

In particular, $\mathscr{H}_t$ is independent of the chosen basis.

Theorem 3.1. *Over $\mathbb{C}(t)$, the quotient $P/\mathscr{H}_t$ has the graded Hilbert series of the complete intersection $P/(f_1,\dots,f_r)$. At every complex specialization, each graded dimension is at least its complete-intersection value. If a finite group acts linearly, preserves the grading and L, and makes ω equivariant, the same statements hold for each graded irreducible multiplicity.*

Proof. Set

$$\mathscr{F}_t : L \to P, \qquad \mathscr{F}_t(X) = t\,\mathrm{div}\,X + (1-t)\omega(X).$$

Since $\mathscr{F}_t(aX) = \mathscr{B}_t(X)(a)$, its image is $\mathscr{H}_t$. Define the $\mathbb{C}$-linear map $\mathscr{G}_t : \bigwedge^2_P L \to L$ by

$$\mathscr{G}_t(X \wedge Y) = \mathscr{F}_t(Y)X - \mathscr{F}_t(X)Y - t[X,Y].$$

It is well defined on the balanced exterior product, since replacing X by aX, or Y by aY, gives in both cases

$$a\mathscr{G}_t(X \wedge Y) + tY(a)X - tX(a)Y.$$

Both divergence and ω satisfy the Lie cocycle identity, so

$$\mathscr{F}_t\mathscr{G}_t(X \wedge Y) = t\big(X\mathscr{F}_t(Y) - Y\mathscr{F}_t(X) - \mathscr{F}_t([X,Y])\big) = 0.$$

At $t = 0$, these are the first two Koszul maps for the regular sequence $f_1,\dots,f_r$. They are exact at L. In each fixed degree all spaces are finite dimensional and the matrices depend polynomially on t. Lemma 2.1 therefore gives the asserted generic equality and specialized lower bounds. Linear changes of coordinates preserve divergence, since the volume form changes by a constant. The maps are consequently equivariant under the stated group hypotheses, and the same lemma applies to multiplicity spaces. □

For ordinary grading, the generic Hilbert series in this theorem is $\prod_i (1-q^{d_i})/(1-q)^N$. For weights $w_1,\dots,w_N$, the denominator is $\prod_j (1-q^{w_j})$. "Generic" here means over the rational function field, or degree by degree after specialization; the theorem does not assert that a finite exceptional set controls all degrees.

3.2. **The universal parameter resolution**

For the weighted power operators, these maps extend to a free resolution over the ring of independent parameters. Arbitrary specializations need not remain exact. At the nonresonant parameters considered above, however, the sharp dimension formula implies local freeness of the universal module, and therefore proves exactness after specialization as well.

Let K be any field, let $n, m \geq 1$, and set

$$S = K[c_1, \ldots, c_n, \lambda], \qquad P = S[z_1, \ldots, z_n], \qquad J = \{m, m+1, \ldots, m+n-1\}.$$

Give the parameters standard degree one. Independently, grade in the z-variables by total degree. Define

$$B_k = \sum_{i=1}^{n} (c_i + \lambda E_i) M_{z_i^k}, \qquad \mathcal{M} = P \Big/ \sum_{k \geq m} B_k(P).$$

Write $\mathcal{M}_d$ for its homogeneous component of z-degree d, retaining the grading in the parameters.

Theorem 3.2. *Each $\mathcal{M}_d$ has a minimal graded free resolution over S, linear in the parameter grading, whose term in homological degree p is*

$$C_p(d) = \bigoplus_{\substack{I \subseteq J \\ |I| = p}} S(-p) \otimes_K K[z_1, \ldots, z_n]_{d - \sum_{k \in I} k}. \tag{3.1}$$

Here a negative-degree polynomial space is zero. Consequently,

$$\beta^S_{p,j}(\mathcal{M}_d) = 0 \quad (j \neq p), \qquad \beta^S_{p,p}(\mathcal{M}_d) = \sum_{\substack{I \subseteq J \\ |I| = p}} \dim_K K[z_1, \ldots, z_n]_{d - \sum_{k \in I} k}.$$

For a nonnegative integer a, the last polynomial dimension is $\binom{n+a-1}{n-1}$.

Proof. The Lie brackets determine the differential; exactness will follow by setting $\lambda = 0$, where the complex becomes an ordinary Koszul complex. We first construct the Lie module on which this reduction takes place. For $k \geq m$, put

$$X_k = \sum_{i=1}^{n} z_i^{k+1} \partial_{z_i}, \qquad L = \sum_{k \in J} P X_k \subseteq \mathrm{Der}_S(P).$$

The coefficient determinant of $X_m, \ldots, X_{m+n-1}$ is, up to sign,

$$(z_1 \cdots z_n)^{m+1} \prod_{i<j} (z_j - z_i),$$

a nonzero polynomial over every field. These vector fields are therefore a free P-basis of L. The characteristic-polynomial recurrence expresses every X_k, $k \geq m$, in this basis. Together with $[X_i, X_j] = (j-i)X_{i+j}$, this proves closure under brackets; closure for polynomial multiples follows from the Leibniz rule.

For $X \in L$, define

$$\eta(X) = \lambda \operatorname{div}(X) + \sum_{i=1}^{n} (c_i - \lambda)\frac{X(z_i)}{z_i}, \qquad \mathcal{B}_X(f) = \lambda X(f) + \eta(X)f.$$

The displayed quotients are polynomials, since every $X(z_i)$ is divisible by z_i^{m+1}. A direct calculation gives $\mathcal{B}_{X_k} = B_k$. Divergence and the logarithmic one-forms dz_i/z_i satisfy the cocycle identities

$$\eta([X,Y]) = X\eta(Y) - Y\eta(X), \qquad \eta(aX) = a\eta(X) + \lambda X(a).$$

Consequently,

$$[\mathcal{B}_X, \mathcal{B}_Y] = \lambda \mathcal{B}_{[X,Y]}, \qquad \mathcal{B}_{aX} = \mathcal{B}_X M_a. \tag{3.2}$$

These formulas hold in every characteristic.

Scale the bracket and anchor of L by λ, and give P the right action $f \cdot X = -\mathcal{B}_X(f)$. Equations (3.2) are exactly the right-module compatibilities. Put $\mathbf{X} = X_1 \wedge \cdots \wedge X_p$, and use $\mathbf{X}_{\widehat{i}}$ and $\mathbf{X}_{\widehat{ij}}$ for the products with the indicated factors omitted. The associated Chevalley–Eilenberg differential, with first map B, is

$$\begin{aligned} d_p(f\mathbf{X}) = &\sum_{i=1}^{p} (-1)^{i-1} \mathcal{B}_{X_i}(f)\mathbf{X}_{\widehat{i}} \\ &+ \lambda \sum_{i<j} (-1)^{i+j+1} f[X_i, X_j] \wedge \mathbf{X}_{\widehat{ij}}. \end{aligned} \tag{3.3}$$

Here the symbols X_i in the formula are arbitrary elements of L, not necessarily the consecutively indexed basis elements. The second identity in (3.2) and the bracket Leibniz rule make this formula balanced over P; the first identity and Jacobi give $d_{p-1}d_p = 0$. For example, in degree two the composite is $-[\mathcal{B}_X, \mathcal{B}_Y] + \lambda \mathcal{B}_{[X,Y]} = 0$. Thus this is a well-defined complex of S-modules.

Give the basis vector X_k z-degree k. The differential preserves this degree, and every coefficient in its matrix is linear in $c_1, \ldots, c_n, \lambda$. Taking z-degree d, and shifting parameter degree in homological degree p by p, gives precisely the finite free modules (3.1) and homogeneous differentials of degree zero. The characteristic-polynomial recurrence on the right shows that the image of d_1 is the joint image of all tail operators, so $H_0(C_\bullet(d)) = \mathcal{M}_d$.

Modulo λ, the complex is the Koszul complex of

$$p_k^c = \sum_{i=1}^{n} c_i z_i^k, \qquad k = m, \dots, m+n-1,$$

over $K[c_1, \dots, c_n, z_1, \dots, z_n]$. We verify that this is a regular sequence globally, not merely at generic coefficients. Passing to an algebraic closure, stratify the common zero set according to the zero coordinates and to the partition of the remaining coordinates into blocks $I_1, \dots, I_r$ having distinct nonzero values $y_1, \dots, y_r$. The first r equations become

$$\sum_{j=1}^{r} \Big(\sum_{i \in I_j} c_i \Big) y_j^m y_j^a = 0, \qquad 0 \le a < r.$$

Vandermonde invertibility forces the r independent linear conditions $\sum_{i \in I_j} c_i = 0$. Conversely, these conditions imply all the equations on the stratum. Its z-coordinates have dimension r, and its coefficient coordinates have dimension $n - r$, so it has dimension n. There are only finitely many such strata, including the dimension-n stratum $z = 0$. The ideal therefore has height n. Its n homogeneous generators form a regular sequence in the Cohen–Macaulay polynomial ring. Regularity descends from the algebraic closure by faithful flatness. Notice that the block-sum constraints have disjoint supports and coefficient one, so the argument is valid in arbitrary characteristic.

It follows that $C_\bullet(d)/\lambda C_\bullet(d)$ is exact in every positive homological degree. Since each chain module is free over S, the short exact sequence of complexes

$$0 \longrightarrow C_\bullet(d)(-1) \xrightarrow{\lambda} C_\bullet(d) \longrightarrow C_\bullet(d)/\lambda C_\bullet(d) \longrightarrow 0$$

is valid. Its long exact homology sequence implies $H_p(C_\bullet(d)) = \lambda H_p(C_\bullet(d))$ for every $p > 0$. These homology modules are graded and bounded below. Multiplication by the positive-degree parameter λ cannot be surjective on a nonzero such module; a nonzero element of least homogeneous degree would have to come from a smaller degree. Hence all these positive homology modules vanish. This proves that (3.1) is a free resolution of $\mathcal{M}_d$.

Every differential entry lies in $(c_1, \dots, c_n, \lambda)$, the homogeneous maximal ideal. Thus the resolution is minimal and has the stated linear Betti numbers. □

At any parameter point with residue field κ, the homology of the specialized complex is

$$H_p(C_\bullet(d) \otimes_S \kappa) \cong \operatorname{Tor}_p^S(\mathcal{M}_d, \kappa).$$

Its degree-zero group is the specialized joint-image quotient, since cokernels commute with tensor product. Exactness after specialization can therefore be read from the quotient dimension.

Corollary 3.3. *Fix a z-degree d, and let*

$$r_d = [q^d] \prod_{j=m}^{m+n-1} (1 + q + \cdots + q^{j-1}).$$

At a parameter point $\mathfrak{p} \in \operatorname{Spec} S$, with residue field κ, the following conditions are equivalent

(i) *The specialized joint-image quotient has degree-d dimension r_d.*

(ii) *$(\mathcal{M}_d)_{\mathfrak{p}}$ is a free $S_{\mathfrak{p}}$-module.*

(iii) $\operatorname{Tor}_1^S(\mathcal{M}_d, \kappa) = 0$.

(iv) *The specialized finite Lie–Rinehart complex $C_\bullet(d) \otimes_S \kappa$ is exact in every positive homological degree.*

In particular, if the specialized parameters satisfy

$$c_I + \lambda s \neq 0 \qquad \big(\varnothing \neq I \subseteq [n], \quad s \in \mathbb{Z}, \quad s \geq m|I|\big),$$

then the specialized finite Lie–Rinehart complex is exact in every positive homological degree, in every z-degree.

Proof. The generic rank of $\mathcal{M}_d$ is r_d; tensor its free resolution with $\operatorname{Frac}(S)$ and take the Euler characteristic. Indeed, the generating series of the alternating ranks is $\prod_{k=m}^{m+n-1}(1-q^k)/(1-q)^n$. This argument is valid in every characteristic. Write $A = S_{\mathfrak{p}}$ and $N = (\mathcal{M}_d)_{\mathfrak{p}}$. This is a finitely generated module of rank r_d over the local domain A.

If $\dim_\kappa N \otimes_A \kappa = r_d$, choose lifts of a basis of this fiber. Nakayama's lemma gives a surjection $A^{r_d} \to N$. Its kernel has rank zero, because the source and target have equal rank. As a submodule of a free module over a domain, that kernel is torsion-free, so it must be zero. Thus N is free. This argument also covers $r_d = 0$. Conversely, a free localization necessarily has rank r_d, giving the stated fiber dimension. This proves the equivalence of (i) and (ii).

A free localization has no positive Tor. Conversely, tensor a minimal presentation $0 \to K' \to A^t \to N \to 0$ with κ. If $\operatorname{Tor}_1^A(N, \kappa) = 0$, minimality gives $K' \otimes_A \kappa = 0$, and Nakayama gives $K' = 0$. This proves (ii)$\Longleftrightarrow$(iii); the free resolution identifies these conditions with (iv). Under nonresonance, Theorem 2.2 gives the fiber dimension r_d, and hence all four conditions in every degree. □

3.3. **Differential-form quotients**

The diagonal conjugation used in Theorem 1.1 also applies to Lie derivatives on differential forms. Write

$$X_k = \sum_i z_i^{k+1}\partial_i, \qquad \Omega^\bullet_{\log} = P \otimes_{\mathbb{C}} \bigwedge^\bullet \langle \theta_1, \dots, \theta_n \rangle, \qquad \theta_i = \frac{dz_i}{z_i}.$$

Assign polynomial degree zero to each θ_i, and let S_n permute the variables and their differentials. On this space set

$$\mathscr{D}_{\alpha,k} = p_k + \alpha \mathcal{L}_{X_k}, \qquad \mathscr{H}^p_\alpha = \sum_{k\geq 1} \mathscr{D}_{\alpha,k}(\Omega^p_{\log}).$$

Here $\mathcal{L}$ denotes the usual Lie derivative.

Theorem 3.4. *For every $\alpha \notin \mathcal{E}_n$ and every $0 \leq p \leq n$,*

$$\Omega^p_{\log}/\mathscr{H}^p_\alpha \cong C_n \otimes_{\mathbb{C}} \bigwedge^p \mathbb{C}^n$$

as graded S_n-modules, with $\bigwedge^p \mathbb{C}^n$ placed in polynomial degree zero.

The integrable differential

$$\nabla_\alpha = \alpha d + \left(\sum_i \theta_i\right) \wedge = \sum_i (1 + \alpha E_i)\, \theta_i \wedge$$

preserves the hit subspaces and makes their quotient into an acyclic complex. An S_n-equivariant contracting homotopy is induced by

$$h_\alpha = \frac{1}{n} \sum_i (1 + \alpha E_i)^{-1} \iota_i,$$

where ι_i contracts the exterior generator θ_i.

Proof. For an ordered subset $S \subseteq [n]$, write $\theta_S = \bigwedge_{i\in S} \theta_i$. Since $\mathcal{L}_{X_k}\theta_i = k z_i^k \theta_i$, the operator on the coefficient of θ_S is

$$\sum_i z_i^k \big(1 + \alpha E_i + \alpha k\, \mathbf{1}_{i\in S}\big).$$

Define the diagonal operator on this sector by

$$T_S = \prod_{i\notin S} (1 + \alpha E_i).$$

All its eigenvalues are nonzero at the stated parameters. If $i \notin S$, conjugation changes $z_i^k(1+\alpha E_i)$ into $(1+\alpha E_i)z_i^k$; if $i \in S$, the operator already has the latter form. Hence the direct sum $T = \bigoplus_S T_S$ is graded and S_n-equivariant, and

$$T\mathscr{D}_{\alpha,k}T^{-1} = B_{\alpha,k} \otimes 1$$

on every exterior degree. The scalar theorem proves the module isomorphism.

The operators $1 + \alpha E_i$ commute, so $\nabla_\alpha^2 = 0$. Moreover,

$$T\nabla_\alpha T^{-1} = \rho\wedge, \qquad \rho = \sum_i \theta_i.$$

Indeed, adding $i \notin S$ to S removes the factor $1 + \alpha E_i$ from T_S, exactly canceling the coefficient in the differential. The transformed differential commutes with every $B_{\alpha,k}\otimes 1$, and therefore descends to the quotient. The exterior complex with differential $\rho\wedge$ has contracting homotopy $n^{-1}\sum_i \iota_i$. Conjugating back gives the displayed h_α, which also commutes with the hit operators. Thus $\nabla_\alpha h_\alpha + h_\alpha \nabla_\alpha = 1$ on the quotient. □

There is a parallel statement for ordinary polynomial forms $\Omega_P^p = P \otimes \bigwedge^p \langle dz_1, \dots, dz_n\rangle$. Give dz_i degree one and use the same formula $\mathscr{D}_{\alpha,k} = p_k + \alpha\mathcal{L}_{X_k}$.

Theorem 3.5. *For every $\alpha \notin \mathcal{E}_n$,*

$$\Omega_P^p \Big/ \sum_{k\geq 1} \mathscr{D}_{\alpha,k}(\Omega_P^p) \cong C_n \otimes_{\mathbb{C}} \bigwedge^p \mathbb{C}^n$$

as graded S_n-modules, with the exterior factor placed in degree p.

Proof. Now $\mathcal{L}_{X_k} dz_i = (k+1)z_i^k dz_i$. On the dz_S sector, conjugation by the same T_S therefore gives the weighted scalar family

$$B_k^S = \sum_i (c_i^S + \alpha E_i) M_{z_i^k}, \qquad c_i^S = 1 + \alpha\, \mathbf{1}_{i\in S}.$$

For every nonempty $I \subseteq [n]$ and every integer $s \geq |I|$, its nonresonance scalar is

$$\sum_{i\in I} c_i^S + \alpha s = |I| + \alpha\big(s + |I \cap S|\big),$$

which is nonzero because $\alpha \notin \mathcal{E}_n$. The weighted scalar theorem thus identifies this sector quotient with C_n, restricted to its stabilizer $S_S \times S_{S^c}$.

The exterior generator dz_S contributes the sign representation of S_S and degree p. Inducing over all p-element subsets gives

$$\mathrm{Ind}_{S_p\times S_{n-p}}^{S_n}\Big(\mathrm{Res}_{S_p\times S_{n-p}}^{S_n} C_n \otimes (\mathrm{sgn}_p \boxtimes \mathbf{1})\Big) \cong C_n \otimes \bigwedge^p \mathbb{C}^n,$$

with the asserted degree shift. □

3.4. **Parameter shifts and a finite Witt complex**

Multiplication by a variable shifts one coefficient of the weighted operators. These maps form a Koszul complex, whose homology is computed by a finite truncation of the positive Witt algebra. Fix $m \geq 1$. For a commutative $\mathbb{C}$-algebra R and parameters $c_1, \dots, c_n, \lambda \in R$, write

$$B_k^{\mathbf{c}} = \sum_i (c_i + \lambda E_i) M_{z_i^k}, \qquad Q_{\mathbf{c}} = R[z_1, \dots, z_n] \Big/ \sum_{k \geq m} B_k^{\mathbf{c}}(R[z]).$$

For $S \subseteq [n]$, set $Q_S = Q_{\mathbf{c}+\lambda \mathbf{1}_S}$. The identity

$$B_k^{\mathbf{c}} M_{z_i} = M_{z_i} B_k^{\mathbf{c}+\lambda \mathbf{e}_i}$$

shows that multiplication by z_i induces $Q_S \to Q_{S \setminus i}$ whenever $i \in S$. Define the chain complex

$$\mathcal{K}_p(\mathbf{c}, \lambda) = \bigoplus_{\substack{S \subseteq [n] \\ |S| = p}} Q_S \varepsilon_S, \qquad \kappa([f]\varepsilon_S) = \sum_{i \in S} (-1)^{\mathrm{pos}_S(i)-1} [z_i f] \varepsilon_{S \setminus i}. \tag{3.4}$$

Give ε_S polynomial degree $|S|$; thus κ has degree zero. Commutativity of the multiplications proves $\kappa^2 = 0$.

The parameter shift is exactly what makes this polynomial Koszul differential commute with the operator resolution. The resulting double complex can be read in either direction; resolving the operators gives $\mathcal{K}_\bullet$, while resolving the coordinates sets $z = 0$ and leaves only the constant bracket coefficients. These coefficients form the following finite Lie algebra.

Let $\mathfrak{w}_{m,n}$ be the graded complex Lie algebra with basis $u_m, \dots, u_{m+n-1}$, $\deg u_j = j$, and brackets

$$[u_i, u_j] = \begin{cases} (j-i) u_{i+j}, & i + j \leq m + n - 1, \\ 0, & i + j > m + n - 1. \end{cases} \tag{3.5}$$

It is the quotient of the Lie algebra with basis u_j, $j \geq m$, by the ideal spanned by u_j, $j \geq m + n$. Write ∂ for its Chevalley–Eilenberg differential with trivial coefficients, using the sign convention

$$\partial(u_{i_1} \wedge \cdots \wedge u_{i_q}) = \sum_{a<b} (-1)^{a+b+1} [u_{i_a}, u_{i_b}] \wedge u_{i_1} \wedge \cdots \widehat{u_{i_a}} \cdots \widehat{u_{i_b}} \cdots \wedge u_{i_q}.$$

In particular $\partial(u_i \wedge u_j) = [u_i, u_j]$.

Theorem 3.6. *Assume that every scalar*

$$\sum_{i\in I} c_i + \lambda s, \qquad \varnothing \neq I \subseteq [n], \quad s \geq m|I|, \tag{3.6}$$

is a unit of R. There is a natural zigzag of graded quasi-isomorphisms

$$\mathcal{K}_\bullet(\mathbf{c},\lambda) \;\longleftarrow\; \operatorname{Tot}\mathcal{D} \;\longrightarrow\; \Big(R\otimes_{\mathbb{C}} \bigwedge^{\bullet} \mathfrak{w}_{m,n}, \lambda\partial\Big). \tag{3.7}$$

It commutes with all coordinate permutations preserving the parameters. The induced action on the finite Lie algebra $\mathfrak{w}_{m,n}$ is trivial.

With the independent universal parameters $S_0 = \mathbb{C}[c_1,\dots,c_n,\lambda]$, the same quasi-isomorphism holds over S_0 before any localization. The unit condition (3.6) *ensures that it continues to hold for the ordinary quotient complex after arbitrary specialization of parameters.*

Proof. Let $P_R = R[z]$, and let L_R be the free P_R-module with basis

$$X_k = \sum_i z_i^{k+1}\partial_i, \qquad k = m,\dots,m+n-1.$$

For independent parameters this is the Lie–Rinehart module used in Theorem 3.2. The operator for weights $\mathbf{c}$ is

$$\mathcal{B}_X^{\mathbf{c}}(f) = \lambda X(f) + \left(\lambda \operatorname{div} X + \sum_i (c_i-\lambda)\frac{X(z_i)}{z_i}\right) f.$$

For $X = X_k$, it is $B_k^{\mathbf{c}}$. We use the right action and differential of (3.3). Every $X \in L_R$ has $X(z_i)$ divisible by z_i, and

$$\mathcal{B}_X^{\mathbf{c}} M_{z_i} = M_{z_i}\mathcal{B}_X^{\mathbf{c}+\lambda\mathbf{e}_i}. \tag{3.8}$$

Thus the ordinary polynomial Koszul complex

$$K_p^{\mathrm{pol}} = \bigoplus_{|S|=p} P_R\varepsilon_S, \qquad \kappa = \sum_i z_i\iota_i,$$

is a complex of right Lie–Rinehart modules when the sector ε_S has weights $\mathbf{c}+\lambda\mathbf{1}_S$. The differential κ is P_R-linear.

Set

$$\mathcal{D}_{p,q} = K_p^{\mathrm{pol}} \otimes_{P_R} \bigwedge_{P_R}^{q} L_R.$$

Use the Koszul differential in the p-direction and the finite Lie–Rinehart differential in the q-direction, with the usual total-complex sign. Equation (3.8) makes these

differentials commute before the total sign is inserted. Give the exterior generator X_k degree k. The total polynomial degree is therefore the coefficient degree plus $|S|$ plus the sum of the exterior indices k; both differentials preserve it.

First work over S_0. For a fixed S, its q-complex is the universal free resolution from Theorem 3.2, after the polynomial automorphism $c_i \mapsto c_i + \lambda \mathbf{1}_{i\in S}$. It is exact in positive q, with degree-zero homology Q_S. The augmentation to that quotient gives the left arrow in (3.7); it is zero for $q > 0$, and for $q = 0$ it sends each polynomial coefficient to its class in Q_S. The bounded double-complex spectral sequence proves that this arrow is a quasi-isomorphism.

For the other direction, fix q. Since $\bigwedge^q L_R$ is free over P_R, the p-complex is the polynomial Koszul resolution of its fiber at $z = 0$. Its only homology is in degree $p = 0$, where it equals

$$\bigwedge_R^q \big(L_R/(z_1, \dots, z_n)L_R\big).$$

The second augmentation is zero for $p > 0$, and for $p = 0$ evaluates the coefficient at $z = 0$. The action terms in its induced differential vanish; $\mathcal{B}^{\mathbf{c}}_{X_k}(f)(0) = 0$ for every polynomial f, since $k \geq m \geq 1$. Only constant bracket coefficients survive. Since $[X_i, X_j] = (j - i)X_{i+j}$, these are precisely (3.5) when the sum lies in the basis range. If $i + j > m + n - 1$, its expression in that range has coefficients of positive degrees $i + j - k$, which vanish at the origin. The induced differential is consequently $\lambda\partial$. This proves the right-hand quasi-isomorphism over S_0.

Now localize S_0 by all the scalars in (3.6). The shifted sectors have the same nonresonance property, since

$$\sum_{i\in I}(c_i + \lambda \mathbf{1}_{i\in S}) + \lambda s = \sum_{i\in I} c_i + \lambda(s + |I \cap S|).$$

The coordinate reductions in Theorem 2.2 use only the inverted scalars, and the resulting Artin classes are independent over the fraction field. Thus each homogeneous quotient is free over this localization. Its augmented Lie–Rinehart resolution is split exact, and remains so after every base change to R satisfying (3.6). The polynomial Koszul resolution also survives every such base change. The same two augmentations prove (3.7) over R.

All constructions commute with coordinate permutations. Each X_k is itself invariant under those permutations, so the action on its fiber class u_k is trivial. □

The homology of the finite model gives an exact universal torsion description. Put

$$\mathsf{H}_q = H_q(\mathfrak{w}_{m,n}; \mathbb{C}), \qquad \mathsf{B}_q = \operatorname{im}\left(\partial : \bigwedge^{q+1} \mathfrak{w}_{m,n} \to \bigwedge^{q} \mathfrak{w}_{m,n}\right),$$

and set $\mathsf{B}_{-1} = 0$. All of these spaces retain their polynomial grading. Choosing homogeneous complements to cycles and boundaries gives, for any R in Theorem 3.6,

$$H_q(\mathcal{K}_\bullet) \cong (R \otimes_{\mathbb{C}} \mathsf{H}_q) \ \oplus \ ((R/\lambda R) \otimes_{\mathbb{C}} \mathsf{B}_q) \ \oplus \ (\mathrm{Ann}_R(\lambda) \otimes_{\mathbb{C}} \mathsf{B}_{q-1}). \tag{3.9}$$

The decomposition is not canonical, but its summands and gradings are specified. Indeed, in the chosen decomposition the only nonzero blocks of ∂ identify a complement to the cycles in degree q with B_{q-1}; multiplying these blocks by λ gives (3.9) directly. For the universal polynomial ring, or its nonresonance localization, λ is a nonzero divisor, so the last summand is absent. The entire torsion is killed by one power of λ.

Over $\mathbb{C}$, at $\lambda = 0$ the homology is the full exterior space $\bigwedge^q \mathfrak{w}_{m,n}$, whereas at any $\lambda \neq 0$ it is H_q. The first homology is completely explicit

$$[\mathfrak{w}_{m,n}, \mathfrak{w}_{m,n}] = \langle u_{2m+1}, \dots, u_{m+n-1} \rangle,$$

where the span is zero if the lower limit exceeds the upper. For the reverse inclusion one uses $[u_m, u_{j-m}] = (j - 2m)u_j$. Thus, at a nonzero parameter,

$$H_1(\mathcal{K}_\bullet) \cong \bigoplus_{j=m}^{\min(2m,m+n-1)} \mathbb{C}u_j, \qquad \dim H_1 = \min(m+1, n).$$

Over the universal ring, its free summands have these same degrees, and it has one λ-torsion summand in each degree $2m+1, \dots, m+n-1$.

For the original one-parameter problem take $m = 1$, $c_i = 1$, and $\lambda = \alpha$. On the polynomial-form hit quotients of Theorem 3.5, the cube differential becomes

$$\kappa_\alpha = \sum_i (1 + \alpha E_i)^{-1} M_{z_i} \, \iota_{dz_i}. \tag{3.10}$$

This follows by conjugating (3.4) with the sectorwise operators T_S used there. It has total polynomial degree zero, its square is zero, and it descends to the hit quotients. For every $\alpha \notin \mathcal{E}_n$ its complex is quasi-isomorphic to $(\bigwedge^\bullet \mathfrak{w}_{1,n}, \alpha\partial)$. In particular, all its homology carries the trivial S_n-action. At $\alpha = 0$ it recovers the exterior classes in degrees $1, \dots, n$; at every allowed $\alpha \neq 0$, its first homology has one class in each of degrees $1, 2$ when $n \geq 2$, and only the degree-1 class when $n = 1$.

Corollary 3.7 (Stable pentagonal degrees)**.** *For the original polynomial-form complex* (3.10)*, let* $\alpha \notin \mathcal{E}_n$ *and* $\alpha \neq 0$*. For every homological degree* $p \geq 1$ *and every polynomial degree* $0 \leq d \leq n$*,*

$$\dim_{\mathbb{C}} H_p(\kappa_\alpha)_d = \begin{cases} 1, & d = (3p^2 - p)/2 \text{ or } d = (3p^2 + p)/2, \\ 0, & \text{otherwise.} \end{cases}$$

Every nonzero group in this formula is the trivial S_n*-module.*

Proof. In total polynomial degree $d \leq n$, every exterior basis index and every bracket output is at most n. The complex of $\mathfrak{w}_{1,n}$ is therefore exactly the degree-d part of the Chevalley–Eilenberg complex of the infinite positive Witt algebra, with basis u_j, $j \geq 1$. Goncharova's theorem gives its positive-degree homology in precisely the two displayed weights [9]; see also [10, Theorem 1.4] for the stated grading convention. Cohomology and homology have identical dimensions in each fixed weight, since their finite-dimensional complexes are dual. The result now follows from Theorem 3.6. □

4. **Monomial reflection groups and residues**

We return to the coordinate filtration. Arithmetic progressions gave the $G(h, 1, n)$ quotients; the product invariant of $G(h, p, n)$ requires a thicker coordinate boundary. Let K be a field, put $P = K[z_1, \ldots, z_n]$, and fix $p \mid h$ and $a = h/p$. Over $\mathbb{C}$, the group $G(h, p, n)$ consists of monomial matrices whose nonzero entries are h-th roots of unity and whose product of nonzero entries has a-th power one. Set $\pi = z_1 \cdots z_n$, retain the operators $B_{hk}(c, \lambda)$, $k \geq 1$, and add

$$J^B = (c_{[n]} + \lambda E) M_{\pi^a}.$$

Theorem 4.1. *Suppose*

$$\begin{aligned} c_I + \lambda s &\neq 0 && \big(\varnothing \neq I \subsetneq [n], \quad s \geq h|I|\big), \\ c_{[n]} + \lambda s &\neq 0 && \big(s \geq an\big), \end{aligned}$$

where s is integral. Then the quotient by the joint images of B_{hk}, $k \geq 1$, and J^B has Hilbert series

$$[an]_q \prod_{j=1}^{n-1} [hj]_q.$$

A basis is the disjoint union, over $1 \leq i \leq n$, of the monomials z^b with

$$0 \leq b_j < hj \ (j < i), \qquad 0 \leq b_i < a, \qquad a \leq b_j < a + h(j-1) \ (j > i).$$

Over $\mathbb{C}$, for equal coefficients, the quotient is the graded $G(h, p, n)$ coinvariant module.

Over $\mathbb{C}$, the same conclusions hold for $A_{hk}(c, \lambda)$ together with

$$J^A = M_{\pi^a}(c_{[n]} + \lambda E)$$

if the proper-subset condition above holds and, in addition,

$$c_i + \lambda s \neq 0 \quad (1 \leq i \leq n, \ s \geq 0), \qquad c_{[n]} + \lambda s \neq 0 \quad (s \geq 0).$$

Proof. For the lower bound in the B case, use the free module of derivations with basis

$$X_{hk} = \sum_i z_i^{hk+1} \partial_i \quad (1 \leq k < n), \qquad Y = \pi^a E.$$

Its coefficient determinant is, up to sign, $\pi^{a+1} \prod_{i<j} (z_j^h - z_i^h)$, hence is nonzero. It is closed under brackets. Besides the Witt brackets, one has

$$[X_{hk}, Y] = a p_{hk} Y - hk \pi^a X_{hk}.$$

The characteristic identity, with e_j evaluated in $z_1^h, \ldots, z_n^h$, gives

$$\sum_{k=1}^{n} (-1)^{k-1} e_{n-k} X_{hk} = \pi^{h-a} Y.$$

It expresses X_{hn}, and then every higher X_{hk}, in this basis.

Use the cocycle and operators from the preceding universal construction,

$$\eta(X) = \lambda \operatorname{div} X + \sum_i (c_i - \lambda) \frac{X(z_i)}{z_i}, \qquad \mathcal{B}_X = \lambda X + M_{\eta(X)}.$$

They give $\mathcal{B}_{X_{hk}} = B_{hk}$ and $\mathcal{B}_Y = J^B$. Right balance and the commutator identity give the first two Koszul deformation maps exactly as before. Over the field of independent coefficients $K(c_1, \dots, c_n)$, the comparison sequence at $\lambda = 0$ is

$$\sum_i c_i z_i^h, \quad \dots, \quad \sum_i c_i z_i^{(n-1)h}, \quad c_{[n]}\pi^a.$$

It is regular; at a common zero, $\pi = 0$, and the remaining at most $n-1$ distinct nonzero values among z_i^h are ruled out by the Vandermonde argument, since every nonempty block sum of the independent coefficients is nonzero. The rank lemma identifies the generic quotient dimension with the complete-intersection value. Specialization cannot increase matrix rank, so this is a lower bound at every parameter. For equal coefficients over $\mathbb{C}$, apply the same argument at $c_i = 1$, $\lambda = 0$ on each multiplicity space. Its comparison sequence is $p_h, \dots, p_{(n-1)h}, n\pi^a$.

For the upper bound, the image of J^B is exactly $\pi^a P$, because on a monomial of degree d its scalar coefficient is $c_{[n]} + \lambda(d + an)$. Filter $P/\pi^a P$ by

$$F_i = (z_{i+1} \cdots z_n)^a (P/\pi^a P), \qquad 0 \le i \le n.$$

The i-th layer is the direct sum of a copies, shifted by $a(n-i)+b$, $0 \le b < a$, of the $(n-1)$-variable arithmetic-progression module, with coefficients c_j for $j < i$ and $c_j + \lambda a$ for $j > i$. The omitted term raises the i-th exponent by at least $h \ge a$ and vanishes in the layer. The proper-subset hypotheses ensure nonresonance on all these layers. Thus right exactness gives

$$\operatorname{Hilb} Q \le [a]_q \sum_{i=1}^{n} q^{a(n-i)} \prod_{j=1}^{n-1} [hj]_q = [an]_q \prod_{j=1}^{n-1} [hj]_q.$$

The lifted bases are precisely the displayed disjoint union. The lower bound makes them independent and proves the character statement.

Over $\mathbb{C}$, the invariant ring of $G(h, p, n)$ is

$$\mathbb{C}[e_1(z^h), \dots, e_{n-1}(z^h), \pi^a].$$

Indeed invariance under its diagonal subgroup gives $\bigoplus_{b=0}^{p-1} \pi^{ab}\mathbb{C}[z_1^h, \dots, z_n^h]$; taking symmetric invariants and using $\pi^{ap} = e_n(z^h)$ gives the displayed ring. Newton's identities identify its positive-generator ideal with the comparison ideal used above.

Finally, under the hypotheses for A, both $J^A(P)$ and $J^B(P)$ equal $\pi^a P$. The invertible diagonal map $T = \prod_i (c_i + \lambda E_i)$ preserves this monomial ideal and conjugates every A_{hk} to B_{hk}. It therefore identifies the two quotients, and preserves each monomial line and the group action in the equal-coefficient case. This proves all assertions. □

Corollary 4.2. *Over $S = K[c_1, \dots, c_n, \lambda]$, let $\mathcal{M}^{h,p}$ be the universal quotient by B_{hk}, $k \geq 1$, and J^B. Put*

$$(\delta_1, \dots, \delta_n) = (h, 2h, \dots, (n-1)h, an).$$

The entries are indexed degrees and may repeat. For each z-degree d, $\mathcal{M}_d^{h,p}$ has a minimal linear free resolution over S with term

$$\bigoplus_{\substack{I \subseteq [n] \\ |I| = r}} S(-r) \otimes_K K[z_1, \dots, z_n]_{d - \sum_{i \in I} \delta_i}$$

in homological degree r. At every parameter point satisfying the nonvanishing assumptions of Theorem 4.1, this finite Lie–Rinehart complex remains exact in all positive homological degrees after specialization.

Proof. The derivation basis, bracket formulas, and cocycle in the preceding proof are valid in every characteristic. Its determinant is still a nonzero polynomial. At $\lambda = 0$, the comparison sequence over $K[c, z]$ is now

$$\sum_i c_i z_i^h, \quad \sum_i c_i z_i^{2h}, \quad \dots, \quad \sum_i c_i z_i^{(n-1)h}, \quad c_{[n]} \pi^a.$$

This sequence is globally regular in arbitrary characteristic. To see this, pass to an algebraic closure and stratify by zero coordinates and by the distinct nonzero values of z_i^h. If $\pi = 0$, their number r is at most $n - 1$; the first r power-sum equations force the r disjoint block sums of the coefficients to be zero. If $\pi \neq 0$, the last equation gives $c_{[n]} = 0$, supplying the zeroth power-sum equation. Together with the first $r - 1$ other equations, this again forces all r block sums to be zero by Vandermonde invertibility. In either case there are r independent coefficient constraints, while the z-stratum has dimension r. The latter statement remains valid when the characteristic divides h, since the power map is finite even when inseparable. Each stratum therefore has dimension n. The common zero set has dimension n, so the n generators have height n and form a regular sequence.

The proof of Theorem 3.2 now applies without change; the complex modulo λ is this Koszul complex, and positive-degree universal homology vanishes by graded Nakayama. All matrix entries are linear in the parameters, giving the stated minimal resolution. Its generic rank in degree d is the coefficient of $[an]_q \prod_{j=1}^{n-1} [hj]_q$.

At every nonresonant point, Theorem 4.1 gives this same fiber dimension. The local-freeness argument of Corollary 3.3 then gives exactness of the specialized complex. □

4.1. **Flat braid residues**

The coordinate flag also respects the fusion of logarithmic residues that occurs when a coordinate is deleted. This allows noncommuting residues along the braid hyperplanes, provided their flatness relations survive the fusion. Multiplication and differentiation still have the same degree here; Section 5 will treat differential terms of lower degree.

Let V be a finite-dimensional complex vector space. Assign matrices $R_{ab} = R_{ba} \in \mathrm{End}_{\mathbb{C}}(V)$ to the edges of the complete graph on $\{0, 1, \dots, n\}$. Assume the relations

$$[R_{ab}, R_{cd}] = 0 \quad (a, b, c, d \text{ distinct}), \qquad [R_{ab}, R_{ac} + R_{bc}] = 0 \quad (a, b, c \text{ distinct}). \tag{4.1}$$

Write $c_i = R_{0i}$, $b_{ij} = R_{ij}$, and let $\lambda \in \mathbb{C}$. On $P \otimes_{\mathbb{C}} V$, define

$$B_k = \sum_i (c_i + \lambda E_i) M_{z_i^k} + \sum_{i<j} b_{ij} M_{h_k(z_i, z_j)}, \qquad h_k(x, y) = \frac{x^{k+1} - y^{k+1}}{x - y}.$$

The displayed divided difference is a polynomial. For a nonempty $J \subseteq [n]$, put

$$C_J = \sum_{\substack{0 \le a < b \le n \\ \{a,b\} \cap J \neq \varnothing}} R_{ab}.$$

Theorem 4.3. *Suppose that*

$$C_J + \lambda s \, 1_V \quad \textit{is invertible for every } \varnothing \neq J \subseteq [n] \textit{ and every integer } s \ge |J|. \tag{4.2}$$

Then the classes of

$$z_1^{a_1} \cdots z_n^{a_n} \otimes v, \qquad 0 \le a_i < i,$$

where v ranges through any basis of V, form a basis of

$$(P \otimes_{\mathbb{C}} V) \Big/ \sum_{k \ge 1} B_k (P \otimes_{\mathbb{C}} V).$$

Its Hilbert series is $(\dim V)[n]_q!$.

If a finite group $G \subseteq S_n$ acts on V and satisfies

$$\rho(g) R_{ab} \rho(g)^{-1} = R_{g(a) g(b)}, \qquad g(0) = 0,$$

the quotient is isomorphic as a graded G-module to $\mathrm{Res}^{S_n}_G C_n \otimes_{\mathbb{C}} V$*, with diagonal action.*

Proof. We first verify the operator identities used for the lower bound. The matrix-valued logarithmic form

$$\omega = \sum_i (c_i - \lambda 1_V)\frac{dz_i}{z_i} + \sum_{i<j} b_{ij}\frac{d(z_i - z_j)}{z_i - z_j}$$

is closed and flat. Indeed, (4.1) are the rank-two residue relations for the hyperplanes $z_i = 0$ and $z_i = z_j$, and subtracting the central matrices $\lambda 1_V$ does not change them. The rank-two curvature calculation in Section 5.2 proves flatness directly. For the free Lie module with basis $X_k = \sum_i z_i^{k+1}\partial_i$, $1 \le k \le n$, set

$$\mathcal{B}_X = \lambda X + \lambda \operatorname{div} X + \omega(X).$$

Then $\mathcal{B}_{X_k} = B_k$, and

$$\mathcal{B}_{aX} = \mathcal{B}_X M_a, \qquad [\mathcal{B}_X, \mathcal{B}_Y] = \lambda \mathcal{B}_{[X,Y]}. \tag{4.3}$$

In particular, the characteristic-polynomial recurrence reduces all B_k to the joint images of $B_1, \dots, B_n$, and the same two-map syzygy construction as in the main proof applies.

For an unconditional lower bound, introduce a scalar u and replace the parameters by

$$\lambda(u) = u\lambda, \qquad c_i(u) = (1-u)1_V + uc_i, \qquad b_{ij}(u) = ub_{ij}.$$

All commutators in (4.1) are multiplied by u^2, so the relations persist. At $u = 0$, the first two maps are the power-sum Koszul maps tensored with V. The polynomial two-map rank lemma gives the classical lower bound in every degree at every u, hence at the desired $u = 1$. For a symmetry group G, this entire path is equivariant, so the lower bound also holds for each irreducible multiplicity.

For the upper bound, write $\pi = z_1 \cdots z_n$. In addition to the scalar characteristic identity, we have

$$\sum_{k=1}^{n} (-1)^{k-1} e_{n-k}\, h_k(z_i, z_j) = \pi.$$

To see this, the characteristic identity gives $\sum_k (-1)^{k-1} e_{n-k} z_i^{k+1} = \pi z_i$; subtract its version for j and divide by $z_i - z_j$. Consequently,

$$\sum_{k=1}^{n} (-1)^{k-1} B_k M_{e_{n-k}} = M_\pi\big(C_{[n]} + \lambda(E+n)1_V\big).$$

In polynomial degree d, the last factor is invertible by (4.2), so $\pi(P \otimes V)$ lies in the hit subspace.

Use the same coordinate-ideal flag on $(P/\pi P)\otimes V$ as in the main proof. On the layer whose largest zero coordinate is i, remove z_i, divide out $z_{i+1}\cdots z_n$, and shift degree by $n-i$. The new residues are

$$c'_j = c_j + b_{ij} + \lambda \mathbf{1}_{j>i}\, 1_V, \qquad b'_{jk} = b_{jk} \quad (j,k\neq i). \tag{4.4}$$

Here $h_k(0,z_j) = z_j^k$ gives the fusion term b_{ij}; the derivative of the divided-out monomial gives the scalar λ-shift.

The relations survive this fusion. Ignoring the central shifts, replace the vertex 0 by the block $\{0,i\}$; $R'_{0j} = R_{0j} + R_{ij}$. For example, the new triangle relation expands as

$$\begin{aligned}[R_{0j}+R_{ij}, R_{0k}+R_{ik}+R_{jk}] &= [R_{0j}, R_{0k}+R_{jk}] + [R_{ij}, R_{ik}+R_{jk}] \\ &\quad + [R_{0j}, R_{ik}] + [R_{ij}, R_{0k}] = 0.\end{aligned}$$

The first two terms are old triangle relations and the last two are disjoint-edge relations. All other new triangle and disjoint-edge relations follow in the same way, or are unchanged. Adding the central shifts in (4.4) preserves them.

For every nonempty remaining subset J, its new matrix C'_J is

$$C'_J = C_J + \lambda\, |J\cap\{i+1,\dots,n\}|\, 1_V.$$

Thus (4.2) is inherited by every layer. Induction on the number of variables, using right exactness of the joint-image quotient, gives the coefficientwise bound

$$\operatorname{Hilb} Q \le (\dim V)\sum_{i=1}^{n} q^{n-i}[n-1]_q! = (\dim V)[n]_q!.$$

The same induction gives precisely the Artin monomial lifts. The lower bound proves equality and their independence. Equality of total dimensions together with the irreducible lower bounds proves the equivariant assertion. □

There is also a tail version. If only $k \ge m$ are used, replace (4.2) by invertibility of $C_J + \lambda s\, 1_V$ for $s \ge m|J|$. Then the quotient has the basis (2.2) tensored with V, and

$$\operatorname{Hilb} Q = (\dim V)\prod_{j=m}^{m+n-1} [j]_q.$$

For a symmetry group G, the comparison is $P/(p_m,\dots,p_{m+n-1})\otimes V$ as a graded G-module. Indeed, fusion and the coordinate flag are unchanged. The interior recursion uses

$$B_\ell M_\pi = \sum_{k=1}^{n}(-1)^{k-1} B_{\ell+k} M_{e_{n-k}}, \qquad B_\ell^{\mathbf{c},\mathbf{b}} M_\pi = M_\pi B_\ell^{\mathbf{c}+\lambda\mathbf{1},\mathbf{b}},$$

which follow from (4.3) and the characteristic relation of the vector fields. These are exactly the two identities used in the tail induction of Theorem 2.2. The lower-bound path now starts at the regular sequence $p_m, \dots, p_{m+n-1}$.

4.1.1. **Young-content parameters for symmetric-group residues.** Put $N = n+1$, label the letters of S_N by $0, 1, \dots, n$, and let V_τ be its irreducible complex representation indexed by a partition $\tau \vdash N$. For a partition ν, write

$$\operatorname{ct}(\nu) = \sum_{(a,b)\in\nu} (b-a).$$

For $c, \kappa, \lambda \in \mathbb{C}$, take

$$c_i = c\,\mathrm{id} + \kappa(1 - (0\,i)), \qquad b_{ij} = \kappa(1-(i\,j)) \quad \text{on } V_\tau.$$

These residues satisfy the required infinitesimal braid relations. Consequently the preceding theorem has the following scalar form; if

$$rc + \kappa\left\{\binom{N}{2} - \binom{N-r}{2} - \operatorname{ct}(\tau) + \operatorname{ct}(\mu)\right\} + \lambda s \neq 0 \tag{4.5}$$

for every $1 \le r \le n$, every partition $\mu \subseteq \tau$ of size $N-r$, and every integer $s \ge r$, then the classes of

$$\{z_1^{a_1} \cdots z_n^{a_n} : 0 \le a_i < i\} \otimes V_\tau$$

form a basis of the joint-image quotient, and that quotient is isomorphic, as a graded S_n-module, to

$$C_n \otimes \operatorname{Res}^{S_N}_{S_n} V_\tau.$$

Here S_n fixes the letter 0 and acts diagonally on the tensor product. For the tail with $k \ge m$, the same statement uses $s \ge mr$ in (4.5), the basis $\mathcal{A}_{n,m} \otimes V_\tau$, and the comparison module $P_n/(p_m, \dots, p_{m+n-1}) \otimes \operatorname{Res}^{S_N}_{S_n} V_\tau$.

Indeed, let $J \subseteq \{1, \dots, n\}$ have size r, and let $\Omega_A = \sum_{\{a,b\}\subseteq A} (a\,b)$ for a set of letters A. Writing $K = \{0, \dots, n\} \setminus J$, the relevant matrix is

$$C_J = rc\,\mathrm{id} + \kappa\left[\left(\binom{N}{2} - \binom{N-r}{2}\right)\mathrm{id} - \Omega_{\{0,\dots,n\}} + \Omega_K\right].$$

The sum of transpositions acts on a Specht representation by its content sum; iterated branching to S_K has precisely the constituents V_μ with $\mu \subseteq \tau$, $|\mu| = N-r$. The displayed scalars are therefore exactly the eigenvalues of $C_J + \lambda s\,\mathrm{id}$. This proves the assertion from the matrix theorem.

For the standard representation $V_{(N-1,1)}$, the expression inside braces has eigenvalue N with multiplicity r, and eigenvalue r with multiplicity $N-r-1$; a value with

zero multiplicity is omitted. To see this directly, on the permutation representation $\sum_{\{a,b\}\cap J\neq\varnothing}(1-(a\,b))$ is the graph Laplacian of the complete graph with the edges inside K removed. Its constant eigenspace has eigenvalue zero. On the sum-zero subspace, vectors supported on K with coordinate sum zero have eigenvalue r, while their complementary r-dimensional subspace has eigenvalue N.

4.2. **Cyclotomic residues**

For the cyclotomic hyperplanes, coordinate deletion again fuses the incident residues, but intermediate powers require a refinement of the coordinate flag. This combines the two preceding constructions and includes the extra invariant of every group $G(h,p,n)$. Let $h\geq 1$, let $p\mid h$, and put $a=h/p$. Let V be a finite-dimensional complex vector space. Choose matrices c_i and $b_{ij,\zeta}$, where $i<j$ and $\zeta^h=1$, such that the rational matrix-valued one-form

$$\omega=\sum_i (c_i-\lambda 1_V)\,d\log z_i+\sum_{i<j}\sum_{\zeta^h=1} b_{ij,\zeta}\,d\log(z_i-\zeta z_j)$$

is flat. Equivalently, one may impose the rank-two residue relations (5.8) for this arrangement. The subtraction of central $\lambda 1_V$ from the coordinate residues does not affect flatness. For $k\geq 1$, define

$$\begin{aligned} B_{hk}=&\sum_i (c_i+\lambda E_i)M_{z_i^{hk}}\\ &+\sum_{i<j}\sum_{\zeta^h=1} b_{ij,\zeta}M_{(z_i^{hk+1}-\zeta z_j^{hk+1})/(z_i-\zeta z_j)}. \end{aligned} \tag{4.6}$$

All the divided differences are polynomials. For a nonempty $J\subseteq[n]$, set

$$C_J=\sum_{i\in J} c_i+\sum_{\substack{i<j\\ \{i,j\}\cap J\neq\varnothing}}\sum_{\zeta^h=1} b_{ij,\zeta},\qquad J^B=(C_{[n]}+\lambda E)M_{\pi^a},\quad \pi=\prod_i z_i.$$

Theorem 4.4. *Assume that the matrices*

$$C_J + \lambda s\, 1_V$$

are invertible for every nonempty proper $J \subsetneq [n]$ and every integer $s \geq h|J|$, and that $C_{[n]} + \lambda s\, 1_V$ is invertible for every integer $s \geq an$. Then the joint-image quotient of $P \otimes V$ by all operators in (4.6) *and by J^B has Hilbert series*

$$(\dim V)[an]_q \prod_{j=1}^{n-1} [hj]_q.$$

A basis is the monomial basis in Theorem 4.1 tensored with any basis of V. If $G(h,p,n)$ acts on V and the residues transform covariantly with their hyperplanes, the quotient is $C_{G(h,p,n)} \otimes V$ as a graded representation, with the diagonal action.

Proof. For a logarithmic vector field X, put

$$\mathcal{B}_X = \lambda X + \lambda \operatorname{div} X + \omega(X).$$

Closedness and flatness of ω give right balance and $[\mathcal{B}_X, \mathcal{B}_Y] = \lambda \mathcal{B}_{[X,Y]}$. On the free Lie module with basis $X_h, \ldots, X_{(n-1)h}, Y = \pi^a E$, from the proof of Theorem 4.1, these are precisely the operators B_{hk} and J^B. The characteristic identity reduces the remaining B_{hk} on the right. The path

$$\lambda(u) = u\lambda, \quad c_i(u) = (1-u)1_V + uc_i, \quad b_{ij,\zeta}(u) = ub_{ij,\zeta}$$

preserves flatness, as its logarithmic form is $u\omega + (1-u)\sum_i d\log z_i\, 1_V$. At zero the first two maps are the Koszul maps for the regular sequence $p_h, \ldots, p_{(n-1)h}, n\pi^a$, tensored with V. The rank lemma proves the required lower bound, including irreducible multiplicities in the equivariant case.

We first establish the upper bound for just the operators B_{hk} in m variables, assuming invertibility of every $C_J + \lambda s 1_V$ with $s \geq h|J|$. The characteristic identity in the variables z_i^h gives

$$\sum_{k=1}^{m} (-1)^{k-1} B_{hk} M_{e_{m-k}(z^h)} = M_{\pi^h}\big(C_{[m]} + \lambda(E + hm)1_V\big).$$

For the divided differences, this follows by subtracting the two identities with numerators z_i^{hk+1} and ζz_j^{hk+1}; the right side is $\pi^h(z_i - \zeta z_j)$ before division. Thus every multiple of π^h is hit.

The divided differences contain intermediate powers of the coordinate being deleted. To isolate their constant term in that coordinate, filter $P/\pi^h P$ by

$(z_{i+1}\cdots z_m)^h(P/\pi^h P)$, and refine each successive layer by the powers z_i^b, $0 \le b < h$, in decreasing order. In the associated layer, terms that raise the exponent of z_i disappear. The remaining divided difference is z_j^{hk}, regardless of ζ. Hence deletion fuses

$$c'_j = c_j + \sum_{\zeta^h=1} b_{ij,\zeta} + \lambda h \mathbf{1}_{j>i}\, 1_V, \tag{4.7}$$

with the natural interpretation if $j < i$, and retains the other braid residues. Flatness is preserved. Indeed, the coefficients of $dz_j \wedge dz_k$, for $j, k \neq i$, in $\omega \wedge \omega$ are regular at a generic point of $z_i = 0$. Their restrictions are exactly the curvature coefficients of the fused form; they vanish. This proves the rational identity on the whole smaller arrangement. The scalar shifts do not change it. Closedness is automatic. For every remaining subset J, fusion changes its C_J only by $\lambda h|J \cap \{i+1,\dots,m\}|1_V$, so all nonresonance conditions are inherited.

Induction and right exactness give

$$\operatorname{Hilb} Q \le (\dim V)[h]_q \sum_{i=1}^{m} q^{h(m-i)} \prod_{j=1}^{m-1} [hj]_q = (\dim V) \prod_{j=1}^{m} [hj]_q.$$

The same induction provides the monomials $0 \le b_j < hj$ as a spanning set. This proves the needed upper bound for the smaller layers.

For the theorem itself, the full-subset hypothesis makes the image of J^B exactly $\pi^a(P \otimes V)$. Apply the thick coordinate filtration with exponent a, refining the i-th layer by z_i^b, $0 \le b < a$, as above. Since $a \le h$, the i-th coordinate-derivative term vanishes in every such layer. Fusion is unchanged, while the central shift in (4.7) is now λa instead of λh. Each remaining layer meets the just-proved hypotheses by the proper-subset assumptions of the theorem. Therefore

$$\operatorname{Hilb} Q \le (\dim V)[a]_q \sum_{i=1}^{n} q^{a(n-i)} \prod_{j=1}^{n-1} [hj]_q = (\dim V)[an]_q \prod_{j=1}^{n-1} [hj]_q.$$

The lifted spanning monomials are exactly those stated in Theorem 4.1. The lower bounds make them independent and prove the representation statement. □

5. **Filtered logarithmic operators**

When a flat differential perturbation has lower degree than the multiplication part, a regular sequence gives normal forms at every parameter. The same argument works with scalar, matrix, or noncommutative coefficients.

Let K be a field, let $P = K[x_1, \dots, x_N]$ have positive integer weights, and let $L \subseteq \mathrm{Der}_K(P)$ be free over P, graded, and closed under Lie brackets. Choose a homogeneous basis $X_1, \dots, X_r$, with $\deg X_i = a_i$. A P-linear map $\omega : L \to P$ is closed if

$$\omega([X,Y]) = X(\omega(Y)) - Y(\omega(X)).$$

Suppose ω is homogeneous of degree $h > 0$ and $f_i = \omega(X_i)$ form a regular sequence of positive degrees $k_i = a_i + h$. Write $I = (f_1, \dots, f_r)$, and fix a graded vector-space complement V_0 to I in P. The principal example will be $\omega(X) = X(\eta)$ for a homogeneous potential η.

Theorem 5.1 (Filtered normal forms)**.** *Let $\mathscr{A}$ be a unital associative K-algebra, let $t \in \mathscr{A}$ be central, and let $a : L \to P \otimes \mathscr{A}$ be P-linear and homogeneous of degree zero. Assume*

$$t\big(X(a(Y)) - Y(a(X)) - a([X,Y])\big) + [a(X), a(Y)] = 0. \tag{5.1}$$

For left multiplication by $a(X)$, define

$$B_X(g) = t\ \mathrm{div}(gX) + \omega(X)g + a(X)g.$$

Then inclusion induces a filtered isomorphism of right $\mathscr{A}$-modules

$$V_0 \otimes \mathscr{A} \ \xrightarrow{\sim}\ (P \otimes \mathscr{A}) \Big/ \sum_i B_{X_i}(P \otimes \mathscr{A}). \tag{5.2}$$

The hit subspace is independent of the basis of L, its associated graded space is $I \otimes \mathscr{A}$, and the finite Lie–Rinehart complex is exact in positive degrees. For every left $\mathscr{A}$-module U, the same assertions hold with coefficient space U; in particular the quotient is $V_0 \otimes U$. All identifications are equivariant for compatible finite-group actions when V_0 is chosen stable.

Proof. Closedness, the divergence identity, and (5.1) give

$$B_{bX} = B_X M_b, \qquad [B_X, B_Y] = tB_{[X,Y]} \qquad (b \in P). \tag{5.3}$$

Thus $-B_X$ defines a right action for the bracket and anchor of L scaled by t. Use the finite exterior differential of (3.3), with first map $+B$. Give an exterior basis vector $X_{i_1} \wedge \cdots \wedge X_{i_p}$ filtration shift $k_{i_1} + \cdots + k_{i_p}$, and give the coefficient algebra degree zero. The leading complex is the Koszul complex of $f_1, \dots, f_r$, tensored over K with $\mathscr{A}$. Indeed, the other action terms lower shifted degree by h, and so do the bracket terms, since $\deg c_{ij}^{\ell} = a_i + a_j - a_\ell$ in $[X_i, X_j] = \sum_\ell c_{ij}^{\ell} X_\ell$.

The remaining issue is cancellation between leading multipliers; such a cancellation might otherwise create a new leading term outside I. Regularity makes each

cancellation a Koszul boundary, and the flat differential lifts that boundary without changing the image. Here is the degree reduction explicitly. For $0 \neq u = d_1 v$, suppose the highest shifted degree of v exceeds $\deg u$. Its leading part is a Koszul cycle, hence a Koszul boundary. Subtracting the corresponding lifted boundary $d_2 w$ leaves $d_1 v$ unchanged and lowers that degree. Iteration terminates, so the leading part of u lies in $I \otimes \mathscr{A}$. Conversely, every homogeneous element of this space is the leading part of an image. Therefore

$$\operatorname{gr} \sum_i B_{X_i}(P \otimes \mathscr{A}) = I \otimes \mathscr{A}.$$

The decomposition $P = I \oplus V_0$ now gives, by degree reduction,

$$P \otimes \mathscr{A} = \sum_i B_{X_i}(P \otimes \mathscr{A}) \ \oplus \ (V_0 \otimes \mathscr{A}).$$

This proves (5.2), including its filtration assertion. For a positive-degree cycle, its leading part is again a Koszul boundary; subtracting a lift and descending in degree proves exactness.

Every step remains valid with coefficients in an arbitrary U; the scalar Koszul complex is exact after tensoring with any K-vector space. The balanced identity in (5.3) gives basis independence. For a stable complement, inclusion makes the identifications equivariant. □

The proof requires only strict degree decrease of the lower terms. It therefore applies unchanged when these terms have several different degrees, provided the identities (5.3) still hold.

Corollary 5.2 (Scalar parameters)**.** *Let $\eta \in P_h$, $h > 0$, and suppose $f_i = X_i\eta$ form a regular sequence of positive degrees. Let $\chi_1, \dots, \chi_s : L \to P$ be closed P-linear maps of degree zero. For every choice of parameters,*

$$\mathscr{B}_{t,\mathbf{b}}(X)(g) = t\ \mathrm{div}(gX) + \left(X\eta + \sum_{\nu=1}^{s} b_\nu \chi_\nu(X) \right) g$$

has a joint-image quotient with basis space V_0, filtered associated graded space P/I, and an exact finite exterior complex in positive degrees. Over $R = K[t, b_1, \dots, b_s]$, the quotient is freely $V_0 \otimes R$, and this identification survives every specialization. Giving the parameters degree h makes it graded. If a finite group preserves L, η, and the cocycles, a stable V_0 gives the same conclusions equivariantly.

Proof. Take $\mathscr{A} = R$, $\omega(X) = X\eta$, and $a(X) = \sum b_\nu \chi_\nu(X)$ in the theorem. □

For matrix coefficients, retain L, η, f_i, I, V_0 from the preceding corollary. Let R be any commutative K-algebra, V a finite free R-module, and $M = P \otimes V$. Suppose $t \in R$ and the degree-zero P_R-linear map $A : L_R \to \mathrm{End}_{P_R}(M)$ satisfies

$$t\big(X(A(Y)) - Y(A(X)) - A([X,Y])\big) + [A(X), A(Y)] = 0. \tag{5.4}$$

For $B_X = t(X + \mathrm{div}\, X) + X\eta + A(X)$, inclusion gives

$$V_0 \otimes V \ \xrightarrow{\sim}\ M/\sum_i B_{X_i}(M). \tag{5.5}$$

Apply Theorem 5.1 to $\mathscr{A} = \mathrm{End}_R(V)$ and its left module V. The isomorphism is filtered with associated graded $(P/I) \otimes V$, and the finite exterior complex is exact in positive degrees. All assertions commute with arbitrary base change and are equivariant when the data and the complement are preserved. In particular, a homogeneous flat connection $\nabla_X = X + C(X)$ gives $A(X) = tC(X)$. Its flatness equation $d_L C + C \wedge C = 0$ implies (5.4); separate closedness of C is unnecessary.

A stable graded complement exists by averaging when the group order is invertible in K.

Flatness cannot be omitted. On $K[x, y]$, the operators $B_1 = M_x + \partial_y$ and $B_2 = M_y$ have regular leading multipliers x, y, but $[B_1, B_2] = 1$. Thus $f = B_1(B_2 f) - B_2(B_1 f)$ is hit for every f, and the quotient is zero.

5.1. **Free arrangements and scalar residues**

For the geometric applications, take $K = \mathbb{C}$. Let $\mathcal{A}$ be a central arrangement in $\mathbb{C}^N$, choose a linear equation ℓ_H for every $H \in \mathcal{A}$, and let

$$D(\mathcal{A}) = \{X \in \mathrm{Der}_{\mathbb{C}}(P) : X(\ell_H) \in \ell_H P \text{ for all } H\}.$$

Assume that $\mathcal{A}$ is free, so this module has a homogeneous basis $X_1, \dots, X_N$. It is closed under brackets. If its usual arrangement exponents are $e_1, \dots, e_N$, then the anchor degrees used above are $a_i = e_i - 1$.

Call a homogeneous polynomial η admissible if its restriction to each intersection flat of $\mathcal{A}$ has no critical point outside the origin. For every degree $h \geq 2$, admissible polynomials form a nonempty Zariski open subset of P_h; restriction to a fixed flat is surjective, a general degree-h form on that flat has a smooth projective zero locus, and there are only finitely many flats.

The regular-sequence consequence can be proved directly. At a point x, let X be the intersection of the hyperplanes containing x. The values at x of logarithmic derivations span exactly X. Containment follows from tangency. For the reverse inclusion, if $v \in X$, then the constant vector field in direction v, multiplied by $\prod_{H:\, x \notin H} \ell_H$, is logarithmic and takes a nonzero scalar multiple of v at x. Thus the common zero locus of all $X_i(\eta)$ is the origin when η is admissible. These N polynomials therefore form a regular sequence.

The maps

$$\chi_H(X) = \frac{X(\ell_H)}{\ell_H}$$

are polynomial, P-linear, homogeneous Lie cocycles of degree zero. Corollary 5.2 consequently applies to

$$\mathscr{B}_{t,\mathbf{b}}(X)(g) = t\, \operatorname{div}(gX) + \left(X(\eta) + \sum_{H \in \mathcal{A}} b_H \frac{X(\ell_H)}{\ell_H} \right) g \tag{5.6}$$

with every t and every collection (b_H). Its associated graded quotient is the Solomon–Terao complete intersection $P/(X_1\eta, \dots, X_N\eta)$; this terminology follows [7]. In particular, for ordinary polynomial degree,

$$\operatorname{Hilb}(\operatorname{gr}(P/\mathscr{H}_{t,\mathbf{b}}); q) = \prod_{i=1}^{N} [e_i + h - 1]_q, \qquad \dim_{\mathbb{C}} P/\mathscr{H}_{t,\mathbf{b}} = \prod_{i=1}^{N} (e_i + h - 1). \tag{5.7}$$

There are no parameter exclusions.

The same conclusion holds for any free multiarrangement with positive integer multiplicities m_H, replacing $D(\mathcal{A})$ by $\{X : X(\ell_H) \in \ell_H^{m_H} P\}$. This module is closed under brackets; if $X(\ell) = \ell^m a$ and $Y(\ell) = \ell^m b$, the terms of order $2m - 1$ cancel in $[X, Y](\ell)$. The evaluation argument uses $\prod_{H:\, x \notin H} \ell_H^{m_H}$ instead, and is otherwise unchanged.

Reflection groups. For every finite complex reflection group, its reflection arrangement is free by Terao's theorem [8, Theorem 2]. Hence (5.6)–(5.7) provide a

family for every such group. If η is group-invariant and the weights b_H are constant on hyperplane orbits, every assertion is equivariant. Such an invariant admissible potential always exists in sufficiently large degree. For example, the invariant forms of degree $2|G|$ have no projective base point; for a nonzero x, choose a linear form nonzero on its finite G-orbit and square the product of all its G-translates. The resulting invariant form does not vanish at x. Restricting this base-point-free linear system to each projective flat and applying Bertini's theorem gives a general invariant admissible potential. Zero-dimensional projective flats are handled by choosing a form nonzero at the corresponding point.

For a general complex reflection group the comparison algebra here is the displayed Solomon–Terao complete intersection, with degrees $e_i + h - 1$. The following specialization gives ordinary coinvariants for every finite real reflection group.

Corollary 5.3. *Let G be a finite real reflection group acting on the complexification of its real reflection representation. Choose orthonormal coordinates and basic homogeneous invariants $p_1, \dots, p_N$, with $\deg p_i = d_i$, and put*

$$X_i = \frac{1}{d_i} \nabla p_i, \qquad \eta = \frac{1}{2} \sum_j x_j^2.$$

For every $t \in \mathbb{C}$ and arbitrary logarithmic weights b_H, the joint-image quotient of the operators

$$g \longmapsto p_i g + t \, \operatorname{div}(g X_i) + \left(\sum_H b_H \frac{X_i(\ell_H)}{\ell_H} \right) g$$

has associated graded vector space equal to the ordinary coinvariant algebra $C_G = P/(p_1, \dots, p_N)$. Every fixed graded complement representing C_G is a basis space for the quotient at all these parameters. If the weights are constant on hyperplane orbits, this is an isomorphism of filtered G-modules. The universal quotient is free over the polynomial ring in all the displayed parameters.

Proof. The gradient fields are logarithmic because reflection invariance of p_i makes its derivative normal to a reflecting hyperplane vanish on that hyperplane. Their coefficient determinant is a nonzero constant times $\prod_H \ell_H$, by the basic-invariant Jacobian formula for real reflection groups. They therefore form a basis of $D(\mathcal{A})$; in Cramer's rule for any logarithmic field, each numerator determinant vanishes on every H, so is divisible by the same squarefree product. Finally, $X_i(\eta) = p_i$ by Euler's identity. Corollary 5.2 applies to the regular sequence of basic invariants. □

5.2. **Matrix and universal residues**

For a free central arrangement $\mathcal{A}$, choose matrices $R_H \in \mathrm{End}_R(V)$, one for each hyperplane. Suppose that for every codimension-two intersection flat Z,

$$\left[R_H, \sum_{H' \supseteq Z} R_{H'}\right] = 0 \qquad (H \supseteq Z). \tag{5.8}$$

Then

$$A(X) = \sum_{H \in \mathcal{A}} R_H \frac{X(\ell_H)}{\ell_H}$$

is polynomial on $L = D(\mathcal{A})$, separately closed, and satisfies $[A(X), A(Y)] = 0$. Consequently (5.4) holds for every t, independently of the matrices' eigenvalues or Jordan forms. The quotient in (5.5) is therefore a simultaneous free family over the coordinate ring defined by the matrix relations (5.8), together with the independent parameter t.

Here is an elementary verification of the curvature assertion. The rational matrix-valued one-form $\omega = \sum_H R_H \, d\log \ell_H$ is closed. Group the terms in $\omega \wedge \omega$ according to the codimension-two flat $Z = H \cap H'$. The group for Z is the square of the logarithmic form for the hyperplanes containing Z, regarded on the two-dimensional quotient by Z. If E is its radial Euler field, contraction of that square is

$$\left[\sum_{H \supseteq Z} R_H, \sum_{H \supseteq Z} R_H \, d\log \ell_H\right] = 0$$

by (5.8). A rational two-form on a two-dimensional vector space with zero contraction by its nonzero radial field is zero. Every group of terms therefore vanishes. Restriction to logarithmic vector fields proves the assertion about A.

For a finite real reflection group G and a representation $\rho : G \to \mathrm{GL}(V)$, a particularly natural choice is

$$R_H = k_H\big(1 - \rho(s_H)\big),$$

where s_H is reflection in H and k_H is constant on G-orbits of hyperplanes. These matrices satisfy (5.8); in the rank-two parabolic subgroup fixing Z, the sum of the weighted reflections is central, since its coefficients are constant on conjugacy classes. With the quadratic potential and gradient basis of Corollary 5.3, the associated graded quotient is

$$C_G \otimes_{\mathbb{C}} V$$

with the diagonal G-action, for every value of t and the orbit parameters k_H. More generally, arbitrary residues satisfying (5.8) give the same vector-space conclusion for every free arrangement, without a semisimplicity assumption.

The residue construction also works for every finite complex reflection group. Let $\mathcal{R}(G)$ be its set of reflections, choose parameters k_s constant on G-conjugacy classes, and set

$$R_H = \sum_{\substack{s\in\mathcal{R}(G)\\ \mathrm{Fix}(s)=H}} k_s\big(1-\rho(s)\big).$$

For a codimension-two flat Z, the sum of the R_H with $H \supseteq Z$ is a scalar minus a conjugation-invariant weighted sum of reflections in the pointwise stabilizer G_Z. It is therefore central in the image of $R[G_Z]$, while each of these R_H belongs to that image. Thus (5.8) holds. Using a G-invariant admissible potential as in Section 5.1 gives the equivariant associated graded quotient

$$\mathrm{ST}(\mathcal{A},\eta)\otimes_{\mathbb{C}} V$$

for all parameters. For real reflection groups the quadratic potential specializes this statement to the ordinary coinvariant algebra.

The universal residue algebra. For a free central arrangement, an explicit universal algebra is

$$\mathscr{A}_{\mathcal{A}} = \frac{\mathbb{C}[t]\langle r_H : H\in\mathcal{A}\rangle}{\Big(\Big[r_H, \sum_{H'\supseteq Z} r_{H'}\Big] : \operatorname{codim} Z = 2,\ H\supseteq Z\Big)},$$

where the coefficient t is central. Define

$$a(X) = \sum_H r_H\,\frac{X(\ell_H)}{\ell_H}.$$

The rank-two curvature calculation above is valid with values in any associative algebra. It proves $d_L a = 0$ and $[a(X), a(Y)] = 0$, hence (5.1). For every admissible potential, the resulting quotient is therefore free over $\mathscr{A}_{\mathcal{A}}$, with any homogeneous Solomon–Terao basis as a right-module basis. Every representation of $\mathscr{A}_{\mathcal{A}}$, including nonsemisimple and infinite-dimensional ones, inherits the same normal forms and positive-degree acyclicity. For a real reflection arrangement with the quadratic potential, the universal right-module basis can be any homogeneous basis representing C_G. The construction is equivariant under arrangement symmetries acting on the algebra by permuting the generators r_H.

5.3. **Connections in the potential parameters**

The uniform normal forms also carry an explicit flat connection. This construction gives a parameter family of finite-dimensional connections for every free arrangement, with no additional nonresonance conditions on its residues.

Retain P, L, η and the regular sequence $X_i\eta$ from Corollary 5.2, and write $h = \deg \eta$. Let $\mathscr{A}$ be a unital associative complex algebra. Fix a homogeneous flat L-connection

$$\nabla_X = X + C(X) \quad \text{on } P \otimes \mathscr{A},$$

where $C(X) \in P \otimes \mathscr{A}$ acts on the left, is P-linear in X, and has the anchor degree of X. Flatness means

$$X(C(Y)) - Y(C(X)) - C([X,Y]) + [C(X), C(Y)] = 0.$$

Choose homogeneous polynomials $\phi_1, \dots, \phi_s$ of degrees strictly smaller than h, allowing constants, and introduce central parameters $t, u_1, \dots, u_s$. Put

$$\eta_{\mathbf{u}} = \eta + \sum_{\nu=1}^{s} u_\nu \phi_\nu, \qquad B_X = t\big(\nabla_X + \operatorname{div} X\big) + X(\eta_{\mathbf{u}}).$$

Differentiating in u_ν changes B_X by $X(\phi_\nu)$. Commuting multiplication by $t^{-1}\phi_\nu$ past its differential part tX produces the opposite term. This determines the connection in the u_ν direction. In the t direction, preservation of the joint image allows the commutator to be $t^{-1}B_X$, which leads to the second operator below.

Theorem 5.4. *Let $V_0 \subseteq P$ be a fixed graded complement to $(X_1\eta, \dots, X_r\eta)$. The quotient by the joint images of B_{X_i} is naturally a free right $\mathscr{A}[t, \mathbf{u}]$-module with basis any complex basis of V_0. Its finite exterior complex is exact in positive homological degrees, over the parameter algebra and after every coefficient-module specialization.*

After inverting t, the operators

$$\mathcal{D}_\nu = \partial_{u_\nu} + t^{-1} M_{\phi_\nu}, \qquad \mathcal{D}_t = \partial_t - t^{-2} M_{\eta_{\mathbf{u}}} \tag{5.9}$$

descend to the quotient. They are mutually commuting connections in the displayed parameter directions. In the fixed normal-form basis, their matrices have coefficients in $\mathscr{A}[t, t^{-1}, \mathbf{u}]$. At $t = 0$, the u_ν-connection matrices have poles of order at most one, and the t-connection matrix has a pole of order at most two. There are no poles at other finite parameter values.

Proof. The leading polynomial-degree part of B_{X_i} is still multiplication by $X_i\eta$. The extra terms $u_\nu X_i \phi_\nu$ have strictly smaller degree, as do the connection and divergence terms. The operators remain right balanced, and their commutators equal $tB_{[X,Y]}$, because $d_L(X \mapsto X\eta_{\mathbf{u}}) = 0$ and the potential is scalar. The degree-reduction proof of Theorem 5.1 and its degree-decreasing variant therefore give the free basis and higher exactness.

On the full polynomial module, direct calculation gives

$$[\mathcal{D}_\nu, B_X] = X(\phi_\nu) - X(\phi_\nu) = 0, \qquad [\mathcal{D}_t, B_X] = t^{-1} B_X.$$

For the second identity, writing $L_X = \nabla_X + \operatorname{div} X$, the derivative term is L_X, while the multiplication commutator contributes $t^{-1} X(\eta_{\mathbf{u}})$. Thus every connection preserves the sum of the images. Moreover,

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = 0, \qquad [\mathcal{D}_\nu, \mathcal{D}_t] = -t^{-2} M_{\phi_\nu} + t^{-2} M_{\phi_\nu} = 0.$$

This proves flatness on the quotient.

The normal-form projection is right $\mathscr{A}[t, \mathbf{u}]$-linear and uses no parameter inverses. Hence the normal forms of $\phi_\nu v$ and $\eta_{\mathbf{u}} v$, for basis elements $v \in V_0$, have polynomial coefficients in the parameters. Equation (5.9) gives exactly the asserted pole orders. The algebraic exactness persists under every coefficient-module specialization. The connection identities persist under every parameter-independent representation of $\mathscr{A}$; more generally, one may use coefficient modules equipped with compatible parameter connections. A parameter derivative is not asserted to descend to a quotient that fixes that same parameter. □

For the universal residue algebra $\mathscr{A}_{\mathcal{A}}$, use $C(X) = \sum_H r_H X(\ell_H)/\ell_H$, regarding the residue generators as independent of the new central parameter t. The rank-two

relations make this a flat connection. The theorem then supplies one universal flat parameter connection whose every representation has the corresponding Solomon–Terao normal forms. For a real reflection group with quadratic potential, the rank of its finite coefficient specialization is $|G| \dim V$, and the underlying associated graded module is $C_G \otimes V$.

5.4. **Constant matrix curvature**

For constant matrix perturbations, flatness can be imposed on a quotient of the coefficient module. The following elementary observation identifies that quotient.

Let A be a commutative $\mathbb{C}$-algebra, let V be an A-module, and put $P = \mathbb{C}[x_1, \dots, x_N]$. All endomorphisms called constant below are independent of the x_i, but may depend on parameters in A.

Lemma 5.5 (Curvature reduction)**.** *Suppose*

$$B_i = tX_i + f_i + A_i \qquad (1 \le i \le s)$$

act on $P \otimes_{\mathbb{C}} V$, where the X_i are scalar polynomial derivations, $f_i \in P$, $t \in A$, and $A_i \in \mathrm{End}_A(V)$. Assume that

$$[B_i, B_j] = \sum_k a_{ij}^k B_k + C_{ij},$$

with scalar coefficients $a_{ij}^k \in A$ and constant endomorphisms C_{ij}. Let $R \subseteq V$ be the smallest submodule stable under all A_i and containing all $C_{ij}V$. If $H = \sum_i B_i(P \otimes V)$, then $P \otimes R \subseteq H$. Consequently the quotient by H is the joint-image quotient of the induced operators on $P \otimes (V/R)$.

Proof. The commutator identity puts $P \otimes C_{ij}V$ in H. If $P \otimes U \subseteq H$, then

$$g \otimes A_i u = B_i(g \otimes u) - (tX_i g + f_i g) \otimes u \in H \qquad (g \in P,\ u \in U).$$

Iteration gives $P \otimes R \subseteq H$. Stability under the A_i makes all operators descend, proving the quotient assertion. □

Theorem 5.6. *Give P a positive integral grading. Let a finite-dimensional graded Lie algebra act on P, with homogeneous derivation basis $X_1, \dots, X_s$ and constant structure coefficients*

$$[X_i, X_j] = \sum_k c_{ij}^k X_k, \qquad \deg X_i = a_i.$$

Suppose $h > 0$ and homogeneous polynomials f_i, of degrees $d_i = a_i + h > 0$, form a regular sequence and satisfy

$$X_i(f_j) - X_j(f_i) = \sum_k c_{ij}^k f_k.$$

For arbitrary constant $A_i \in \mathrm{End}_A(V)$, set

$$B_i = tX_i + f_i + A_i, \qquad C_{ij} = [A_i, A_j] - t\sum_k c_{ij}^k A_k,$$

$$R = \sum_{w,i,j} A_w C_{ij} V, \qquad W = V/R,$$

where w ranges over all words in $1, \dots, s$, including the empty word. For any homogeneous vector-space complement E to $(f_1, \dots, f_s)$, the quotient map induces a filtered isomorphism

$$E \otimes_{\mathbb{C}} W \xrightarrow{\sim} \frac{P \otimes_{\mathbb{C}} V}{\sum_i B_i (P \otimes_{\mathbb{C}} V)}.$$

The Chevalley–Eilenberg complex of the induced operators on $P \otimes W$, with bracket multiplied by t, is exact in positive homological degrees.

Proof. The cocycle identity gives $[B_i, B_j] = t\sum_k c_{ij}^k B_k + C_{ij}$, so Lemma 5.5 reduces the quotient to W. The induced operators satisfy the required Lie relations. Give a basis wedge $X_{i_1} \wedge \cdots \wedge X_{i_p}$ filtration shift $d_{i_1} + \cdots + d_{i_p}$. Multiplication by f_i preserves total filtration degree; tX_i lowers it by h, A_i by d_i, and a nonzero bracket term by $d_i + d_j - d_k = h$. The associated graded differential is the Koszul differential of the regular sequence (f_i), tensored over $\mathbb{C}$ with W. The degree-reduction argument in Theorem 5.1 now gives positive-degree exactness and the normal-form decomposition with complement $E \otimes W$. □

Corollary 5.7. *Let $\eta \in P$ have ordinary polynomial degree $h \geq 2$, and suppose the partial derivatives of its highest homogeneous part η_h form a regular sequence. For constant endomorphisms $A_1, \dots, A_N$, set*

$$B_i = t\partial_i + \partial_i \eta + A_i, \qquad R = \sum_{w,i,j} A_w [A_i, A_j] V, \qquad W = V/R.$$

If E is a homogeneous complement to the Jacobian ideal of η_h, the joint-image quotient is $E \otimes_{\mathbb{C}} W$, and the induced commuting Koszul complex is exact in positive degrees. In particular, when V is finite dimensional over $\mathbb{C}$, its dimension is $(h-1)^N \dim_{\mathbb{C}} W$.

Proof. Since $[B_i, B_j] = [A_i, A_j]$, curvature reduction makes the operators commute on W. The Koszul filtration by polynomial degree plus $p(h-1)$ has leading differential given by the regular sequence $(\partial_i \eta_h)$. The degree-reduction argument applies; the complete intersection has length $(h-1)^N$. □

The module W is the largest quotient on which the induced constant operators satisfy the required Lie relations. Its formation commutes with base change because it is the cokernel of the map with columns $A_w C_{ij}$. In finite dimension it is found by repeatedly adjoining the A_i-images of the initial curvature images. For fixed complex matrices in the gradient case, W is independent of t, and the quotient over $\mathbb{C}[t]$ is free on $E \otimes W$.

There is also a universal form. Take $V = \mathbb{C}[t]\langle a_1, \dots, a_s \rangle$, with A_i acting by left multiplication by a_i. Then

$$W = U_t(\mathfrak{g}) = \frac{\mathbb{C}[t]\langle a_1, \dots, a_s \rangle}{\left(a_i a_j - a_j a_i - t \sum_k c_{ij}^k a_k\right)_{i,j}},$$

and Theorem 5.6 identifies the quotient with the free right $U_t(\mathfrak{g})$-module $E \otimes U_t(\mathfrak{g})$.

6. **Jacobian operators and resonant fibers**

The preceding normal forms have no exceptional parameters because their differential terms lower degree. Jacobian operators instead have terms of equal degree, so resonance can occur. A single Euler identity isolates the possible resonant degree in both scalar and vector-valued families; the remaining task is to measure the defect there.

Let $P = \mathbb{C}[x_1, \dots, x_N]$, with positive integer weights w_i, and put $W = \sum_i w_i$ and $E_w = \sum_i w_i x_i \partial_i$. Let M be a graded P-module and choose $A_i \in M_{-w_i}$. Set

$$\mathbf{F} = \sum_i w_i x_i A_i, \qquad \mathscr{J} = \sum_i A_i P, \qquad Z_A = \left\{ \sum_i g_i \partial_i : \sum_i A_i g_i = 0 \right\}.$$

For $t \in \mathbb{C}$, define

$$B_{t,i}(g) = A_i g + t\mathbf{F}\,\partial_i g, \qquad \mathscr{H}_t = \sum_i B_{t,i}(P).$$

Lemma 6.1 (Euler reduction)**.** *One has $\mathscr{H}_t \subseteq \mathscr{J}$, with equality unless $t = -1/(W + d)$ for some d with $P_d \neq 0$. At that parameter, multiplication by $\mathbf{F}$ identifies $\mathscr{J}/\mathscr{H}_t$ with*

$$(P/\operatorname{div} Z_A)_d,$$

placed entirely in target degree d.

Proof. Since $\mathbf{F} \in \mathscr{J}$, the inclusion is immediate, and $\mathscr{J} = \mathscr{H}_t + \mathbf{F}P$. The identity

$$\sum_i w_i B_{t,i}(x_i g) = \mathbf{F}[1 + t(E_w + W)]g$$

kills $\mathbf{F}P_d$ in the quotient unless $1 + t(W + d) = 0$. This proves equality away from the stated parameters and concentration in degree d at resonance.

Suppose $\mathbf{F}u = \sum_i B_{t,i}(g_i)$, with $u \in P_d$, $g_i \in P_{d+w_i}$, and $t = -1/(W + d)$. Put

$$v = u - t\sum_i \partial_i g_i.$$

Then $\sum_i A_i g_i = \mathbf{F}v$. Subtract the Euler expression $\mathbf{F}v = \sum_i A_i(w_i x_i v)$; the differences $r_i = g_i - w_i x_i v$ form an actual syzygy, $\sum_i A_i r_i = 0$. Their divergence recovers u, since

$$t\sum_i \partial_i r_i = u - [1 + t(W + d)]v = u.$$

Conversely, $\sum_i A_i r_i = 0$ gives $\sum_i B_{t,i}(r_i) = t\mathbf{F}\sum_i \partial_i r_i$. Since $t \neq 0$, the kernel is precisely $(\operatorname{div} Z_A)_d$. □

6.1. **Hypersurfaces**

Theorem 6.2. *Let $F \in P$ be nonzero and weighted homogeneous of positive degree h. Put $S = \{d : P_d \neq 0\}$ and*

$$\mathscr{B}_{t,i}(g) = F_i g + tF\partial_i g, \quad F_i = \partial_i F, \quad \mathscr{H}_t(F) = \sum_i \mathscr{B}_{t,i}(P).$$

If

$$t \notin \left\{ -\frac{h}{W+d} : d \in S \right\},$$

then $\mathscr{H}_t(F) = (F_1, \ldots, F_N)$. An isolated critical point is not required for this equality.

Proof. Apply Lemma 6.1 with $M = Pe$, $\deg e = -h$, $A_i = h^{-1}F_i e$, and parameter t/h. Euler's identity gives $\mathbf{F} = Fe$, and the normalized operators are $h^{-1}\mathscr{B}_{t,i}e$. □

If F has an isolated critical point at the origin, its partial derivatives form a regular sequence, and throughout this range

$$\mathrm{Hilb}(P/\mathscr{H}_t(F); q) = \prod_i \frac{1 - q^{h-w_i}}{1 - q^{w_i}}.$$

For ordinary homogeneity of degree $h \geq 2$, this becomes

$$\mathrm{Hilb}(P/\mathscr{H}_t(F); q) = (1 + q + \cdots + q^{h-2})^N, \qquad \dim(P/\mathscr{H}_t(F)) = (h-1)^N.$$

The equality of subspaces is automatically equivariant for every finite linear group preserving F.

Under the isolated-critical-point hypothesis, the free Lie module $L = F\,\mathrm{Der}(P)$ and cocycle dF/F also place this family in Theorem 3.1, since $\omega(F\partial_i) = \mathrm{div}(F\partial_i) = F_i$. Euler reduction determines the individual specializations.

6.2. **The exact defect at every exceptional parameter**

Now assume $N \geq 2$ and that F has an isolated critical point at the origin. The Euler reduction leaves the divergence quotient of the fields tangent to F. Put

$$K_F = P/J(F), \qquad J(F) = (F_1, \ldots, F_N),$$

and retain the notation above.

Theorem 6.3. *If t is not of the form $-h/(W+d)$ with $d \in S$, then $P/\mathscr{H}_t(F) = K_F$. At $t = -h/(W+d)$, with $d \in S$, one has*

$$\mathrm{Hilb}(P/\mathscr{H}_t(F); q) = \mathrm{Hilb}(K_F; q) + q^{h+d}[u^d]\frac{\mathrm{Hilb}(K_F; u)}{1-u^h}. \tag{6.1}$$

More precisely, the quotient $J(F)/\mathscr{H}_t(F)$ is concentrated in weighted degree $h+d$ and is naturally isomorphic to

$$(\mathcal{V}_F)_d, \qquad \mathcal{V}_F = \frac{P}{\sum_{i<j}(F_j\partial_i - F_i\partial_j)(P)}.$$

The graded $\mathbb{C}[F]$-module $\mathcal{V}_F$ is free, with a homogeneous basis obtained by lifting any homogeneous vector-space basis of K_F.

If a finite linear group G preserves the weights and F, all the natural maps above are G-equivariant. In particular, the exceptional defect, as a G-module, is isomorphic to

$$\bigoplus_{j\geq 0}(K_F)_{d-jh}, \tag{6.2}$$

placed entirely in weighted degree $h+d$. Terms of negative degree are zero.

Proof. The normalized Euler family above has syzygy module $\Theta_F = \{X : X(F) = 0\}$. Since the partial derivatives form a regular sequence, this module is generated by the Koszul fields $F_j\partial_i - F_i\partial_j$, whose divergences vanish. Hence

$$\operatorname{div}\Theta_F = \sum_{i<j}(F_j\partial_i - F_i\partial_j)(P).$$

Lemma 6.1, with parameter t/h, identifies the resonant defect with $(\mathcal{V}_F)_d$, in polynomial degree $h+d$ after undoing the target shift.

We prove the claimed freeness directly. Give dx_i weight w_i, write Ω^k for polynomial differential forms, and identify P with Ω^N after multiplication by the volume form. Under this identification, the denominator defining $\mathcal{V}_F$ is

$$R = dF \wedge d\Omega^{N-2}.$$

It is preserved by multiplication by F, because $F\,dF \wedge d\xi = dF \wedge d(F\xi)$. Let $E_w = \sum_i w_i x_i \partial_i$. For any homogeneous $(N-1)$-form η of form weight L, Cartan's identity gives

$$L\,dF \wedge \eta = dF \wedge d\iota_{E_w}\eta + hF\,d\eta.$$

Since $N \geq 2$ and the weights are positive, $L > 0$. It follows that

$$dF \wedge \Omega^{N-1} = R + F\Omega^N.$$

The reverse inclusion uses Euler's identity for F. Thus $\mathcal{V}_F/F\mathcal{V}_F \cong K_F$, with coefficient degrees unchanged.

Multiplication by F on $\mathcal{V}_F$ is injective. Indeed, suppose $F\omega = dF \wedge d\xi$, with ω a homogeneous top form of weight $\ell > 0$. Then

$$dF \wedge \left(d\xi - \frac{1}{h} \iota_{E_w} \omega \right) = 0.$$

Koszul exactness at Ω^{N-1} gives $d\xi - \iota_{E_w}\omega/h = dF \wedge \zeta$. Differentiating, and using $d\iota_{E_w}\omega = \ell\omega$, gives

$$\omega = \frac{h}{\ell}\, dF \wedge d\zeta \in R.$$

This proves injectivity. Lifts of a homogeneous basis of K_F now generate $\mathcal{V}_F$ over $\mathbb{C}[F]$ by induction on degree. A relation among them reduces modulo F to a relation among the basis of K_F; its constant coefficients therefore vanish. Injectivity of F permits cancellation, and iteration proves linear independence. Hence

$$\mathrm{Hilb}(\mathcal{V}_F; q) = \frac{\mathrm{Hilb}(K_F; q)}{1 - q^h},$$

proving (6.1).

All the maps defining the defect are natural under linear transformations preserving the grading and F. Choose a graded G-stable complement to $J(F)$ in P, which exists by averaging in each degree. Its image provides equivariant Milnor-basis lifts in the preceding freeness proof. Since F is invariant, this gives $\mathcal{V}_F \cong K_F \otimes_{\mathbb{C}} \mathbb{C}[F]$ as graded G-modules, and therefore (6.2). The natural sequence $0 \to J(F)/\mathscr{H}_t(F) \to P/\mathscr{H}_t(F) \to K_F \to 0$ also splits degree by degree as a G-module sequence. □

For ordinary homogeneity and $N \geq 2$, this gives an exact exceptional set as well as the size and character of every defect. If $h \geq 3$, the polynomial $\mathrm{Hilb}(K_F; u) = (1 + u + \cdots + u^{h-2})^N$ has positive coefficients in every degree from 0 to $N(h-2)$, and $N(h-2) \geq h-1$. Thus every $-h/(N+d)$, $d \geq 0$, produces a nonzero defect. If $h = 2$, then $K_F \cong \mathbb{C}$, so a defect occurs exactly at

$$t = -\frac{2}{N+2j} \quad (j \geq 0),$$

and has dimension one, in degree $2 + 2j$.

In one variable there is a different, elementary answer. Write $F = ax^H$, with $a \neq 0$, and give x weight w. Then

$$\mathscr{B}_t(x^\ell) = a(H + t\ell)x^{H+\ell-1}.$$

The quotient equals $P/(x^{H-1})$ unless $t = -H/\ell$ for an integer $\ell \geq 1$. At such a parameter it acquires exactly one additional monomial class, $x^{H+\ell-1}$, of weighted degree $w(H + \ell - 1)$. This also explains why the $N \geq 2$ hypothesis is needed for the Brieskorn-module freeness formula above.

The actual weighted exceptional set can be stated without coefficient extraction. Let $a_1, \ldots, a_\mu$ be the degrees of a homogeneous Milnor basis, counted with multiplicity, and put

$$\sigma_j = \frac{W + a_j}{h} \qquad (1 \leq j \leq \mu).$$

For $N \geq 2$, the parameters at which the quotient differs from K_F are exactly

$$\left\{ -\frac{1}{\sigma_j + k} : 1 \leq j \leq \mu, \quad k \in \mathbb{Z}_{\geq 0} \right\}.$$

At $t = -h/(W+d)$, the defect dimension is the number of pairs (j, k) with $a_j + kh = d$, as read from the homogeneous basis $F^k v_j$ of $\mathcal{V}_F$.

6.3. **Complete intersections**

The Euler argument also applies to a vector-valued family attached to a complete intersection. The Buchsbaum–Rim complex identifies the divergence quotient with a Brieskorn module, whose rank is then given by Greuel's freeness theorem.

Let $P = \mathbb{C}[x_1, \ldots, x_N]$, with positive integer weights w_i, and write $W = \sum_i w_i$. Let $F_1, \ldots, F_r$ be a weighted homogeneous regular sequence, of degrees $h_1, \ldots, h_r > 0$, where $1 \leq r < N$. Assume

$$V\big(F_1, \ldots, F_r, I_r(J_F)\big) = \{0\}, \qquad J_F = (\partial_i F_a)_{\substack{1 \leq a \leq r \\ 1 \leq i \leq N}},$$

where $I_r(J_F)$ is the ideal of maximal minors. Thus the zero fiber is a positive-dimensional complete intersection with an isolated singularity at the origin. Put

$$\Phi_F = P/(F_1, \ldots, F_r, I_r(J_F)), \qquad M = \bigoplus_{a=1}^{r} P e_a, \quad \deg(e_a) = -h_a,$$

$$\mathbf{F} = \sum_a F_a e_a.$$

In particular, $\mathbf{F}$ has degree zero. Define

$$A_i = \sum_{a=1}^{r} \frac{\partial_i F_a}{h_a} e_a, \qquad \mathscr{B}_{t,i}(g) = A_i g + t\mathbf{F}\,\partial_i g,$$

and the graded subspaces

$$\mathscr{J}_F = \sum_i A_i P, \qquad \mathscr{H}_t = \sum_i \mathscr{B}_{t,i}(P), \qquad Q_F = M/\mathscr{J}_F.$$

The map $\mathscr{B}_{t,i} : P \to M$ has degree $-w_i$. The baseline Q_F is a Jacobian cokernel; it need not be finite dimensional.

Theorem 6.4. *For every $t \in \mathbb{C}$, one has $\mathscr{H}_t \subseteq \mathscr{J}_F$, with equality if*

$$t \notin \left\{ -\frac{1}{W+d} : d \in \sum_i w_i \mathbb{Z}_{\geq 0} \right\}.$$

At $t = -1/(W+d)$, the defect $\mathscr{J}_F/\mathscr{H}_t$ is concentrated in degree d, and its dimension is

$$b_d = [u^d]\, \frac{\mathrm{Hilb}(\Phi_F; u)}{\prod_{a=1}^{r}(1-u^{h_a})}.$$

Consequently,

$$\mathrm{Hilb}(M/\mathscr{H}_t; q) = \mathrm{Hilb}(Q_F; q) + b_d q^d.$$

More precisely, there is a canonical identification of the defect with the degree-d part of

$$\mathcal{V}_F = P/\operatorname{div}\Theta_F, \qquad \Theta_F = \{X \in \mathrm{Der}_{\mathbb{C}}(P) : X(F_a) = 0 \text{ for every } a\}.$$

The subspace $\operatorname{div}\Theta_F$ is stable under multiplication by each F_a, and $\mathcal{V}_F$ is a free $\mathbb{C}[F_1, \ldots, F_r]$-module on homogeneous lifts of a basis of Φ_F.

If a finite group G acts linearly, preserves the weights, and fixes each F_a, then the defect, with its entire underlying space placed in degree d, is isomorphic as a G-module to

$$\bigoplus_{(k_1,\ldots,k_r)\in\mathbb{Z}^r_{\geq 0}} (\Phi_F)_{\,d-\sum_a h_a k_a}.$$

The natural sequence with quotient Q_F splits as a graded G-module sequence.

Proof. Weighted Euler gives $\mathbf{F} = \sum_i w_i x_i A_i$, and the syzygy module Z_A is exactly Θ_F. Lemma 6.1 therefore gives the nonresonant equality and identifies every resonant defect with $(\mathcal{V}_F)_d$. Since $\operatorname{div}(F_a X) = F_a \operatorname{div} X$ for $X \in \Theta_F$, this quotient is a module over $\mathbb{C}[F_1, \ldots, F_r]$.

We next identify and compute this module. Set

$$D = dF_1 \wedge \cdots \wedge dF_r, \qquad R = D \wedge d\Omega^{N-r-1}.$$

We claim that multiplication by the volume form identifies $\mathcal{V}_F$ with Ω^N/R. Here and below Ω^j denotes polynomial differential forms.

To justify the syzygy assertion needed for this claim, let $Z = V(I_r(J_F))$. The positive grading and the finite-dimensionality of $P/(F_1, \ldots, F_r, I_r(J_F))$ imply, by induction on degree, that $P/I_r(J_F)$ is finite over the subalgebra generated by the F_a. Thus $F|_Z$ is finite. On each reduced irreducible component of Z, the differential of F has rank at most $r-1$; in characteristic zero its image therefore has dimension at most $r-1$. Finiteness gives $\dim Z \leq r-1$, and consequently

$$\operatorname{grade} I_r(J_F) \geq N - r + 1.$$

The Buchsbaum–Rim exactness theorem [5, Corollary 2.6] now says that $\ker J_F$ is generated by the alternating maximal-minor vectors associated to the $(r+1)$-element subsets of the columns. For each associated vector field δ, its contraction with the volume form is, up to sign, $D \wedge \beta$, where β is a constant coordinate $(N-r-1)$-form. These vector fields have zero divergence, since $d(D \wedge \beta) = 0$. It follows that

$$(\operatorname{div} \Theta_F)\, dx_1 \wedge \cdots \wedge dx_N = D \wedge d\Omega^{N-r-1} = R.$$

In particular, this identity concerns the image of the divergence map as a vector subspace, without taking an ideal closure.

The quotient by the base parameters is elementary. Give dx_i weight w_i. For a homogeneous $(N-r)$-form η of form weight L, Cartan's identity implies

$$LD \wedge \eta = D \wedge d\iota_{E_w}\eta + (-1)^{r+1}(\iota_{E_w} D) \wedge d\eta.$$

Since $N - r > 0$, $L > 0$. Every term of $\iota_{E_w} D$ is a multiple of some F_a, by weighted homogeneity. Since $D \wedge \Omega^{N-r} = I_r(J_F)\Omega^N$, this proves

$$R + (F_1, \ldots, F_r)\Omega^N = \big(I_r(J_F) + (F_1, \ldots, F_r)\big)\Omega^N. \tag{6.3}$$

Hence, with coefficient degrees unchanged, $\mathcal{V}_F/(F_1, \ldots, F_r)\mathcal{V}_F \cong \Phi_F$. Homogeneous lifts of a basis of Φ_F generate $\mathcal{V}_F$ over $\mathbb{C}[F_1, \ldots, F_r]$ by induction on weighted degree.

For independence of these lifts, we use Greuel's freeness theorem [6, Proposition 4.8, p. 261]. In the present smooth ambient space and positive fiber dimension it states that the analytic Brieskorn module

$$\frac{\Omega^N_{\mathbb{C}^N,0}}{dF_1 \wedge \cdots \wedge dF_r \wedge d\Omega^{N-r-1}_{\mathbb{C}^N,0}}$$

is a finite free module over $\mathbb{C}\{y_1, \dots, y_r\}$, with y_a acting by F_a. The assumptions hold because the zero fiber is an isolated complete intersection singularity and the ambient space is smooth. The same Euler homotopy proves (6.3) for convergent forms; division of a homogeneous component by its positive form weight preserves convergence near the origin. Its fiber is therefore Φ_F; a finite-colength polynomial quotient agrees with its analytic version, since a sufficiently high power of the maximal ideal is already zero. Nakayama's lemma and analytic freeness show that the chosen homogeneous lifts form an analytic basis.

Any polynomial relation among the same lifts in Ω^N/R would be an analytic relation among that basis. Its coefficients must vanish as germs in the independent base variables y_a, and hence as polynomials. The lifts are therefore independent over $\mathbb{C}[F_1, \dots, F_r]$. This proves polynomial freeness and the Hilbert-series formula.

Finally, the coefficient-space constructions, including divergence, are equivariant for a linear group fixing the F_a. Choose a graded G-stable complement representing Φ_F. Its lifts provide an equivariant free basis in the preceding argument, and each F_a is G-invariant. Thus

$$\mathcal{V}_F \cong \Phi_F \otimes_{\mathbb{C}} \mathbb{C}[F_1, \dots, F_r]$$

as graded G-modules. The natural sequence

$$0 \longrightarrow \mathscr{J}_F/\mathscr{H}_t \longrightarrow M/\mathscr{H}_t \longrightarrow Q_F \longrightarrow 0$$

splits degree by degree by averaging. □

If $a_1, \dots, a_\mu$ are the degrees of a homogeneous basis of Φ_F, counted with multiplicity, then the actual exceptional parameters are exactly

$$\left\{ -\frac{1}{W + a_j + \sum_{a=1}^{r} h_a k_a} : 1 \le j \le \mu, \quad (k_1, \dots, k_r) \in \mathbb{Z}^r_{\ge 0} \right\}.$$

At $t = -1/(W + d)$, the defect dimension is the number of pairs $(j, (k_1, \dots, k_r))$ with $a_j + \sum_a h_a k_a = d$. For $r = 1$, multiplying the operators by h_1, replacing the parameter by $h_1 t$, and undoing the target grading shift recovers Theorem 6.3.

The remaining codimension $r = N$ has a simpler answer and does not require a singularity-theoretic input.

Proposition 6.5. *Let $F_1, \dots, F_N$ be a weighted homogeneous regular sequence of positive degrees, and use the same normalized operators and target grading as above. Then $\mathscr{H}_t = \mathscr{J}_F$ except at the parameters*

$$t = -\frac{1}{W + d}, \qquad P_d \neq 0.$$

At such a parameter the defect is canonically P_d, placed in target degree d. In particular,

$$\mathrm{Hilb}(M/\mathscr{H}_t; q) = \mathrm{Hilb}(Q_F; q) + (\dim_{\mathbb{C}} P_d) q^d.$$

For a finite linear group preserving the weights and every F_a, this is also the exact G-module defect.

Proof. The regular sequence has finite-dimensional quotient, so P is finite over $\mathbb{C}[F_1, \dots, F_N]$. Dimension therefore forces the F_a to be algebraically independent. In characteristic zero their Jacobian determinant is nonzero, and consequently $\Theta_F = \ker J_F = 0$. Lemma 6.1 applies without $r < N$ and gives the asserted defect $P_d/(\operatorname{div}\Theta_F)_d = P_d$. □

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